

## FINAL EXAM (BERGMAN) - ANSWER KEY

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(1) (a) Characteristic polynomial:  $\lambda^3 + 3\lambda^2 - 4 = (\lambda - 1)(\lambda + 2)^2$

Eigenvalues:  $\lambda = 1, -2$

Eigenvectors:

$$\lambda = 1 : \text{Span} \left\{ \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} \right\}$$

$$\lambda = -2 : \text{Span} \left\{ \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} \right\}$$

(b)

$$\mathbf{x}(t) = Ae^t \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} + Be^{-2t} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + C \left( te^{-2t} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + e^{-2t} \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} \right)$$

**Note:** For the last vector, use generalized eigenvectors!

(c) Define  $w = u'$ , then we get:

$$\begin{cases} u' = w \\ w' = u'' = -u' - u + v = -u + w + v \\ v' = 3u' + 6u - 2v = 6u + 3w - 2v \end{cases} \Rightarrow \begin{cases} u' = w \\ w' = -u + w + v \\ v' = 6u + 3w - 2v \end{cases}$$

which gives us the system in (a).

(d)

$$\mathbf{x}(t) = \frac{1}{4}e^t \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix} - \frac{1}{6}e^{-2t} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + \frac{1}{4} \left( te^{-2t} \begin{bmatrix} 1 \\ -2 \\ 3 \end{bmatrix} + e^{-2t} \begin{bmatrix} -\frac{1}{3} \\ \frac{5}{3} \\ 1 \end{bmatrix} \right)$$

(2) (a)

$$W(f, g) = \begin{vmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{vmatrix} = f(t)g'(t) - f'(t)g(t)$$

(b) Suppose  $f$  and  $g$  are linearly dependent.Then there are constants  $a$  and  $b$  such that  $af(t) + bg(t) = 0$  for all  $t$ Now differentiating this, we get  $af'(t) + bg'(t) = 0$  for all  $t$ In other words, the following equation has a solution for every  $t$ :

$$\begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix} \begin{bmatrix} a \\ b \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

But this means that (for every  $t$ ), the above matrix is not invertible, hence its determinant is  $= 0$  (for every  $t$ ). That is, for every  $t$ :

$$f(t)g'(t) - f'(t)g(t) = 0$$

Thus, for every  $t$ :

$$W(f, g) = 0$$

(3) (a)

$$\begin{cases} b''(y) = \lambda b(y) \\ a'(x) = \lambda p(x)a(x) \end{cases}$$

(b)

$$u(x, y) = e^{\frac{-x^2}{2}} \cos(y)$$

(just choose  $\lambda = -1$  and solve the corresponding equation! Also, you're allowed to set your constants to whatever you want them to be!)

(4)

$$\frac{1}{\sqrt{2}} \begin{bmatrix} 1 \\ -i \\ 0 \end{bmatrix}, \frac{1}{\sqrt{6}} \begin{bmatrix} 1 \\ i \\ 2 \end{bmatrix}$$

(5) Ignore this question, it is outside of the scope of this course!

(6) (a)

$$b_n = \frac{4}{\pi n} \left( 1 - \cos\left(\frac{\pi n}{2}\right) \right)$$

(b)

$$u(x, t) = \sum_{m=1}^{\infty} b_m e^{-\alpha \pi m t} \sin(\pi m x) = \sum_{m=1}^{\infty} \left( \frac{2}{\pi m} \left( 1 - \cos\left(\frac{\pi n}{2}\right) \right) \right) e^{-\alpha \pi m t} \sin(\pi m x)$$

(c) 54/Practice Exams/Bergmangraph.png

