

Solutions

August 14th, 2009
Math 54
Instructor: Alan Tarr

Math 54 Final Exam

Problem 1: _____/10 points

Problem 2: _____/12 points

Problem 3: _____/8 points

Problem 4: _____/15 points

Problem 5: _____/15 points

Problem 6: _____/15 points

Problem 7: _____/15 points

Total: _____/90 points

- The only materials allowed are pencil/pen and paper! No calculators, books, notes, electronics of any kind, etc.
- In order to receive full credit, you need to show your work (except on problem 1). Also, when possible explain what steps you are doing with words along with your formulas.
- Please use the space provided to give your answers, if you need more space, use the back of a page FIRST, and indicate that you did so on the front. If you still need more space then ask me to use extra pages.
- When possible use the techniques we discussed in class to check your work as you go along!
- You're almost done with math 54! Great work this summer everyone!

1. (2 pts each) Mark each of the following statements TRUE, FALSE, or NONSENSE, where NONSENSE indicates that one or more vocabulary terms has been misused in the statement. For example, in the phrase "the matrix A has dimension 2", is nonsense, because the word dimension has been used incorrectly. NO JUSTIFICATION IS REQUIRED.

- (a) If W is any subspace of an inner product space V , then $W^\perp \cap W = \{\vec{0}_V\}$, where \cap indicates intersection.

True If $\vec{w} \in W^\perp \cap W$, then $\langle \vec{w}, \vec{w} \rangle = 0 = \|\vec{w}\|^2$ so $\vec{w} = \vec{0}$.

- (b) If $T: \mathbb{P}_2 \rightarrow M^{2 \times 2}(\mathbb{R})$ is a one-to-one and onto linear transformation, then the standard matrix A_T of T is invertible.

Nonsense Standard matrix is only for $\mathbb{R}^n \rightarrow \mathbb{R}^m$!

- (c) If S is a linearly independent set of vectors in a vector space V , then S is a basis of $\text{Span}(S)$.

True S is L.I. and spans $\text{Span}(S)$ clearly.

- (d) If $P(t)$ is an $n \times n$ matrix whose columns are solutions to the system of differential equations $\vec{x}'(t) = A\vec{x}(t)$, then $P(t)$ is a fundamental matrix of the system.

False The ~~columns~~ columns need to form a basis

- (e) Every solution to the heat equation

$$u_t = \alpha^2 u_{xx}$$

$$u(0, t) = u(1, t) = 0$$

can be written in the form $u(x, t) = T(t)X(x)$.

False They may be a sum of solutions like that:

$$\sum_{n=1}^{\infty} a_n e^{-n^2 \pi^2 \alpha^2 t} \sin n\pi x$$

2. (3 pts each) Mark each of the following statements TRUE or FALSE. If you marked the answer TRUE, provide a short explanation. If FALSE, provide a counter-example. NO CREDIT WILL BE GIVEN FOR AN UNJUSTIFIED ANSWER.

(a) If $A, B,$ and C are symmetric $n \times n$ matrices, then $AB + C$ is symmetric.

False $A = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}$ $B = \begin{bmatrix} 1 & 1 \\ 1 & 1 \end{bmatrix}$, $C = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$

$$AB + C = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix} \leftarrow \text{not symmetric}$$

(b) If A and B are matrices such that $AB = I_n$, then A and B are both invertible.

False e.g. $A = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$, $B = \begin{bmatrix} 1 & 0 \end{bmatrix}$

$$AB = \begin{bmatrix} 1 \end{bmatrix}. \quad A, B \text{ not invertible}$$

(c) If $S = \{\vec{v}_1, \vec{v}_2, \dots, \vec{v}_n\}$ is a set of vectors in some vector space, and

$$a_1 \vec{v}_1 + a_2 \vec{v}_2 + \dots + a_n \vec{v}_n = \vec{0}$$

for some choice of scalars $a_1, a_2, \dots, a_n \in \mathbb{R}$, then S is a linearly dependent set.

False e.g. $S = \left\{ \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right\} \subseteq \mathbb{R}^2$

Then $0 \begin{bmatrix} 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$, but S is L.I.

(d) If A is a matrix, then

$$\text{Row}(A) = \text{Row}(A^t A),$$

where $\text{Row}(B)$ indicates the row space of a matrix B , and B^t indicates transpose.

True

$$\begin{aligned} \text{Row}(A) &= (\text{Row}(A)^\perp)^\perp = (\text{Nul } A)^\perp \stackrel{\text{from HW problem in 6.5}}{=} (\text{Nul } A^t A)^\perp \\ &= \text{Row}(A^t A) \quad \text{tough problem!} \end{aligned}$$

3. (8 pts) Compute

$$\det \begin{bmatrix} 1 & 2 & 4 & 2 \\ 3 & 3 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 3 & 2 & 1 & 0 \end{bmatrix} \leftarrow \text{cofactor exp}$$

$$= (-1) \det \begin{bmatrix} 1 & 4 & 2 \\ 3 & -1 & 0 \\ 3 & 1 & 0 \end{bmatrix}$$

↑
cofactor exp.

$$= (-1)(+1)(2) \det \begin{bmatrix} 3 & -1 \\ 3 & 1 \end{bmatrix} = \boxed{-12}$$

4. Consider the matrix

$$A = \begin{bmatrix} -2 & 2 & -2 \\ -1 & 2 & -3 \\ 0 & 1 & -2 \end{bmatrix}$$

(a) (7 pts) Find all the eigenvalues of A , and for each eigenvalue, find a basis of the associated eigenspace. *Hint: A is not invertible.*

$$\det(A - \lambda I) = \begin{vmatrix} -2-\lambda & 2 & -2 \\ -1 & 2-\lambda & -3 \\ 0 & 1 & -2-\lambda \end{vmatrix} = \begin{vmatrix} -2-\lambda & 2 \\ -1 & 2-\lambda \end{vmatrix} \begin{vmatrix} -2-\lambda & 2 \\ 0 & 1 \end{vmatrix} = (\lambda+2)^2(2-\lambda) + 2 - 3(2+\lambda) - 2(2+\lambda)$$

e. values are $0, -1, -1$

$$E_0 = \text{Nul} \begin{bmatrix} -2 & 2 & -2 \\ -1 & 2 & -3 \\ 0 & 1 & -2 \end{bmatrix} = \text{Nul} \begin{bmatrix} +1 & -1 & +1 \\ 0 & 1 & -2 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} \right\}$$

$$E_{-1} = \text{Nul} \begin{bmatrix} -1 & 2 & -2 \\ -1 & 3 & -3 \\ 0 & 1 & -1 \end{bmatrix} = \text{Nul} \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix} \right\}$$

(b) (8 pts) Find a fundamental matrix of the system $\vec{x}'(t) = A\vec{x}(t)$, and use it to compute e^{tA} .

Need generalized e. vector for e. value -1 :

$$\text{Nul} \begin{bmatrix} -1 & 2 & -2 \\ -1 & 3 & -3 \\ 0 & 1 & -1 \end{bmatrix}^2 = \text{Nul} \begin{bmatrix} -1 & 2 & -2 \\ -2 & 4 & -4 \\ -1 & 2 & -2 \end{bmatrix} = \text{Nul} \begin{bmatrix} +1 & -2 & +2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} = \text{Span} \left\{ \begin{bmatrix} 0 \\ 1 \\ 1 \end{bmatrix}, \begin{bmatrix} 2 \\ +1 \\ 0 \end{bmatrix} \right\}$$

We get solution to $\vec{x}'(t) = A\vec{x}(t)$: $e^{-t} \begin{bmatrix} +2 \\ +1 \\ 0 \end{bmatrix} + te^{-t} \begin{bmatrix} -1 & 2 & -2 \\ -1 & 3 & -3 \\ 0 & 1 & -1 \end{bmatrix} \begin{bmatrix} 2 \\ 1 \\ 0 \end{bmatrix} = \begin{bmatrix} 2e^{-t} + 0 \\ e^{-t} + te^{-t} \\ te^{-t} \end{bmatrix}$

$$P(t) = \begin{bmatrix} e^0 & 0 & 2e^t \\ 2e^0 & e^{-t} & e^{-t} + te^{-t} \\ e^0 & e^{-t} & te^{-t} \end{bmatrix} = \begin{bmatrix} 1 & 0 & 2e^{-t} \\ 2 & e^{-t} & e^{-t} + te^{-t} \\ 1 & e^{-t} & te^{-t} \end{bmatrix} \quad P(0) = \begin{bmatrix} 1 & 0 & 2 \\ 2 & 1 & 1 \\ 1 & 1 & 0 \end{bmatrix}$$

$$P(0)^{-1} : \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 2 & 1 & 1 & | & 0 & 1 & 0 \\ 1 & 1 & 0 & | & 0 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & 2 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & 1 & -2 & | & -1 & 0 & 1 \end{bmatrix} \sim \begin{bmatrix} 1 & 0 & +2 & | & 1 & 0 & 0 \\ 0 & 1 & -3 & | & -2 & 1 & 0 \\ 0 & 0 & 1 & | & 1 & -1 & 1 \end{bmatrix}$$

$$\sim \begin{bmatrix} 1 & 0 & 0 & | & +2 & -2 \\ 0 & 1 & 0 & | & 1 & -2 & 3 \\ 0 & 0 & 1 & | & 1 & -1 & 1 \end{bmatrix}$$

$$e^{tA} = \begin{bmatrix} 1 & 0 & 2e^{-t} \\ 2 & e^{-t} & e^{-t} + te^{-t} \\ 1 & e^{-t} & te^{-t} \end{bmatrix} \begin{bmatrix} -1 & 2 & -2 \\ -1 & -2 & 3 \\ 1 & -1 & 1 \end{bmatrix} = \begin{bmatrix} 2e^{-t} - 1 & 2 - 2e^{-t} & 2e^{-t} - 2 \\ 2 + 2e^{-t} & 4 - 3e^{-t} - te^{-t} & -4 + 4e^{-t} + te^{-t} \\ -1 + e^{-t} + te^{-t} & 2 - 2e^{-t} - te^{-t} & -2 + 3e^{-t} + te^{-t} \end{bmatrix}$$

5. (a) (10 pts) Find the Fourier Series expansion of the function

$$f(x) = \begin{cases} 0 & \text{for } -1 \leq x < 0 \\ 1 & \text{for } 0 < x \leq 1 \end{cases}$$

On the interval $[-1, 1]$. As usual, write your answer in the form

$$f(x) \sim \frac{a_0}{2} + \sum_{n=1}^{\infty} a_n \cos(n\pi x) + \sum_{m=1}^{\infty} b_m \sin(m\pi x)$$

$$a_0 = \int_{-1}^1 f(x) dx = \int_0^1 dx = 1$$

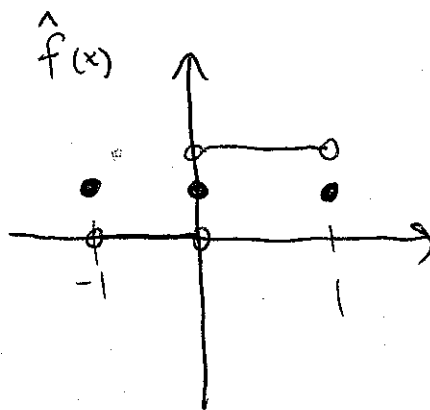
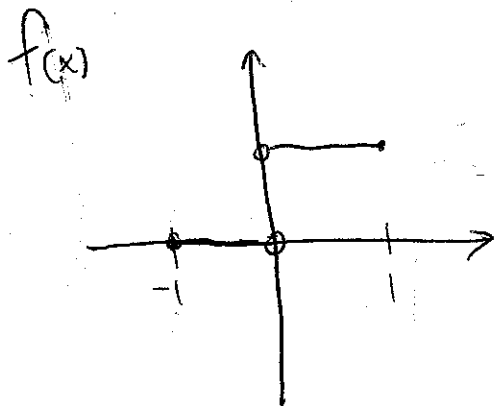
$$a_n = \int_{-1}^1 f(x) \cos(n\pi x) dx = \int_0^1 \cos(n\pi x) dx = \frac{1}{n\pi} [\sin(n\pi x)]_0^1 = 0$$

$$b_n = \int_{-1}^1 f(x) \sin(n\pi x) dx = \int_0^1 \sin(n\pi x) dx = -\frac{1}{n\pi} [\cos(n\pi x)]_0^1 = \frac{1}{n} - \frac{(-1)^n}{n}$$

$$f(x) \sim \frac{1}{2} + \sum_{n=1}^{\infty} \frac{(1 - (-1)^n)}{n\pi} \sin(n\pi x) = \frac{1}{2} + \frac{2}{\pi} \sin(\pi x) + \frac{2}{3\pi} \sin(3\pi x) + \dots$$

$$= \frac{1}{2} + \sum_{n=0}^{\infty} \frac{2}{(2n+1)\pi} \sin((2n+1)\pi x)$$

- (b) (5 pts) On separate axes, on the interval $[-1, 1]$, draw the functions f and \hat{f} , where \hat{f} is the function that the Fourier Series of f converges to. Indicate with a thick dot the values of these functions where there is some kind of discontinuity.



6. (15 pts) Consider the following partial differential equation.

$$\begin{aligned} u_{xx} + 2u_x &= u_t \\ u(0, t) = u(\pi, t) &= 0 \\ u(x, 0) &= f(x), \end{aligned}$$

where $u(x, t)$ is defined for $t > 0$ and $0 < x < \pi$.

(a) (5 pts) What would a function of the form $u(x, t) = X(x)T(t)$ need to satisfy in order for it to be a solution to the PDE above? State your answer in terms of ODEs in terms of the functions $X(x)$ and $T(t)$, as well as any boundary conditions they may need to satisfy.

Plug in $X(x)T(t)$:

$$\begin{aligned} X''T + 2X'T &= XT' \\ \Rightarrow \frac{T'}{T} &= \frac{X'' + 2X'}{X} = \text{constant } k \end{aligned}$$

since each side depends on a separate variable.

$$\Rightarrow \boxed{\begin{aligned} T' - kT &= 0 \quad \text{no condition} \\ X'' + 2X' - kX &= 0 \quad \text{where } X(0) = X(\pi) = 0 \end{aligned}}$$

(b) (10 pts) Find a solution to the PDE that satisfies the initial condition

$$u(x, 0) = e^{-x} \sin x - e^{-x} \sin 2x$$

Solution to X equation above:

char. equation is $r^2 + 2r - k = 0 \implies r = \frac{-2 \pm \sqrt{4 + 4k}}{2} = -1 \pm \sqrt{1+k}$

Since we want an initial condition with trig functions, we want $1+k < 0$, i.e. $k < -1$. This gives solutions

$$X_g(x) = e^{-x} (C_1 \cos(\sqrt{-1-k}x) + C_2 \sin(\sqrt{-1-k}x)) \quad \text{now boundary conditions}$$

$$X(0) = 0 \implies e^{-0} (C_1 \cos 0 + C_2 \sin 0) = C_1 = 0.$$

So $X_g(x) = C_2 e^{-x} \sin(\sqrt{-1-k}x)$ and $X(\pi) = 0 \implies C_2 e^{-\pi} \sin(\pi\sqrt{-1-k}) = 0$

Thus we need $\sqrt{-1-k} \pi = n\pi \implies \boxed{k = -1 - n^2}$ w/ solution $(e^{-x} \sin(nx))$.
 T equations: $T' + (1+n^2)T = 0 \implies T = ce^{-(1+n^2)t}$. The initial condition we want

comes from $n=1, n=2$. So we get
$$\boxed{u(x, t) = e^{-x} \sin x e^{-2t} - e^{-x} \sin 2x e^{-5t}}$$

7. (15 points) Let $T: V \rightarrow W$ be a one-to-one and onto linear transformation. Show that $\dim V = \dim W$.

Rank Theorem for linear transformations:

$$\begin{aligned} \dim T(V) + \dim N(T) &= \dim V \\ \text{" } T \text{ is onto" } \quad \text{" } T \text{ is 1-1" } & \\ \dim W + 0 &= \dim V \end{aligned}$$

Alternatively, let $\{\vec{v}_1, \dots, \vec{v}_n\}$ be a basis for V .

Claim: $\{T\vec{v}_1, \dots, T\vec{v}_n\}$ is a basis for W .

Proof: L.I.: Suppose

$$a_1 T(\vec{v}_1) + \dots + a_n T(\vec{v}_n) = \vec{0}_W \quad \text{Then}$$

$$\vec{0}_W = T(a_1 \vec{v}_1 + \dots + a_n \vec{v}_n), \quad \text{and since } T \text{ is 1-1,}$$

$$a_1 \vec{v}_1 + \dots + a_n \vec{v}_n = \vec{0}_V, \quad \text{so } a_1 = a_2 = \dots = a_n = 0.$$

Spans: Suppose $\vec{w} \in W$. Then, since T is onto, there is

a $\vec{v} \in V$ such that $T(\vec{v}) = \vec{w}$. Also, \vec{v} can be written

$$\vec{v} = b_1 \vec{v}_1 + b_2 \vec{v}_2 + \dots + b_n \vec{v}_n \quad \text{since } \{\vec{v}_1, \dots, \vec{v}_n\} \text{ is a basis}$$

Therefore $\vec{w} = T(\vec{v}) = b_1 T(\vec{v}_1) + b_2 T(\vec{v}_2) + \dots + b_n T(\vec{v}_n)$ so $\{T\vec{v}_1, \dots, T\vec{v}_n\}$ spans W .

Since V and W have bases of the same size, $\dim V = \dim W$.