Math 54 Cheat Sheet

Vector spaces

Subspace: If $\mathbf{u}$ and $\mathbf{v}$ are in W, then $\mathbf{u}+\mathbf{v}$ are in W, and $c\mathbf{u}$ is in W
Nul(A): Solutions of $A\mathbf{x}=\mathbf{0}$. Row-reduce A.
Row(A): Space spanned by the rows of A: Row-reduce A and choose the rows that contain the pivots.
Col(A): Space spanned by columns of A: Row-reduce A and choose the **columns of** A that contain the pivots
Rank(A): = $Dim(Col(A))$ = number of pivots
Rank-Nullity theorem: $Rank(A) + dim(Nul(A)) = n$, where A is $m \times n$
Linear transformation: $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v}), T(c\mathbf{u}) = cT(\mathbf{u})$, where c is a number.
T is one-to-one if $T(\mathbf{u}) = \mathbf{0} \Rightarrow \mathbf{u} = \mathbf{0}$
T is onto if $Col(T) = \mathbb{R}^m$.
Linearly independence: $a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n} = \mathbf{0} \Rightarrow a_1 = a_2 = \cdots = a_n = 0$.
To show lin. ind, form the matrix of the vectors, and show that $Nul(A) = \{\mathbf{0}\}$
Linear dependence: $a_1\mathbf{v_1} + a_2\mathbf{v_2} + \cdots + a_n\mathbf{v_n} = \mathbf{0}$ for $a_1, a_2, \cdots, a_n$, not all zero.
Span: Set of linear combinations of $\mathbf{v_1}, \cdots \mathbf{v_n}$
Basis $\mathcal{B}$ for V: A linearly independent set such that $Span(\mathcal{B}) = V$
To show sthg is a basis, show it is linearly independent and spans.
To find a basis from a collection of vectors, form the matrix A of the vectors, and find $Col(A)$.
To find a basis for a vector space, take any element of that v.s. and express it as a linear combination of 'simpler' vectors. Then show those vectors form a basis.
Dimension: Number of elements in a basis.
To find dim, find a basis and find num. elts.
Theorem: If V has a basis of vectors, then every basis of V must have n vectors.
Basis theorem: If V is an $n-$dim v.s., then any lin. ind. set with n elements is a basis, and any set of n elts. which spans V is a basis.
Matrix of a lin. transf T with respect to bases $\mathcal{B}$ and $\mathcal{C}$: For every vector $\mathbf{v}$ in $\mathcal{B}$, evaluate $T(\mathbf{v})$, and express $T(\mathbf{v})$ as a linear combination of vectors in $\mathcal{C}$. Put the **coefficients** in a column vector, and then form the matrix of the column vectors you found!
Coordinates: To find $[\mathbf{x}]_{\mathcal{B}}$, express $\mathbf{x}$ in terms of the vectors in $\mathcal{B}$.
$\mathbf{x} = P_{\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$, where $P_{\mathcal{B}}$ is the matrix whole columns are the vectors in $\mathcal{B}$.
Invertible matrix theorem: If A is invertible, then: A is row-equivalent to I, A has n pivots, $T(\mathbf{x}) = A\mathbf{x}$ is one-to-one and onto, $A\mathbf{x} = \mathbf{b}$ has a unique solution for every $\mathbf{b}$, A^T is invertible, $det(A) \neq 0$, the columns of A form a basis for $\mathbb{R}^n$, $Nul(A) = \{\mathbf{0}\}$, $Rank(A) = n$

$$\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$$

$$[A \mid \quad I] \to \left[I \mid \quad A^{-1}\right]$$

Change of basis: $[\mathbf{x}]_{\mathcal{C}} = P_{\mathcal{C}\leftarrow\mathcal{B}}[\mathbf{x}]_{\mathcal{B}}$ (think of $\mathcal{C}$ as the new, cool basis)
$[\mathcal{C} \mid \mathcal{B}] \to [I \mid P_{\mathcal{C}\leftarrow\mathcal{B}}]$
$P_{\mathcal{C}\leftarrow\mathcal{B}}$ is the matrix whose columns are $[\mathbf{b}]_{\mathcal{C}}$, where $\mathbf{b}$ is in $\mathcal{B}$

Diagonalization

Diagonalizability: A is **diagonalizable** if $A = PDP^{-1}$ for some diagonal D and invertible P.
A and B are similar if $A = PBP^{-1}$ for P invertible
Theorem: A is diagonalizable $\Leftrightarrow$ A has n linearly independent **eigenvectors**
Theorem: **IF** A has n distinct eigenvalues, **THEN** A is diagonalizable, but the opposite is not always true!!!!
Notes: A can be diagonalizable even if it's not invertible (Ex: $A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}$). Not all matrices are diagonalizable (Ex: $\begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}$)
Consequence: $A = PDP^{-1} \Rightarrow A^n = PD^nP^{-1}$
How to diagonalize: To find the eigenvalues, calculate $det(A - \lambda I)$, and find the roots of that.
To find the eigenvectors, for each λ find a basis for $Nul(A - \lambda I)$, which you do by row-reducing
Rational roots theorem: If $p(\lambda) = 0$ has a rational root $r = \frac{a}{b}$, then a divides the constant term of p, and b divides the leading coefficient.
Use this to guess zeros of p. Once you have a zero that works, use long division! Then $A = PDP^{-1}$, where D= diagonal matrix of eigenvalues, P = matrix of eigenvectors
Complex eigenvalues If $\lambda = a + bi$, and $\mathbf{v}$ is an eigenvector, then
$A = PCP^{-1}$, where $P = [Re(\mathbf{v}) \quad Im(\mathbf{v})]$, $C = \begin{bmatrix} a & b \\ -b & a \end{bmatrix}$
C is a scaling of $\sqrt{det(A)}$ followed by a rotation by θ, where:

$$\frac{1}{\sqrt{det(A)}}C = \begin{bmatrix} \cos(\theta) & \sin(\theta) \\ -\sin(\theta) & \cos(\theta) \end{bmatrix}$$

Orthogonality

$\mathbf{u}, \mathbf{v}$ orthogonal if $\mathbf{u} \cdot \mathbf{v} = 0$.
$\|\mathbf{u}\| = \sqrt{\mathbf{u} \cdot \mathbf{u}}$
$\{\mathbf{u_1} \cdots \mathbf{u_n}\}$ is orthogonal if $\mathbf{u_i} \cdot \mathbf{u_j} = 0$ if $i \neq j$, orthonormal if $\mathbf{u_i} \cdot \mathbf{u_i} = 1$
$W^{\perp}$: Set of $\mathbf{v}$ which are orthogonal to every $\mathbf{w}$ in W.
If $\{\mathbf{u_1} \cdots \mathbf{u_n}\}$ is an orthogonal basis, then:

$$\mathbf{y} = c_1\mathbf{u_1} + \cdots c_n\mathbf{u_n} \Rightarrow c_j = \frac{\mathbf{y} \cdot \mathbf{u_j}}{\mathbf{u_j} \cdot \mathbf{u_j}}$$

Orthogonal matrix Q has ortho**normal** columns! Consequence: $Q^TQ = I$, QQ^T = Orthogonal projection on $Col(Q)$.
$\|Q\mathbf{x}\| = \|\mathbf{x}\|$
$(Q\mathbf{x}) \cdot (Q\mathbf{y}) = \mathbf{x} \cdot \mathbf{y}$
Orthogonal projection: If $\{\mathbf{u_1} \cdots \mathbf{u_k}\}$ is a basis for W, then orthogonal projection of $\mathbf{y}$ on W is: $\hat{\mathbf{y}} = \left(\frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_1}\mathbf{u_1}}\right)\mathbf{u_1} + \cdots + \left(\frac{\mathbf{y} \cdot \mathbf{u_1}}{\mathbf{u_k}\mathbf{u_k}}\right)\mathbf{u_k}$
$\mathbf{y} - \hat{\mathbf{y}}$ is orthogonal to $\hat{\mathbf{y}}$, shortest distance btw $\mathbf{y}$ and W is $\|\mathbf{y} - \hat{\mathbf{y}}\|$
Gram-Schmidt: Start with $\mathcal{B} = \{\mathbf{u_1}, \cdots \mathbf{u_n}\}$. Let:

$$\mathbf{v_1} = \mathbf{u_1}$$

$$\mathbf{v_2} = \mathbf{u_2} - \left(\frac{\mathbf{u_2} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1}$$

$$\mathbf{v_3} = \mathbf{u_3} - \left(\frac{\mathbf{u_3} \cdot \mathbf{v_1}}{\mathbf{v_1} \cdot \mathbf{v_1}}\right)\mathbf{v_1} - \left(\frac{\mathbf{u_3} \cdot \mathbf{v_2}}{\mathbf{v_2} \cdot \mathbf{v_2}}\right)\mathbf{v_2}$$

Then $\{\mathbf{v_1} \cdots \mathbf{v_n}\}$ is an orthogonal basis for $Span(\mathcal{B})$, and if $\mathbf{w_i} = \frac{\mathbf{v_i}}{\|\mathbf{v_i}\|}$, then $\{\mathbf{w_1} \cdots \mathbf{w_n}\}$ is an orthonormal basis for $Span(\mathcal{B})$.
QR-factorization: To find Q, apply G-S to columns of A. Then $R = Q^TA$
Least-squares: To solve $A\mathbf{x} = \mathbf{b}$ in the least squares-way, solve $A^TA\mathbf{x} = A^T\mathbf{b}$.
Least squares solution makes $\|A\mathbf{x} - \mathbf{b}\|$ smallest.
$\hat{\mathbf{x}} = R^{-1}Q^T\mathbf{b}$, where $A = QR$.
Inner product spaces $f \cdot g = \int_a^b f(t)g(t)dt$. G-S applies with this inner product as well.
Cauchy-Schwarz: $|\mathbf{u} \cdot \mathbf{v}| \leq \|\mathbf{u}\| \, \|\mathbf{v}\|$
Triangle inequality: $\|\mathbf{u} + \mathbf{v}\| \leq \|\mathbf{u}\| + \|\mathbf{v}\|$

Symmetric matrices ($A = A^T$)

Has n real eigenvalues, always diagonalizable, orthogonally diagonalizable ($A = PDP^T$, P is an orthogonal matrix, equivalent to symmetry!).
Theorem: If A is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.
How to orthogonally diagonalize: First diagonalize, then apply G-S on each eigenspace and normalize. Then P = matrix of (orthonormal) eigenvectors, D = matrix of eigenvalues.
Quadratic forms: To find the matrix, put the x_i^2-coefficients on the diagonal, and evenly distribute the other terms. For example, if the x_1x_2-term is 6, then the $(1,2)$th and $(2,1)$th entry of A is 3.
Then orthogonally diagonalize $A = PDP^T$.
Then let $\mathbf{y} = P^T\mathbf{x}$, then the quadratic form becomes $\lambda_1y_1^2 + \cdots + \lambda_ny_n^2$, where λ_i are the eigenvalues.
Spectral decomposition: $\lambda_1\mathbf{u_1}\mathbf{u_1}^T + \lambda_2\mathbf{u_2}\mathbf{u_2}^T + \cdots + \lambda_n\mathbf{u_n}\mathbf{u_n}^T$

Second-order and Higher-order differential equations

Homogeneous solutions: Auxiliary equation: Replace equation by polynomial, so y''' becomes r^3 etc. Then find the zeros (use the rational roots theorem and long division, see the 'Diagonalization-section). 'Simple zeros' give you e^{rt}, Repeated zeros (multiplicity m) give you $Ae^{rt} + Bte^{rt} + \cdots Zt^{m-1}e^{rt}$, Complex zeros $r = a + bi$ give you $Ae^{at}\cos(bt) + Be^{at}\sin(bt)$.
Undetermined coefficients: $y(t) = y_0(t) + y_p(t)$, where y_0 solves the hom. eqn. (equation $= 0$), and y_p is a particular solution. To find y_p:
If the inhom. term is Ct^me^{rt}, then:
$y_p = t^s(A_mt^m \cdots + A_1t + 1)e^{rt}$, where if r is a root of aux with multiplicity m, then $s = m$, and if r is not a root, then $s = 0$.
If the inhom term is $Ct^me^{at}\sin(\beta t)$, then: $y_p = t^s(A_mt^m \cdots + A_1t + 1)e^{at}\cos(\beta t) + t^s(B_mt^m \cdots + B_1t + 1)e^{rt}\sin(\beta t)$, where $s = m$, if $a + bi$ is also a root of aux with multiplicity m ($s = 0$ if not). $\cos$ **always goes with** $\sin$ **and vice-versa**, also, you have to look at $a + bi$ as one entity.
Variation of parameters: **First, make sure the leading coefficient (usually the coeff. of y'') is** $= 1$.. Then $y = y_0 + y_p$ as above. Now suppose $y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$, where y_1 and y_2 are your hom. solutions. Then $\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix}\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \begin{bmatrix} 0 \\ f(t) \end{bmatrix}$. Invert the matrix and solve for v_1' and v_2', and integrate to get v_1 and v_2, and finally use:
$y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t)$.
Useful formulas: $\begin{bmatrix} a & b \\ c & d \end{bmatrix}^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$
$\int \sec(t) = \ln|\sec(t) + \tan(t)|$, $\int \tan(t) = \ln|\sec(t)|$,
$\int \tan^2(t) = \tan(x) - x$, $\int \ln(t) = t\ln(t) - t$
Linear independence: f, g, h are linearly independent if $af(t) + bg(t) + ch(t) = 0 \Rightarrow a = b = c = 0$. To show linear dependence, do it directly. To show linear independence, form the Wronskian:
$\widetilde{W}(t) = \begin{bmatrix} f(t) & g(t) \\ f'(t) & g'(t) \end{bmatrix}$ (for 2 functions),
$\widetilde{W}(t) = \begin{bmatrix} f(t) & g(t) & h(t) \\ f'(t) & g'(t) & h'(t) \\ f''(t) & g''(t) & h''(t) \end{bmatrix}$ (for 3 functions). Then pick a point t_0 where $det(\widetilde{W}(t_0))$ is easy to evaluate. If $det \neq 0$, then f, g, h are linearly independent! Try to look for simplifications before you differentiate.
Fundamental solution set: If f, g, h are solutions **and** linearly independent.
Largest interval of existence: First make sure the leading coefficient equals to 1. Then look at the domain of each term. For each domain, consider the part of the interval which contains the initial condition. Finally, intersect the intervals and change any brackets to parentheses. Harmonic oscillator: $my'' + by' + ky = 0$ (m = inertia, b = damping, k = stiffness)

Systems of differential equations

To solve $\mathbf{x}' = A\mathbf{x}$: $\mathbf{x}(t) = Ae^{\lambda_1t}\mathbf{v_1} + Be^{\lambda_2t}\mathbf{v_2} + e^{\lambda_3t}\mathbf{v_3}$ (λ_i are your eigenvalues, $\mathbf{v_i}$ are your eigenvectors)
Fundamental matrix: Matrix whose columns are the solutions, without the constants (the columns are solutions and linearly independent)
Complex eigenvalues If $\lambda = \alpha + i\beta$, and $\mathbf{v} = \mathbf{a} + i\mathbf{b}$. Then:

$$\mathbf{x}(t) = A\left(e^{\alpha t}\cos(\beta t)\mathbf{a} - e^{\alpha t}\sin(\beta t)\mathbf{b}\right) + B\left(e^{\alpha t}\sin(\beta t)\mathbf{a} + e^{\alpha t}\cos(\beta t)\mathbf{b}\right)$$

Notes: You only need to consider one complex eigenvalue. For real eigenvalues, use the formula above. Also, $\frac{1}{a+bi} = \frac{a-bi}{a^2+b^2}$
Generalized eigenvectors If you only find one eigenvector $\mathbf{v}$ (even though there are supposed to be 2), then solve the following equation for $\mathbf{u}$: $(A - \lambda I)(\mathbf{u}) = \mathbf{v}$ (one solution is enough).
Then: $\mathbf{x}(t) = Ae^{\lambda t}\mathbf{v} + B\left(te^{\lambda t}\mathbf{v} + e^{\lambda t}\mathbf{u}\right)$

Undetermined coefficients First find hom. solution. Then for $\mathbf{x_p}$, just like regular undetermined coefficients, except that instead of guessing $\mathbf{x_p}(t) = ae^t + b\cos(t)$, you guess $\mathbf{a}e^t + \mathbf{b}\cos(t)$, where $\mathbf{a} = \begin{bmatrix} a_1 \\ a_2 \end{bmatrix}$ is a vector. Then plug into $\mathbf{x}' = A\mathbf{x} + \mathbf{f}$ and solve for $\mathbf{a}$ etc.

Variation of parameters First hom. solution $\mathbf{x_h}(t) = A\mathbf{x_1}(t) + B\mathbf{x_2}(t)$. Then sps $\mathbf{x_p}(t) = v_1(t)\mathbf{x_1}(t) + v_2(t)\mathbf{x_2}(t)$, then solve $\widetilde{W}(t)\begin{bmatrix} v_1' \\ v_2' \end{bmatrix} = \mathbf{f}$, where $\widetilde{W}(t) = \left[\mathbf{x_1}(t) \mid \quad \mathbf{x_2}(t)\right]$. Multiply both sides by $\left(\widetilde{W}(t)\right)^{-1}$, integrate and solve for $v_1(t), v_2(t)$, and plug back into $\mathbf{x_p}$. Finally, $\mathbf{x} = \mathbf{x_h} + \mathbf{x_p}$

Matrix exponential $e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$. To calculate e^{At}, either diagonalize: $A = PDP^{-1} \Rightarrow e^{At} = Pe^{Dt}P^{-1}$, where e^{Dt} is a diagonal matrix with diag. entries $e^{\lambda_i t}$. Or if A only has one eigenvalue λ with multiplicity m, use $e^{At} = e^{\lambda t}\sum_{n=0}^{m-1} \frac{(A-\lambda I)^n t^n}{n!}$. Solution of $\mathbf{x}' = A\mathbf{x}$ is then $\mathbf{x}(t) = e^{At}\mathbf{c}$, where $\mathbf{c}$ is a constant vector.

Coupled mass-spring system

Case $N = 2$

Equation: $\mathbf{x}'' = A\mathbf{x}$, $A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$

Proper frequencies: Eigenvalues of A are: $\lambda = -1, -3$, then proper frequencies $\boxed{\pm i, \pm\sqrt{3}i}$ ($\pm$ square roots of eigenvalues)

Proper modes: $\mathbf{v_1} = \begin{bmatrix} \sin\left(\frac{\pi}{3}\right) \\ \sin\left(2\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ \frac{\sqrt{3}}{2} \end{bmatrix}$,

$\mathbf{v_2} = \begin{bmatrix} \sin\left(2\frac{\pi}{3}\right) \\ \sin\left(4\frac{\pi}{3}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{3}}{2} \\ -\frac{\sqrt{3}}{2} \end{bmatrix}$

Case $N = 3$

Equation: $\mathbf{x}'' = A\mathbf{x}$, $A = \begin{bmatrix} -2 & 1 & 0 \\ 1 & -2 & 1 \\ 0 & 1 & -2 \end{bmatrix}$

Proper frequencies: Eigenvalues of A: $\lambda = -2, -2-\sqrt{2}, -2+\sqrt{2}$, then proper frequencies $\boxed{\pm\sqrt{2}i, \pm\left(\sqrt{2+\sqrt{2}}\right)i, \pm\left(\sqrt{2-\sqrt{2}}\right)i}$

Proper modes: $\mathbf{v_1} = \begin{bmatrix} \sin\left(\frac{\pi}{4}\right) \\ \sin\left(2\frac{\pi}{4}\right) \\ \sin\left(3\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ 1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$, $\mathbf{v_2} = \begin{bmatrix} \sin\left(2\frac{\pi}{4}\right) \\ \sin\left(4\frac{\pi}{4}\right) \\ \sin\left(6\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \\ -1 \end{bmatrix}$, $\mathbf{v_3} = \begin{bmatrix} \sin\left(3\frac{\pi}{4}\right) \\ \sin\left(6\frac{\pi}{4}\right) \\ \sin\left(9\frac{\pi}{4}\right) \end{bmatrix} = \begin{bmatrix} \frac{\sqrt{2}}{2} \\ -1 \\ \frac{\sqrt{2}}{2} \end{bmatrix}$

General case (just in case!)

Equation: $\mathbf{x}'' = A\mathbf{x}$,

$$A = \begin{bmatrix} -2 & 1 & 0 & \cdots & 0 & \\ 1 & -2 & 1 & 0 & \cdots & 0 \\ 0 & 1 & -2 & 1 & 0 & \cdots \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 & -2 & 1 \\ 0 & 0 & 0 & 0 & 1 & -2 \end{bmatrix}$$

Proper frequencies: $\boxed{\pm 2i\sin\left(\frac{k\pi}{2(N+1)}\right)}$, $k = 1, 2, \cdots N$

Proper modes: $\boxed{\mathbf{v_k} = \begin{bmatrix} \sin\left(\frac{k\pi}{N+1}\right) \\ \sin\left(\frac{2k\pi}{N+1}\right) \\ \vdots \\ \sin\left(\frac{Nk\pi}{N+1}\right) \end{bmatrix}}$

Partial differential equations

Full Fourier series: f defined on $(-T, T)$:

$f(x) \sim \sum_{m=0}^{\infty}\left(a_m\cos\left(\frac{\pi m x}{T}\right) + b_m\sin\left(\frac{\pi m x}{T}\right)\right)$, where:

$a_0 = \frac{1}{2T}\int_{-T}^{T} f(x)dx$

$a_m = \frac{1}{T}\int_{-T}^{T} f(x)\cos\left(\frac{\pi m x}{T}\right)$

$b_0 = 0$

$b_m = \frac{1}{T}\int_{-T}^{T} f(x)\sin\left(\frac{\pi m x}{T}\right)$

Cosine series: f defined on $(0, T)$: $f(x) \sim \sum_{m=0}^{\infty} a_m\cos\left(\frac{\pi m x}{T}\right)$, where:

$a_0 = \frac{2}{2T}\int_0^T f(x)dx$ (not a typo)

$a_m = \frac{2}{T}\int_0^T f(x)\cos\left(\frac{\pi m x}{T}\right)$

Sine series: f defined on $(0, T)$: $f(x) \sim \sum_{m=0}^{\infty} b_m\sin\left(\frac{\pi m x}{T}\right)$, where:

$b_0 = 0$

$b_m = \frac{2}{T}\int_0^T f(x)\sin\left(\frac{\pi m x}{T}\right)$

Tabular integration: (IBP: $\int f'g = fg - \int fg'$) To integrate $\int f(t)g(t)dt$ where f is a polynomial, make a table whose first row is $f(t)$ and $g(t)$. Then differentiate f as many times until you get 0, and antidifferentiate as many times until it aligns with the 0 for f. Then multiply the diagonal terms and do $+$ first term $-$ second term etc.

Orthogonality formulas: $\int_{-T}^{T}\cos\left(\frac{\pi m x}{T}\right)\sin\left(\frac{\pi n x}{T}\right)dx = 0$

$\int_{-T}^{T}\cos\left(\frac{\pi m x}{T}\right)\cos\left(\frac{\pi n x}{T}\right)dx = 0$ if $m \neq n$

$\int_{-T}^{T}\sin\left(\frac{\pi m x}{T}\right)\sin\left(\frac{\pi n x}{T}\right)dx = 0$ if $m \neq n$

Convergence: Fourier series $\mathcal{F}$ goes to $f(x)$ is f is continuous at x, and if f has a jump at x, $\mathcal{F}$ goes to the average of the jumps. Finally, at the endpoints, $\mathcal{F}$ goes to average of the left/right endpoints.

Heat/Wave equations:

Step 1: Suppose $u(x,t) = X(x)T(t)$, plug this into PDE, and group X-terms and T-terms. Then $\frac{X''(x)}{X(x)} = \lambda$, so $X'' = \lambda X$. Then find a differential equation for T. **Note:** If you have an α-term, put it with T.

Step 2: Deal with $X'' = \lambda X$. Use boundary conditions to find $X(0)$ etc. (if you have $\frac{\partial u}{\partial x}$, you might have $X'(0)$ instead of $X(0)$).

Step 3: Case 1: $\lambda = \omega^2$, then $X(x) = Ae^{\omega x} + Be^{-\omega x}$, then find $\omega = 0$, contradiction. Case 2: $\lambda = 0$, then $X(x) = Ax + B$, then eihter find $X(x) = 0$ (contradiction), or find $X(x) = A$. Case 3: $\lambda = -\omega^2$, then $X(x) = A\cos(\omega x) + B\sin(\omega x)$. Then solve for ω, usually $\omega = \frac{\pi m}{T}$. Also, if case 2 works, should find $\cos$, if case 2 doesn't work, should find $\sin$. Finally, $\lambda = -\omega^2$, and $X(x) =$ whatever you found in 2) w/o the constant.

Step 4: Solve for $T(t)$ with the λ you found. Remember that for the heat equation: $T' = \lambda T \Rightarrow T(t) = \widetilde{A_m}e^{\lambda t}$. And for the wave equation: $T'' = \lambda T \Rightarrow T(t) = \widetilde{A_m}\cos(\omega t) + \widetilde{B_m}\sin(\omega t)$.

Step 5: Then $u(x,t) = \sum_{m=0}^{\infty} T(t)X(x)$ (if case 2 works), $u(x,t) = \sum_{m=1}^{\infty} T(t)X(x)$ (if case 2 doesn't work!)

Step 6: Use $u(x,0)$, and plug in $t = 0$. Then use Fourier cosine or sine series or just 'compare', i.e. if $u(x,0) = 4\sin(2\pi x) + 3\sin(3\pi x)$, then $\widetilde{A_2} = 4$, $\widetilde{A_3} = 3$, and $\widetilde{A_m} = 0$ if $m \neq 2, 3$.

Step 7: (only for wave equation): Use $\frac{\partial u}{\partial t}u(x,0)$: Differentiate Step 5 with respect to t and set $t = 0$. Then use Fourier cosine or series or 'compare'

Nonhomogeneous heat equation:

$$\begin{cases} \frac{\partial u}{\partial t} = \beta\frac{\partial^2 u}{\partial x^2} + P(x) \\ u(0,t) = U_1, \qquad u(L,t) = U2 \\ u(x,0) = f(x) \end{cases}$$

Then $u(x,t) = v(x) + w(x,t)$, where:

$v(x) =$

$\left[U_2 - U_1 + \int_0^L\int_0^z \frac{1}{\beta}P(s)dsdz\right]\frac{x}{L} + U_1 - \int_0^x\int_0^z \frac{1}{\beta}P(s)dsdz$

and $w(x,t)$ solves the hom. eqn:

$$\begin{cases} \frac{\partial w}{\partial t} = \beta\frac{\partial^2 w}{\partial x^2} \\ w(0,t) = 0, \qquad w(L,t) = 0 \\ u(x,0) = f(x) - v(x) \end{cases}$$

D'Alembert's formula: ONLY works for wave equation and $-\infty < x < \infty$:

$u(x,t) = \frac{1}{2}\left(f(x+\alpha t) + f(x-\alpha t)\right) + \frac{1}{2\alpha}\int_{x-\alpha t}^{x+\alpha t} g(s)ds$,

where $u_{tt} = \alpha^2 u_{xx}$, $u(x,0) = f(x)$, $\frac{\partial u}{\partial t}u(x,0) = g(x)$. The integral just means 'antidifferentiate and plug in'.

Laplace equation:

Same as for Heat/Wave, but $T(t)$ becomes $Y(y)$, and we get $Y''(y) = -\lambda Y(y)$. Also, instead of writing $Y(y) = \widetilde{A_m}e^{\omega y} + \widetilde{B_m}e^{-\omega y}$, write $Y(y) = \widetilde{A_m}\cosh(\omega y) + \widetilde{B_m}\sinh(\omega y)$. Remember $\cosh(0) = 1$, $\sinh(0) = 0$