Math 54 Professor K. A. Ribet

Final Exam December 17, 2005

This was a 3-hour exam, 5–8 PM (ugh!). There were 9 problems, worth 4, 7, 8, 7, 7, 6, 8, 7, and 6 points, respectively. The point values total 60 for this exam, which is intended to represent 45% of the course. My guess is that students will find that there is a good mix of easy and hard problems and that time will not be a big factor. (I hope that a lot of you finish early and run off to have fun.)

*Please put away all books and electronic devices. You may refer to a single 2-sided sheet of notes. Your paper is your ambassador when it is graded. Correct answers without appropriate supporting work will be regarded skeptically. Incorrect answers without appropriate supporting work will receive no partial credit. This exam has 10 pages (and 9 problems). Please write your name on each page. At the conclusion of the exam, please hand in your paper to your GSI. The notations "***DE***" and "***FS***" are provided for Math 49 students. If you are one of those students, write "Math 49" prominently on the cover of your exam.*

Oh, yeah: problems 4 and 6 were "DE" problems that were supposed to be on differential equations, while problems 5 and 9 were the "FS" problems—Fourier series.

1. *Determine bases for the row and column spaces of* $\begin{bmatrix} 1 & 2 & -3 \\ 2 & 4 & 5 \\ -3 & -6 & 0 \end{bmatrix}$.

This 3×3 matrix has rank at most 2 because the second column is twice the first. The rank is 2 because the third column is not a multiple of the first column. A basis for the column space is the set consisting of the first and third columns, for example. The row space has dimension equal to the dimension of the column space, i.e., to 2. A basis consists of any pair of rows, since no two rows are proportional. One might ask for a linear dependence relation among the three rows; there has to be one, if you believe in linear algebra, but none is obvious to me. Well, OK: 15 times row #1 + 9 times row #2 + 11 times row #3 seems to be 0.

2. *Let V be the vector space of 3×3 real matrices. Let W be the set of matrices $A \in V$ such that $A^{\mathrm{T}} = -A$. Is W a subspace of V? If so, find a basis for W.*

Yes, W is closed under addition and scalar multiplication, and it's non-empty (because it contains the 0-matrix). Hence W is a subspace. There is an obvious basis with three elements. The matrices in the basis have a single 1 in one of the three positions above the diagonal and a corresponding -1 in the position below the diagonal gotten by reflecting the chosen position through the diagonal. All other entries are 0. The dimension of the space is 3.

Find the inverse of the matrix $\begin{bmatrix} 1 & 2 & 3 & 1 \\ 1 & 3 & 3 & 2 \\ 2 & 4 & 3 & 3 \\ 1 & 1 & 1 & 1 \end{bmatrix}$.

The answer is $\begin{bmatrix} 1 & -2 & 1 & 0 \\ 1 & -2 & 2 & -3 \\ 0 & 1 & -1 & 1 \\ -2 & 3 & -2 & 3 \end{bmatrix}$, as I see on my computer. I hope that you'll all get this by applying a sequence of elementary row operations to the matrix gotten by writing the identity matrix to the right of given initial matrix.

3. *Let* $v = [1, -1, -1, 1]^{\mathrm{T}}$ *and* $w = [1, 1, 1, 1]^{\mathrm{T}}$. *For* $x \in \mathbf{R}^4$, *let* $T(x) = (x \cdot v)v + (x \cdot w)w$, *where "·" is the usual dot product of vectors. Show that* T *is a linear transformation from* $\mathbf{R}^4$ *to* $\mathbf{R}^4$. *Find two eigenvectors of* T *and one non-zero vector* x *such that* $T(x) = 0$. *(The last sentence will be amended to read: "Find two eigenvectors of* T, *one with eigenvalue 0 and one with a non-zero eigenvalue.")*

That T is linear is easy to show, using the linearity of the dot product in each variable. I won't write down the details. Note now that $v \cdot w = 1 - 1 - 1 + 1 = 0$; v and w are perpendicular. Thus $T(v) = (\|v\|^2)v$, which means that v is an eigenvector with the non-zero eigenvalue $\|v\|^2$. Similarly, w is an eigenvector. To complete the answer to the question, we need to find an x with $T(x) = 0$, which means an x that's perpendicular to the plane spanned by v and w. There is a whole plane of such vectors x. One possible x is $[1, -1, 1, -1]^{\mathrm{T}}$.

4. *Let* A *be a* 2×2 *matrix such that*

$$A \begin{bmatrix} -1 \\ 4 \end{bmatrix} = -4 \begin{bmatrix} -1 \\ 4 \end{bmatrix}, \quad A \begin{bmatrix} 0 \\ 1 \end{bmatrix} = -4 \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \begin{bmatrix} -1 \\ 4 \end{bmatrix}.$$

Find functions $x(t)$ *and* $y(t)$ *with initial values* $x(0) = -2$, $y(0) = 11$ *that satisfy the system of differential equations* $\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} = A \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$.

There is a unique matrix A with the indicated properties, since $v := \begin{bmatrix} -1 \\ 4 \end{bmatrix}$ and $w := \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ form a basis of $\mathbf{R}^2$. The matrix is $\begin{bmatrix} -8 & -1 \\ 16 & 0 \end{bmatrix}$. In fact, we are talking about the exact same system that was in one of the practice problems. The vector v is an eigenvector for A with eigenvalue 0 while w is a pseudo-eigenvector. The general solution is $X(t) = C_1 e^{-4t} v + C_2 e^{-4t}(tv + w)$, where $X(t) = \begin{bmatrix} x(t) \\ y(t) \end{bmatrix}$. We are given that $\begin{bmatrix} -2 \\ 11 \end{bmatrix} = X(0) = C_1 v + C_2 w = \begin{bmatrix} -C_1 \\ 4C_1 + C_2 \end{bmatrix}$. This leads to the values $C_1 = 2$, $C_2 = 3$.

5. *Suppose that $f(x) = 0$ for $-\pi < x < 0$, $f(x) = 1$ for $0 \le x \le \pi$, and $f(x + 2\pi) = f(x)$ for $x \in \mathbf{R}$. As usual, write the Fourier series for $f(x)$ as $\frac{a_0}{2} + \sum_{m=1}^{\infty} (a_m \cos mx + b_m \sin mx)$, Calculate the numbers a_m $(m \ge 0)$ and b_m $(m > 0)$.*

In class, we studied the function $g(x)$ that's -1 from $-\pi$ to 0 and $+1$ from 0 to π. We have $f(x) = (g(x) + 1)/2$, so that the Fourier series for f will be a simple variant of the Fourier series for g, which we have already calculated. The number a_0 is $\frac{1}{\pi}\int_0^\pi 1\,dx = 1$. The a_n with $n > 0$ are all 0, as you can see by integrating $\cos nx\,dx$ or by noting that $f(x)$ is a constant function plus an odd function. We have $b_m = \frac{1}{\pi}\int_0^\pi \sin mx\,dx = \frac{1}{m\pi}(1-\cos m\pi)$. The number $1 - \cos m\pi$ is 0 for m even and 2 for m odd.

6. *Describe all pairs of numbers (y_0, y_0') such that the solution $y(t)$ to the initial value problem $y'' - 2y' - 3y = 0$, $y(0) = y_0$, $y'(0) = y_0'$ satisfies $y(t) \to 0$ as $t \to +\infty$.*

This is a totally standard second-order homogeneous ordinary linear differential equation with constant coefficients. The associated characteristic equation, $r^2 - 2r - 3 = 0$, has roots $+3$ and -1. The general solution is $y(t) = C_1e^{-t} + C_2e^{3t}$. Clearly, $y(t) \to 0$ for large t if and only if $C_2 = 0$. If $y(t)$ satisfies the initial value conditions, then $y_0 = y(0) = C_1 + C_2$ and $y_0' = y'(0) = -C_1 + 3C_2$. We have $C_2 = 0$ if and only if $y_0' = -y_0$. The pairs (y_0, y_0') that make y tend to 0 are those of the form $(a, -a)$.

7. *Let A be a matrix whose null space is $\{0\}$. Explain carefully why each of the following statements is true: The rank of A equals the number of columns of A; The rows of A are linearly independent if and only if A is a square matrix; The product $A^{\mathrm{T}}A$ of the transpose of A and A is an invertible matrix.*

Let's say that A is an $m \times n$ matrix: n columns, m rows. A general theorem, which I hope that you feel free to quote, is that n is the sum of the rank of A and the dimension of the null space of A. If the latter number is 0, then the rank of A is n, which is the number of columns. This gives the first statement. The rank is also the "row rank," i.e., the dimension of the row space. This is the space spanned by the m rows. Since it has dimension n, we must have $m \ge n$. In general, a spanning set is linearly independent if and only if the number of elements in the set is the dimension of the space being spanned. Here, we see that the rows are linearly independent if and only if $m = n$; this gives the second statement. The third statement follows from the stuff that we did when we talked about least squares and such. Namely, $A^{\mathrm{T}}A$ and A have the same null space (Theorem 4.18 on page 258 of Hill). In this case, the null space is 0. The matrix $A^{\mathrm{T}}A$ is thus a square $(n \times n)$ matrix with 0 null space. Accordingly, it is invertible (Theorem 1.50 on page 47 of Hill).

8. *Let V be the vector space of all continuous functions on the real line. Consider the inner product $f \cdot g = \int_0^1 f(x)g(x)\,dx$ on V. Find a non-zero function that is orthogonal to the constant function 1 and to the functions x and x^2.*

Think about applying the Gram–Schmidt process to the sequence of functions $1, x, x^2, \ldots$. The fourth element of the sequence spat out by this process will be orthogonal to the first three elements and therefore to the span of the first three elements. This span will include 1, x, and x^2. Therefore, the fourth element of the sequence is an answer to this problem. That element will be a polynomial x^3+ax^2+bx+c. This suggests a very direct way to do the problem: write down x^3+ax^2+bx+c and view a, b and c as numbers that are determined by the vanishing of three integrals. This leads to three equations in three unknowns: $3+4a+6b+12c=0$, $12+15a+20b+30c=0$, $10+12a+15b+20c=0$. Solving, we get the polynomial $x^3-\frac{3}{2}x^2+\frac{3}{5}x-\frac{1}{20}$ as a non-zero function that is orthogonal to 1, x and x^2. Clearing denominators, we can give $20x^3-30x^2+12x-1$ as an alternative answer to the question. I started the G–S process on the string $1, x, x^2, \ldots$ and got as far as 1, $x-\frac{1}{2}$ and $x^2-x+\frac{1}{6}$. I didn't work out the next element of the series, which will be $x^3-\frac{3}{2}x^2+\frac{3}{5}x-\frac{1}{20}$. I checked my results with `http://mathworld.wolfram.com/LegendrePolynomial.html`, which was validating. Conclusion: I'm going to amend this problem so that you need only be orthogonal to 1 and x.

9. *Solve the partial differential equation $100u_{xx} = u_t$ on the region $0 < x < 1$, $t > 0$, subject to the boundary conditions $u(0,t) = u(1,t)$ for $t > 0$ and $u(x,0) = \sin 2\pi x - \sin 5\pi x$ for $0 \le x \le 1$.*

This exam has its share of misprints! I wanted to say $u(0,t) = u(1,t) = 0$, and I'll add that at the board.

Suppose that we had this problem with the simpler condition $u(x,0) = \sin 2\pi x$. I'd take $u(x,t) = e^{at}\sin 2\pi x$, with a to be determined. Then $u_t = au$ while $u_{xx} = -4\pi^2 u$. To have $u_t = 100u_{xx}$, we need $a = -400\pi^2$, and we'd get $u(x,t) = e^{-400\pi^2 t}\sin 2\pi x$.

If we had $\sin 2\pi x$ instead of $\sin 5\pi x$, the 400 would turn into 2500, and we'd have $u(x,t) = e^{-2500\pi^2 t}\sin 5\pi x$.

The answer to this question is then the difference between the functions u in the previous two paragraphs.