Coupled Harmonic Oscillators

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This handout is meant to summarize everything you need to know about the coupled harmonic oscillators for the final exam.

Note: In what follows we will assume that all masses $m = 1$ and all spring constants $k = 1$.

1 Case $N = 2$ (two harmonic oscillators)

Question: Find the proper frequencies and eigenvectors / proper modes of two coupled harmonic oscillators.

Equation:

$\mathbf{x}'' = A\mathbf{x}$, where:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$$

Proper frequencies:

Find the eigenvalues of the matrix A: $\lambda = -1, -3$

Fact: The proper frequencies are $\pm\sqrt{\lambda}$.

Hence the proper frequencies are: $\pm\sqrt{-1} = \pm i$, $\pm\sqrt{-3} = \pm\sqrt{3}i$.

Proper modes:

To find the modes, use the following trick: Since $N = 2$, $N + 1 = 3$, hence all the modes will involve $\sin\left(q\frac{\pi}{3}\right)$, where q is some number. Then:

$$\mathbf{v_1} = \begin{bmatrix} \sin\left(1\frac{\pi}{3}\right) \\ \sin\left(2\frac{\pi}{3}\right) \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} \sin\left(2\frac{\pi}{3}\right) \\ \sin\left(4\frac{\pi}{3}\right) \end{bmatrix}$$

Note: The way you get the other values is by using multiples, i.e. the multiples of 1 are 1 and 2, the multiples of 2 are 2 and 4.

Hence the proper modes are:

$$\mathbf{v_1} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$$

Note: If you ever get $\begin{bmatrix} 0 \\ 0 \end{bmatrix}$, then you probably used $N = 2$ instead of $N+1 = 3$.

2 Case $N = 3$ (three harmonic oscillators)

Question: Find the proper frequencies and eigenvectors / proper modes of three coupled harmonic oscillators.

Equation:

$\mathbf{x}'' = A\mathbf{x}$, where:

$$\mathbf{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}, \quad A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$$

Proper frequencies:

Find the eigenvalues of the matrix A: $\lambda = -2, -2 - \sqrt{2}, -2 + \sqrt{2}$

Fact: The proper frequencies are $\pm\sqrt{\lambda}$.

Hence the proper frequencies are:

$$\pm\sqrt{-2} = \pm\sqrt{2}i, \quad \pm\sqrt{-2-\sqrt{2}} = \pm\left(\sqrt{2+\sqrt{2}}\right)i, \quad \pm\sqrt{-2+\sqrt{2}} = \pm\left(\sqrt{2-\sqrt{2}}\right)i$$

Proper modes:

To find the modes, use the following trick: Since $N = 3$, $N + 1 = 4$, hence all the modes will involve $\sin\left(q\frac{\pi}{4}\right)$, where q is some number. Then:

$$\mathbf{v_1} = \begin{bmatrix} \sin\left(1\frac{\pi}{4}\right) \\ \sin\left(2\frac{\pi}{4}\right) \\ \sin\left(3\frac{\pi}{4}\right) \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} \sin\left(2\frac{\pi}{4}\right) \\ \sin\left(4\frac{\pi}{4}\right) \\ \sin\left(6\frac{\pi}{4}\right) \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} \sin\left(3\frac{\pi}{4}\right) \\ \sin\left(6\frac{\pi}{4}\right) \\ \sin\left(9\frac{\pi}{4}\right) \end{bmatrix}$$

Note: The way you get the other values is by using multiples, i.e. the multiples of 1 are 1, 2 and 3, the multiples of 2 are 2, 4 and 6, the multiples of 3 are 3, 6, and 9.

Hence the proper modes are:

$$\mathbf{v_1} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}, \mathbf{v_2} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}, \mathbf{v_3} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$$

Note: If you get $\begin{bmatrix} 0 \\ 0 \\ 0 \end{bmatrix}$, then you probably used $N = 3$ instead of $N + 1 = 4$.

3 General case

Equation: $\mathbf{x}'' = A\mathbf{x}$

$$A = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Proper frequencies: $\boxed{\pm 2i \sin\left(\frac{k\pi}{2(N+1)}\right)}$, $k = 1, 2, \cdots N$

Proper modes: $\boxed{\mathbf{v_k} = \begin{bmatrix} \sin\left(\frac{k\pi}{N+1}\right) \\ \sin\left(\frac{2k\pi}{N+1}\right) \\ \vdots \\ \sin\left(\frac{Nk\pi}{N+1}\right) \end{bmatrix}}$