FINAL EXAM (RIBET)

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(1) TRUE/FALSE
(a) If $A$ is a square invertible matrix, then $A$ and $A^{-1}$ have the same rank.
(b) If $A$ is an $m \times n$ matrix and if $b$ is in $\mathbb{R}^m$, there is a unique $x \in \mathbb{R}^n$ for which $\|Ax - b\|$ is smallest.
(c) If $A$ is an $n \times n$ matrix, and if $v$ and $w$ satisfy $Av = 2v$, $Aw = 3w$, the $v \times w = 0$.
(d) If the dimensions of the null spaces of a matrix and its transpose are equal, then the matrix is square.
(e) If $A$ is a $2 \times 2$ matrix, then $-1$ cannot be an eigenvalue of $A^2$.
(f) I likes the linear algebra portion of this course more than the differential equations portion.
(g) If 4 linearly independent vectors lie in $\text{Span} \{w_1, \cdots, w_n\}$, then $n$ must be at least 4.
(h) If $B$ is invertible, then the column spaces of $A$ and $AB$ are equal.
(i) If $A$ is a matrix, then the row spaces of $A$ and $A^T A$ are equal.
(j) If 2 symmetric $n \times n$ matrices $A$ and $B$ have the same eigenvalues, then $A = B$.
(k) If the characteristic polynomial of $A$ is $p(\lambda) = (\lambda - 1)(\lambda + 1)(\lambda - 3)^2$, then $A$ has to be diagonalizable.

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(2) Consider the following vectors:

\[ v_1 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 0 \end{bmatrix}, \quad v_2 = \begin{bmatrix} 0 \\ 1 \\ 1 \\ 0 \end{bmatrix}, \quad v_3 = \begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix} \]

Find \( w_1, w_2, w_3 \) such that \( \{w_1, w_2, w_3\} \) is an orthogonal basis for \( \text{Span} \{v_1, v_2, v_3\} \).

(3) Solve the following system of differential equations:

\[
\begin{align*}
  x_1'(t) &= -2x_1(t) + 2x_2(t) \\
  x_2'(t) &= 2x_1(t) + x_2(t)
\end{align*}
\]

and \( x_1(0) = -1, x_2(0) = 3 \).

(4) Find bases for \( \text{Nul}(A), \text{Row}(A), \text{Col}(A) \), where:

\[
A = \begin{bmatrix} 1 & 1 & 3 & 2 \\ 3 & 1 & 1 & 0 \\ 4 & 2 & 4 & 2 \end{bmatrix}
\]

(5) Find the first 4 terms \( A_0, A_1, A_2, A_3 \) of the Fourier cosine series of \( f(x) = |\sin(x)| \).

**Hint:** \( \sin(A) \cos(B) = \frac{1}{2} [\sin(A + B) + \sin(A - B)] \)

(6) Solve the following PDE:

\[
\begin{align*}
  \frac{\partial u}{\partial t} &= 20 \frac{\partial^2 u}{\partial x^2} & 0 < x < \pi, \quad t > 0 \\
  u(0, t) &= u(\pi, t) = 0 & t > 0 \\
  u(x, 0) &= \sin(3x) - \sin(4x) & 0 < x < \pi
\end{align*}
\]

(7) Suppose \( v_1, \ldots, v_n \) are vectors in \( \mathbb{R}^n \) and that \( A \) is an \( n \times n \) matrix.

If \( Av_1, \ldots, Av_n \) form a basis for \( \mathbb{R}^n \), show that \( v_1, \ldots, v_n \) form a basis of \( \mathbb{R}^n \) and that \( A \) is invertible.
(8) Let \( v_1 = \begin{bmatrix} 0 \\ 5 \\ -2 \end{bmatrix}, \ v_2 = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}, \ v_3 = \begin{bmatrix} 9 \\ 8 \\ 7 \end{bmatrix} \)

Suppose \( A \) is the \( 3 \times 3 \) matrix for which \( Av_1 = v_1, \ Av_2 = 0, \ Av_3 = 5v_3 \).
Find an invertible matrix \( P \) and a diagonalizable matrix \( D \) such that \( A = PDP^{-1} \).