Partial Differential Equations Handout

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This handout is meant to give you a couple more examples of all the techniques discussed in chapter 10, to counterbalance all the dry theory and complicated applications in the differential equations book! Enjoy! :)

1 Boundary-Value Problems

Find the values of λ (eigenvalues) for which the following differential equations has a nonzero solution. Also find the corresponding solutions (eigenfunctions)

$$y'' + \lambda y = 0$$

$$y(0) = 0, y'(\pi) = 0$$

The auxiliary polynomial is $r^2 + \lambda = 0$, which gives $r = \pm\sqrt{-\lambda}$. Now we need to proceed with 3 cases:

Case 1: $\lambda < 0$

Then $\lambda = -\omega^2$, where $\omega > 0$, so: $r = \pm\omega$, and the general solution is:

$$y(t) = Ae^{\omega t} + Be^{-\omega t}$$

Then $y(0) = 0$ gives $A + B = 0$, so $B = -A$, whence:

$$y(t) = Ae^{\omega t} - Ae^{-\omega t}$$

Then:

$$y'(t) = A\omega e^{\omega t} + A\omega e^{-\omega t}$$

Then $y'(\pi) = 0$ gives:

$$A\omega e^{\omega\pi} + A\omega e^{-\omega\pi} = 0$$

Cancelling out $A \neq 0$ (otherwise $B = 0$ and $Y(y) = 0$), we get:

$$e^{\omega\pi} + e^{-\omega\pi} = 0$$

Multiply by $e^{\omega\pi}$:

$$e^{2\omega\pi} + 1 = 0$$

$$e^{2\omega\pi} = -1$$

However, this doesn't have a solution because $e^{2\omega\pi} > 0$, contradiction.

Case 2: $\lambda = 0$. Then we have a double-root $r = 0$, and:

$$y(t) = Ae^{0t} + Bte^{0t} = A + Bt$$

Then $y(0) = 0$ gives $A = 0$, and so $y(t) = Bt$. And $y'(\pi) = 0$ gives $B = 0$, but then $y(t) = 0$, contradiction.

Case 3: $\lambda > 0$. Then $\lambda = \omega^2$, where $\omega > 0$.

Then we get $r = \pm\omega i$, so:

$$y(t) = A\cos(\omega t) + B\sin(\omega t)$$

Then: $y(0) = 0$ gives $A = 0$, so:

$$y(t) = B\sin(\omega t)$$

Then

$$y'(t) = \omega B\cos(\omega t)$$

So $y'(\pi) = 0$ gives:

$$\omega B\cos(\omega\pi) = 0$$

Cancelling out ω and B (because $\omega > 0$, and because $B \neq 0$, otherwise $B = 0$ and $Y(y) = 0$), we get:

$$\cos(\omega\pi) = 0$$

Which tells you that $\omega\pi = \frac{\pi}{2} + \pi M$, where M is an integer, so:

$$\omega = M + \frac{1}{2}, (M = 0, 1, 2\cdots)$$

Answer:
This tells you that the eigenvalues are:

$$\lambda = \omega^2 = \left(M + \frac{1}{2}\right)^2, (M = 0, 1, 2, \cdots)$$

And the corresponding eigenfunctions are:

$$y(t) = B\sin(\omega t) = B_M \sin\left(\left(M + \frac{1}{2}\right)t\right)$$

2 Separation of variables

Use the method of separation of variables to $u_t = u_{xx}$ to convert the PDE into two differential equations

Suppose $u(x,t) = X(x)T(t)$

Then plug this back into $u_t = u_{xx}$:

$$(X(x)T(t))_t = (X(x)T(t))_{xx}$$

$$X(x)T'(t) = X''(x)T(t)$$

Now group the X and the T:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)}$$

Now notice that $\frac{X''(x)}{X(x)}$ *only* depends on x, but also, by the above equation *only* depends on t, hence it is a constant:

$$\frac{X''(x)}{X(x)} = \lambda$$

which gives $\boxed{X''(x) = \lambda X(x)}$.

Moreover: $\frac{T'(t)}{T(t)} = \frac{X''(x)}{X(x)} = \lambda$, so $\boxed{T'(t) = \lambda T(t)}$.

3 Fourier series

3.1 Find the Fourier series of $f(x) = x^2$ on the interval $(-3, 3)$

Here $(-T, T) = (-3, 3)$, so $\boxed{T = 3}$

$$f(x) = \sum_{M=0}^{\infty} A_M \cos\left(\frac{\pi M x}{3}\right) + B_M \sin\left(\frac{\pi M x}{3}\right)$$

Now calculate A_M and B_M:

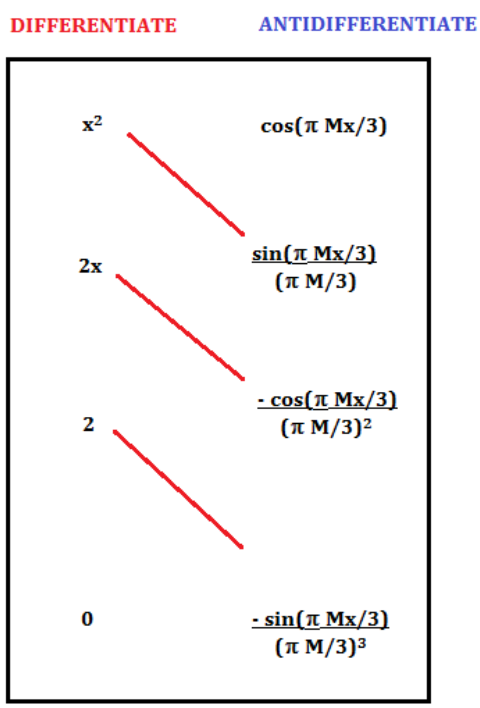
$$A_0 = \frac{\int_{-3}^{3} f(x)dx}{\int_{-3}^{3} 1dx} = \frac{\int_{-3}^{3} x^2 dx}{6} = \frac{\frac{54}{3}}{6} = 3$$

$$A_M = \frac{\int_{-3}^{3} f(x)\cos\left(\frac{\pi M x}{3}\right)dx}{\int_{-3}^{3} \cos^2\left(\frac{\pi M x}{3}\right)dx} = \frac{\int_{-3}^{3} x^2\cos\left(\frac{\pi M x}{3}\right)dx}{3} = \frac{2}{3}\int_0^3 x^2\cos\left(\frac{\pi M x}{3}\right)dx$$

where we used the fact that $x^2\cos\left(\frac{\pi M x}{3}\right)$ is even!

Now, to evaluate the integral, use tabular integration:

54/Handouts/Tabular Integration.png



$$\frac{2}{3}\int_0^3 x^2 \cos\left(\frac{\pi M x}{3}\right) dx = \frac{2}{3}\left[+x^2\left(\frac{\sin\left(\frac{\pi M x}{3}\right)}{\frac{\pi M}{3}}\right) - 2x\left(\frac{-\cos\left(\frac{\pi M x}{3}\right)}{\left(\frac{\pi M}{3}\right)^2}\right) + 2\left(\frac{-\sin\left(\frac{\pi M x}{3}\right)}{\left(\frac{\pi M}{3}\right)^3}\right)\right]_0^3$$

$$= \frac{2}{3}(-6)\left(\frac{-\cos(\pi M)}{\left(\frac{\pi M}{3}\right)^2}\right)$$

$$= \frac{2}{3}(6)\left(\frac{9(-1)^M}{(\pi M)^2}\right)$$

$$= \frac{36(-1)^M}{\pi^2 M^2}$$

Now for B_M: First set $\boxed{B_0 = 0}$ (this is just by definition), and:

$$B_M = \frac{\int_{-3}^{3} f(x) \sin\left(\frac{\pi M x}{3}\right) dx}{\int_{-3}^{3} \sin^2\left(\frac{\pi M x}{3}\right) dx} = \frac{\int_{-3}^{3} x^2 \sin\left(\frac{\pi M x}{3}\right) dx}{3} = 0$$

because the numerator is the integral of an odd function over $(-3, 3)$, hence 0.

Putting everything together, we get:

$$f(x) = 3 + \sum_{M=1}^{\infty} \frac{36(-1)^M}{\pi^2 M^2} \cos\left(\frac{\pi M x}{3}\right)$$

3.2 To which function does the Fourier series of f converge to?

$$f(x) = \begin{cases} x & -2 < x < 0 \\ 1 & 0 \leq x < 2 \end{cases}$$

Fact: The Fourier series converges to $f(x)$ whenever f is **continuous** at x, and to $\frac{f(x-)+f(x+)}{2}$ whenever f is discontinuous at x. As for the endpoints, the Fourier series converges to $\frac{f(L+)+f(R-)}{2}$, where R is the rightmost endpoint, and L is the leftmost endpoint.

Discontinuity: Here the only discontinuity is at 0, hence at 0, the F.S. converges to:

$$\frac{f(0-)+f(0+)}{2} = \frac{0+1}{2} = \frac{1}{2}$$

Endpoints: $L = -2$, $R = 2$, so at -2 and 2, the F.S. converges to:

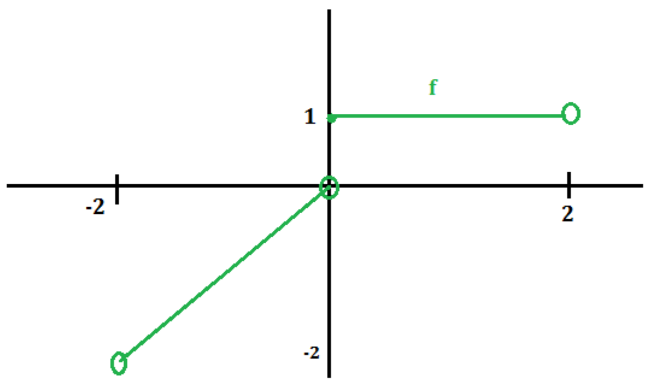
$$\frac{f((-2)^{+})+f(2^{-})}{2} = \frac{-2+1}{2} = -\frac{1}{2}$$

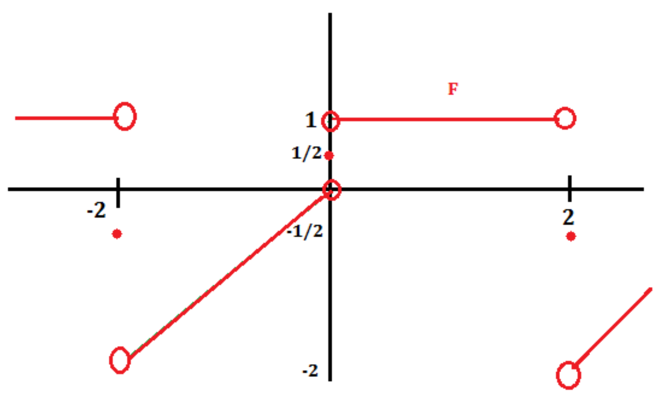
Putting everything together, we find that the F.S. converges to $\mathcal{F}$, where:

$$\mathcal{F}(x) = \begin{cases} -\frac{1}{2} & x = -2 \\ x & -2 < x < 0 \\ \frac{1}{2} & x = 0 \\ 1 & 0 < x < 2 \\ -\frac{1}{2} & x = 2 \end{cases}$$

Note: Technically, $\mathcal{F}$ is a periodic fuction of period 4, so you'd have to 'repeat' the graph, just like the picture below!

54/Handouts/Convergence.png


f
1
-2
2
-2


F
1
1/2
-2
-1/2
2
-2

4 Fourier cosine and sine series

Same thing as before, except that we're expressing a function **only** in terms of $\cos$ or **only** in terms of $\sin$. The formulas are almost the same, except that we need to multiply things by 2 and we only integrate from 0 to T.

4.1 Calculate the Fourier *cosine* series of $f(x) = x$ on $(0, \pi)$

Notice that it doesn't matter that the function f is odd, because we're only focusing on the half-interval $(0, \pi)$ and not on the full interval $(-\pi, \pi)$.

Here $T = \pi$, and our goal is to find A_m ($m = 0, 1, 2, \cdots$) such that:

$$\sum_{m=1}^{\infty} A_m \cos(mx) = x$$

As usual, always treat the case $m = 0$ separately, and notice the changes:

$$A_0 = \frac{2}{2\pi} \int_0^{\pi} x dx = \left(\frac{2}{2\pi}\right)\left(\frac{\pi^2}{2}\right) = \frac{\pi}{2}$$

And if $m \neq 0$:

$$\begin{aligned} A_m &= \frac{2}{\pi} \int_0^{\pi} x \cos(mx) dx \\ &= \frac{2}{\pi}\left(\left[x\frac{\sin(mx)}{m}\right]_0^{\pi} - \int_0^{\pi} \frac{\sin(mx)}{m} dx\right) \\ &= \frac{2}{\pi}\left(0 - \left[\frac{-\cos(mx)}{m^2}\right]_0^{\pi}\right) \\ &= \frac{2}{\pi}\left(\frac{\cos(m\pi)}{m^2} - \frac{1}{m^2}\right) \\ &= \frac{2}{\pi m^2}\left((-1)^m - 1\right) \end{aligned}$$

Hence $\boxed{A_m = \frac{2}{\pi m^2}\left((-1)^m - 1\right)}$

$$x \text{ “}=\text{” } \frac{\pi}{2} + \sum_{m=1}^{\infty} \frac{2}{\pi m^2}\left((-1)^m - 1\right)\cos(mx)$$

Now notice that if m is even, then $(-1)^m - 1 = 0$, and hence $A_m = 0$. And if m is odd,then $(-1)^m - 1 = -2$, so $A_m = \frac{-4}{\pi m^2}$

Therefore:

$$x\text{“}=\text{”}\frac{\pi}{2} + \sum_{m=1, modd}^{\infty} \frac{-4}{\pi m^2}\cos(mx)$$

$$x\text{“}=\text{”}\frac{\pi}{2} - \frac{4}{\pi}\sum_{k=1}^{\infty} \frac{-4}{\pi(2k-1)^2}\cos((2k-1)x)$$

This is because every odd number $m \geq 1$ can be written as $m = 2k-1$, where $k = 1, 2, \cdots$.

4.2 Calculate the Fourier *sine* series of $f(x) = x$ on $(0, \pi)$

Here $T = \pi$, and our goal is to find B_m ($m = 0, 1, 2, \cdots$) such that:

$$\sum_{m=1}^{\infty} B_m \sin(mx) = x$$

As usual, always treat the case $m = 0$ separately, namely set $B_0 = 0$.

And if $m \neq 0$:

$$\begin{aligned}
B_m &= \frac{2}{\pi}\int_0^{\pi} x\sin(mx)dx \\
&= \frac{2}{\pi}\left(\left[-x\frac{\cos(mx)}{m}\right]_0^{\pi} - \int_0^{\pi}\frac{-\cos(mx)}{m}dx\right) \\
&= \frac{2}{\pi}\left(-\pi\cos(m\pi) + \left[\frac{\sin(mx)}{m^2}\right]_0^{\pi}\right) \\
&= \frac{2}{\pi}\left(-\pi(-1)^m\right) \\
&= 2(-1)^{m+1}
\end{aligned}$$

Hence $\boxed{B_m = 2(-1)^{m+1}}$, and:

$$x``=" \sum_{m=1}^{\infty} 2(-1)^{m+1} \sin(mx)$$

5 The Heat equation

Problem: Solve the following **heat equation**:

$$\begin{cases} \dfrac{\partial u}{\partial t} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < 1, \quad t > 0 \\ u(0,t) = u(1,t) = 0 & t > 0 \\ u(x,0) = x & 0 < x < 1 \end{cases} \tag{5.1}$$

Step 1: Separation of variables

Suppose:

$$u(x,t) = X(x)T(t) \tag{5.2}$$

Plug (5.2) into the differential equation (5.1), and you get:

$$\begin{aligned} (X(x)T(t))_t =& (X(x)T(t))_{xx} \\ X(x)T'(t) =& X''(x)T(t) \end{aligned}$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{T'(t)}{T(t)} \tag{5.3}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on x, but by (5.3) *only* depends on t, hence it is constant:

$$\begin{aligned} \frac{X''(x)}{X(x)} =& \lambda \\ X''(x) =& \lambda X(x) \end{aligned} \tag{5.4}$$

Also, we get:

$$\begin{aligned} \frac{T'(t)}{T(t)} =& \lambda \\ T'(t) =& \lambda T(t) \end{aligned} \tag{5.5}$$

but we'll only deal with that later (Step 4)

Step 2:

Consider (5.4):

$$X''(x) = \lambda X(x)$$

Note: Always start with $X(x)$, do **NOT** touch $T(t)$ until right at the end!

Now use the **boundary conditions** in (5.1):

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(1,t) = X(1)T(t) = 0 \Rightarrow X(1)T(t) = 0 \Rightarrow X(1) = 0$$

Hence we get:

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(1) = 0 \end{cases} \tag{5.6}$$

Step 3: Eigenvalues/Eigenfunctions

The auxiliary polynomial of (5.6) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

Now use $X(0) = 0$ and $X(1) = 0$:

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$$X(1) = 0 \Rightarrow Ae^{\omega} - Ae^{-\omega} = 0 \Rightarrow Ae^{\omega} = Ae^{-\omega} \Rightarrow e^{\omega} = e^{-\omega} \Rightarrow \omega = -\omega \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$$

$$X(1) = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$$

Again, a **contradiction** (we want $X \not\equiv 0$, because otherwise $u(x,t) \equiv 0$)

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm\omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Again, using $X(0) = 0$, $X(1) = 0$, we get:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(\omega x)$$

$$X(1) = 0 \Rightarrow B\sin(\omega) = 0 \Rightarrow \sin(\omega) = 0 \Rightarrow \omega = \pi m, \quad (m = 1, 2, \cdots)$$

This tells us that:

$$\begin{aligned} \text{Eigenvalues:} \lambda &= -\omega^2 = -(\pi m)^2 \quad (m = 1, 2, \cdots) \\ \text{Eigenfunctions:} X(x) &= \sin(\omega x) = \sin(\pi m x) \end{aligned} \tag{5.7}$$

Step 4:

Deal with (5.5), and remember that $\lambda = -(\pi m)^2$:

$$T'(t) = \lambda T(t) \Rightarrow T(t) = Ae^{\lambda t} = T(t) = \widetilde{A_m} e^{-(\pi m)^2 t} \qquad m = 1, 2, \cdots$$

Note: Here we use $\widetilde{A_m}$ to emphasize that $\widetilde{A_m}$ depends on m.

Step 5:

Take linear combinations:

$$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} \widetilde{A_m} e^{-(\pi m)^2 t} \sin(\pi m x) \tag{5.8}$$

Step 6:

Use the initial condition $u(x,0) = x$ in (5.1):

$$u(x,0) = \sum_{m=1}^{\infty} \widetilde{A_m} \sin(\pi m x) = x \qquad \text{on}(0,1) \tag{5.9}$$

Now we want to express x as a linear combination of sines, so we have to use a **sine series** (that's why we used $\widetilde{A_m}$ instead of A_m):

$$\begin{aligned}
\widetilde{A_m} &= \frac{2}{1} \int_0^1 x \sin(\pi m x) dx \\
&= 2 \left(\left[-x \frac{\cos(\pi m x)}{\pi m} \right]_0^1 - \int_0^1 -\frac{\cos(\pi m x)}{\pi m} dx \right) \\
&= 2 \left(-\frac{\cos(\pi m)}{\pi m} + \int_0^1 \frac{\cos(\pi m x)}{\pi m} dx \right) \\
&= 2 \left(-\frac{(-1)^m}{\pi m} + \left[\frac{\sin(\pi m x)}{(\pi m)^2} \right]_0^1 \right) \\
&= \frac{2(-1)^{m+1}}{\pi m} \qquad (m = 1, 2, \cdots)
\end{aligned}$$

Step 7:

Conclude using (5.10)

$$u(x,t) = \sum_{m=1}^{\infty} \frac{2(-1)^{m+1}}{\pi m} e^{-(\pi m)^2 t} \sin(\pi m x) \tag{5.10}$$

6 The Wave equation

Problem: Solve the following **wave equation**:

$$
\begin{cases}
\dfrac{\partial^2 u}{\partial t^2} = \dfrac{\partial^2 u}{\partial x^2} & 0 < x < \pi, \quad t > 0 \\
u(0,t) = u(\pi,t) = 0 & t > 0 \\
u(x,0) = \sin(4x) + 7\sin(5x) & 0 < x < \pi \\
\dfrac{\partial u}{\partial t}(x,0) = 2\sin(2x) + \sin(3x) & 0 < x < \pi
\end{cases}
\tag{6.1}
$$

Step 1: Separation of variables

Suppose:

$$u(x,t) = X(x)T(t) \tag{6.2}$$

Plug (6.2) into the differential equation (6.1), and you get:

$$
\begin{aligned}
(X(x)T(t))_{tt} &= (X(x)T(t))_{xx} \\
X(x)T''(t) &= X''(x)T(t)
\end{aligned}
$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{T''(t)}{T(t)} \tag{6.3}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on x, but by (6.3) *only* depends on t, hence it is constant:

$$
\begin{aligned}
\frac{X''(x)}{X(x)} &= \lambda \\
X''(x) &= \lambda X(x)
\end{aligned}
\tag{6.4}
$$

Also, we get:

$$\begin{aligned} \frac{T''(t)}{T(t)} &= \lambda \\ T''(t) &= \lambda T(t) \end{aligned} \tag{6.5}$$

but we'll only deal with that later (Step 4)

Step 2:

Consider (6.4):

$$X''(x) = \lambda X(x)$$

Note: Always start with $X(x)$, do **NOT** touch $T(t)$ until right at the end!

Now use the **boundary conditions** in (6.1):

$$u(0,t) = X(0)T(t) = 0 \Rightarrow X(0)T(t) = 0 \Rightarrow X(0) = 0$$

$$u(\pi,t) = X(\pi)T(t) = 0 \Rightarrow X(\pi)T(t) = 0 \Rightarrow X(\pi) = 0$$

Hence we get:

$$\begin{cases} X''(x) = \lambda X(x) \\ X(0) = 0 \\ X(\pi) = 0 \end{cases} \tag{6.6}$$

Step 3: Eigenvalues/Eigenfunctions

The auxiliary polynomial of (6.6) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

Now use $X(0) = 0$ and $X(\pi) = 0$:

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$$X(\pi) = 0 \Rightarrow Ae^{\omega\pi} - Ae^{-\omega\pi} = 0 \Rightarrow Ae^{\omega\pi} = Ae^{-\omega\pi} \Rightarrow e^{\omega\pi} = e^{-\omega\pi} \Rightarrow \omega\pi = -\omega\pi \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$$

$$X(\pi) = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$$

Again, a **contradiction** (we want $X \not\equiv 0$, because otherwise $u(x,t) \equiv 0$)

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm\omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Again, using $X(0) = 0$, $X(\pi) = 0$, we get:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(\omega x)$$

$$X(\pi) = 0 \Rightarrow B\sin(\omega\pi) = 0 \Rightarrow \sin(\omega\pi) = 0 \Rightarrow \omega = m, \quad (m = 1, 2, \cdots)$$

This tells us that:

$$\begin{aligned} \text{Eigenvalues:} \lambda = -\omega^2 = -m^2 \quad (m = 1, 2, \cdots) \\ \text{Eigenfunctions:} X(x) = \sin(\omega x) = \sin(mx) \end{aligned} \tag{6.7}$$

Step 4:

Deal with (6.5), and remember that $\lambda = -m^2$:

$$T''(t) = \lambda T(t)$$

Aux: $r^2 = -m^2 \Rightarrow r = \pm mi \qquad (m = 1, 2, \cdots)$

$$T(t) = \widetilde{A_m}\cos(mt) + \widetilde{B_m}\sin(mt)$$

Step 5:

Take linear combinations:

$$u(x,t) = \sum_{m=1}^{\infty} T(t)X(x) = \sum_{m=1}^{\infty} \left(\widetilde{A_m}\cos(mt) + \widetilde{B_m}\sin(mt)\right)\sin(mx) \tag{6.8}$$

Step 6:

Use the initial condition $u(x,0) = \sin(4x) + 7\sin(5x)$ in (6.1):

Plug in $t = 0$ in (6.8), and you get:

$$u(x,0) = \sum_{m=1}^{\infty} \widetilde{A_m}\sin(mx) = \sin(4x) + 7\sin(5x) \qquad \text{on}(0,\pi) \tag{6.9}$$

Note: At this point you would *usually* have to find the sine series of a function (see section 4). But here we're very lucky because we're already given a linear combination of sines!

Equating coefficients, you get:

$$\begin{aligned}
\widetilde{A_4} &= 1 && \text{(coefficient of } \sin(4x)) \\
\widetilde{A_5} &= 7 && \text{(coefficient of } \sin(5x)) \\
\widetilde{A_m} &= 0 && \text{(for all other } m)
\end{aligned}$$

Step 7:

Use the initial condition: $\frac{\partial u}{\partial t}(x,0) = 2\sin(2x) + \sin(3x)$ in (6.1)

First differentiate (6.8) with respect to t:

$$\frac{\partial u}{\partial t}(x,t) = \sum_{m=1}^{\infty} \left(-m\widetilde{A_m}\sin(mt) + m\widetilde{B_m}\cos(mt) \right) \sin(mx) \tag{6.10}$$

Now plug in $t = 0$ in (6.10):

$$\frac{\partial u}{\partial t}(x,0) = \sum_{m=1}^{\infty} m\widetilde{B_m}\sin(mx) = 2\sin(2x) + \sin(3x) \tag{6.11}$$

Again, *usually* you'd have to calculate Fourier sine series, but again we're lucky because the right-hand-side is already a linear combination of sines!

Equating coefficients, you get:

$$\begin{aligned}
2\widetilde{B_2} &= 2 && \text{(coefficient of } \sin(2x)) \\
3\widetilde{B_3} &= 1 && \text{(coefficient of } \sin(3x)) \\
\widetilde{B_m} &= 0 && \text{(for all other } m)
\end{aligned}$$

That is:

$$\widetilde{B_2} = 1 \qquad \text{(coefficient of } \sin(2x))$$
$$\widetilde{B_3} = \frac{1}{3} \qquad \text{(coefficient of } \sin(3x))$$
$$\widetilde{B_m} = 0 \qquad \text{(for all other } m)$$

Step 8:

Conclude using (6.8) and the coefficients A_m and B_m you found:

$$u(x,t) = \sum_{m=1}^{\infty} \left(\widetilde{A_m} \cos(mt) + \widetilde{B_m} \sin(mt) \right) \sin(mx) \tag{6.12}$$

where:

$$\widetilde{A_4} = 1$$
$$\widetilde{A_5} = 7$$
$$\widetilde{A_m} = 0 \qquad \text{(for all other } m)$$

and

$$\widetilde{B_2} = 1$$
$$\widetilde{B_3} = \frac{1}{3}$$
$$\widetilde{B_m} = 0 \qquad \text{(for all other } m)$$

Note: In this *special* case, you can write $u(x,t)$ in the following nice form:

$$u(x,t) = \sin(2t)\sin(2x) + \frac{1}{3}\sin(3t)\sin(3x) + \cos(4t)\sin(4x) + 7\cos(5t)\sin(5x) \tag{6.13}$$

But in general, you'd have to leave your answer in the form of (6.8)

7 Laplace's equation

Problem: Solve the following **Laplace equation**:

$$\begin{cases} \dfrac{\partial^2 u}{\partial x^2} + \dfrac{\partial^2 u}{\partial y^2} = 0 & 0 < x < 1, \quad 0 < y < 1 \\ u(0,y) = u(1,y) = 0 & 0 \le y \le 1 \\ u(x,0) = 6\sin(5\pi x) & 0 \le x \le 1 \\ u(x,1) = 0 & 0 \le x \le 1 \end{cases} \tag{7.1}$$

Step 1: Separation of variables

Suppose:

$$u(x,y) = X(x)Y(y) \tag{7.2}$$

Plug (7.2) into the differential equation (7.1), and you get:

$$\begin{aligned} (X(x)Y(y))_{xx} + (X(x)Y(y))_{yy} &= 0 \\ X''(x)Y(y) + X(x)Y''(y) &= 0 \\ X''(x)Y(y) &= -X(x)Y''(y) \end{aligned}$$

Rearrange and get:

$$\frac{X''(x)}{X(x)} = \frac{-Y''(y)}{Y(y)} \tag{7.3}$$

Now $\frac{X''(x)}{X(x)}$ *only* depends on x, but by (7.3) *only* depends on y, hence it is constant:

$$\begin{aligned} \frac{X''(x)}{X(x)} &= \lambda \\ X''(x) &= \lambda X(x) \end{aligned} \tag{7.4}$$

Also, we get:

$$\begin{aligned} \frac{-Y''(y)}{Y(y)} =& \lambda \\ Y''(y) =& -\lambda Y(y) \end{aligned} \tag{7.5}$$

but we'll only deal with that later (Step 4)

Note: Careful about the $-$ sign!!!

Step 2:

Consider (7.4):

$$X''(x) = \lambda X(x)$$

Note: Always start with $X(x)$, do **NOT** touch $Y(y)$ until right at the end!

Now use the **boundary conditions** in (7.1):

$$u(0, y) = X(0)Y(y) = 0 \Rightarrow X(0)Y(y) = 0 \Rightarrow X(0) = 0$$

$$u(1, y) = X(1)Y(1) = 0 \Rightarrow X(1)Y(y) = 0 \Rightarrow X(1) = 0$$

Hence we get:

$$\begin{cases} X''(x) =& \lambda X(x) \\ X(0) =& 0 \\ X(1) =& 0 \end{cases} \tag{7.6}$$

Step 3: Eigenvalues/Eigenfunctions

The auxiliary polynomial of (7.6) is $p(\lambda) = r^2 - \lambda$

Now we need to consider 3 cases:

<u>Case 1:</u> $\lambda > 0$, then $\lambda = \omega^2$, where $\omega > 0$

Then:

$$r^2 - \lambda = 0 \Rightarrow r^2 - \omega^2 = 0 \Rightarrow r = \pm\omega$$

Therefore:

$$X(x) = Ae^{\omega x} + Be^{-\omega x}$$

Now use $X(0) = 0$ and $X(1) = 0$:

$$X(0) = 0 \Rightarrow A + B = 0 \Rightarrow B = -A \Rightarrow X(x) = Ae^{\omega x} - Ae^{-\omega x}$$

$$X(1) = 0 \Rightarrow Ae^{\omega} - Ae^{-\omega} = 0 \Rightarrow Ae^{\omega} = Ae^{-\omega} \Rightarrow e^{\omega} = e^{-\omega} \Rightarrow \omega = -\omega \Rightarrow \omega = 0$$

But this is a **contradiction**, as we want $\omega > 0$.

Case 2: $\lambda = 0$, then $r = 0$, and:

$$X(x) = Ae^{0x} + Bxe^{0x} = A + Bx$$

And:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = Bx$$

$$X(1) = 0 \Rightarrow B = 0 \Rightarrow X(x) = 0$$

Again, a **contradiction** (we want $X \not\equiv 0$, because otherwise $u(x, y) \equiv 0$)

Case 3: $\lambda < 0$, then $\lambda = -\omega^2$, and:

$$r^2 - \lambda = 0 \Rightarrow r^2 + \omega^2 = 0 \Rightarrow r = \pm\omega i$$

Which gives:

$$X(x) = A\cos(\omega x) + B\sin(\omega x)$$

Again, using $X(0) = 0$, $X(1) = 0$, we get:

$$X(0) = 0 \Rightarrow A = 0 \Rightarrow X(x) = B\sin(\omega x)$$

$$X(1) = 0 \Rightarrow B\sin(\omega) = 0 \Rightarrow \sin(\omega) = 0 \Rightarrow \omega = \pi m, \quad (m = 1, 2, \cdots)$$

This tells us that:

$$\begin{aligned} \text{Eigenvalues:} \lambda = -\omega^2 = -(\pi m)^2 \quad (m = 1, 2, \cdots) \\ \text{Eigenfunctions:} X(x) = \sin(\omega x) = \sin(\pi m x) \end{aligned} \tag{7.7}$$

Step 4:

Deal with (7.5), and remember that $\lambda = -(\pi m)^2$:

$$Y''(y) = -\lambda Y(y)$$

Aux: $r^2 = (\pi m)^2 \Rightarrow r = \pm \pi m \qquad (m = 1, 2, \cdots)$

$$Y(y) = \widetilde{A_m} e^{\pi m y} + \widetilde{B_m} e^{-\pi m y} \tag{7.8}$$

IMPORTANT REMARK: If you leave your answer like that, your algebra becomes messy! Instead, use the following nice formulas:

$$\frac{e^w + e^{-w}}{2} = \cosh(w)$$
$$\frac{e^w - e^{-w}}{2} = \sinh(w)$$

And you get:

$$Y(y) = \widetilde{A_m} \cosh(\pi m y) + \widetilde{B_m} \sinh(\pi m y) \tag{7.9}$$

Note: The constants $\widetilde{A_m}$ and $\widetilde{B_m}$ are different in (7.8) and (7.9), but it doesn't matter because they are only (general) constants!

Step 5:

Take linear combinations:

$$u(x,t) = \sum_{m=1}^{\infty} Y(y)X(x) = \sum_{m=1}^{\infty} \left(\widetilde{A_m} \cosh(\pi m y) + \widetilde{B_m} \sinh(\pi m y) \right) \sin(\pi m x) \tag{7.10}$$

Step 6:

Use the initial condition $u(x,0) = 6\sin(5\pi x)$ in (7.1):

Plug in $y = 0$ in (7.10), and using $\cosh(0) = 1$, $\sinh(0) = 0$, you get:

$$u(x,0) = \sum_{m=1}^{\infty} \widetilde{A_m} \sin(\pi m x) = 6\sin(5\pi x) \qquad \text{on}(0,1) \tag{7.11}$$

Note: At this point you would *usually* have to find the sine series of a function (see the heat equation example). But here again we're very lucky because we're already given a linear combination of sines!

Equating coefficients (notice this is why we used $\cosh$ and $\sinh$ instead of exponential functions), you get:

$$\begin{aligned} \widetilde{A_5} &= 6 \qquad \text{(coefficient of } \sin(5\pi x)\text{)} \\ \widetilde{A_m} &= 0 \qquad \text{(for all other } m\text{)} \end{aligned} \tag{7.12}$$

Step 7:

Use the initial condition: $u(x,1) = 0$ in (7.1)

Plug in $y = 1$ in (7.8), and you get:

$$u(x,1) = \sum_{m=1}^{\infty} \left(\widetilde{A_m} \cosh(\pi m) + \widetilde{B_m} \sinh(\pi m) \right) \sin(\pi m x) = 0 \qquad \text{on}(0,1) \tag{7.13}$$

Again, usually you'd have to use a Fourier sine series, but again you're lucky because the function is 0, so if you equate the coefficients, you get:

$$\cosh(\pi m)\widetilde{A_m} + \sinh(\pi m)\widetilde{B_m} = 0 \qquad (m = 1, 2, \cdots) \tag{7.14}$$

But now combining (7.10) and (7.18), we get:

For $m \neq 5$ $A_m = 0$, so:

$$\sinh(\pi m)\widetilde{B_m} = 0 \tag{7.15}$$

which gives you $\boxed{\widetilde{B_m} = 0}$ for $m \neq 5$.

For $m = 5$:

$$\cosh(5\pi)6 + \sinh(5\pi)\widetilde{B_5} = 0 \tag{7.16}$$

which gives you:

$$\widetilde{B_5} = -\frac{6\cosh(5\pi)}{\sinh(5\pi)} = -6\coth(5\pi)$$

Step 8:

Conclude using (7.10) and the coefficients A_m and B_m you found:

$$u(x,y) = \sum_{m=1}^{\infty} Y(y)X(x) = \sum_{m=1}^{\infty} \left(\widetilde{A_m}\cosh(\pi m y) + \widetilde{B_m}\sinh(\pi m y)\right)\sin(\pi m x) \tag{7.17}$$

where: $\boxed{\widetilde{A_m} = \widetilde{B_m} = 0}$ if $m \neq 5$, and $\boxed{\widetilde{A_5} = 6, \widetilde{B_5} = -6\coth(5\pi)}$.

Note: In this *special* case, you can write $u(x,t)$ in the following nice form:

$$u(x,y) = (6\cosh(5\pi y) - 6\coth(5\pi)\sinh(5\pi y))\sin(\pi m x) \tag{7.18}$$

But in general, you'd have to leave your answer in the general form (7.10).