ABEL'S FORMULA

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Abel's formula: Suppose $y'' + P(t)y' + Q(t)y = 0$. Then:

$$W[y_1, y_2](t) = Ce^{-\int P(t)dt}$$

Proof: This is actually **MUCH** easier than you think! First of all, by definition:

$$W[y_1, y_2](t) = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1 y_2' - y_1' y_2$$

Now differentiate:

$$(W[y_1, y_2](t))' = \cancel{y_1' y_2'} + y_1 y_2'' - y_1'' y_2 - \cancel{y_1' y_2'} = y_1 y_2'' - y_1'' y_2$$

Now since y_1 and y_2 both satisfy the differential equation above, we have:

$$y_1'' + P(t)y_1' + Q(t)y_1 = 0 \qquad \Rightarrow \qquad y_1'' = -P(t)y_1' - Q(t)y_1$$

$$y_2'' + P(t)y_2' + Q(t)y_2 = 0 \qquad \Rightarrow \qquad y_2'' = -P(t)y_2' - Q(t)y_2$$

Hence:

$$\begin{aligned} y_1 y_2'' - y_1'' y_2 =& y_1 \left(-P(t)y_2' - Q(t)y_2\right) - \left(-P(t)y_1' - Q(t)y_1\right) y_2 \\ =& - P(t)y_1 y_2' - \cancel{Q(t)y_1 y_2} + P(t)y_1' y_2 + \cancel{Q(t)y_1 y_2} \\ =& - P(t)\left(y_1 y_2' - y_1' y_2\right) \end{aligned}$$

But this is just equal to $-P(t)W[y_1, y_2](t)$, hence:

$$(W[y_1, y_2](t))' = -P(t)W[y_1, y_2](t)$$

All we need to do is to solve this differential equation, but this is very similar to Math $1B$:

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$$\begin{aligned}\frac{(W[y_1,y_2](t))'}{W[y_1,y_2](t)} =& - P(t)\\(\ln|W[y_1,y_2](t)|)' =& - P(t)\\\ln|W[y_1,y_2](t)| =& - \int P(t)\\W[y_1,y_2](t) =& Ce^{-\int P(t)}\end{aligned}$$

Then we get what we want: $\boxed{W[y_1,y_2](t) = Ce^{-\int P(t)}}$.

Note: To apply this to $ay'' + by' + cy = 0$, divide by a to get $y'' + \frac{b}{a}y' + \frac{c}{a}y = 0$ and let $P = \frac{b}{a}$ and $Q = \frac{c}{a}$. Finally, you can solve for C by setting $t = t_0$ in both sides of the equation. That's how you get the more complicated formula:

$$W[y_1,y_2](t) = W[y_1,y_2](t_0)e^{-\int_{t_0}^{t} P(s)ds}$$