Math 54 Cheat Sheet

Vector spaces
Subspace: If \( u \) and \( v \) are in \( W \), then \( u + v \) are in \( W \), and \( cu \) is in \( W \).
Nul(A): Solutions of \( Ax = 0 \). Row-reduce A.
Row(A): Space spanned by the rows of \( A \). Row-reduce A and choose the rows that contain the pivots.
Col(A): Space spanned by columns of \( A \). Row-reduce A and choose the columns of \( A \) that contain the pivots.

Rank(A): \( \dim(\text{Col}(A)) = \) number of pivots

Rank-Nilpotency theorem: \( \text{Rank}(A) + \dim(\text{Nul}(A)) = n \), where \( A \) is \( m \times n \).
Linear transformation: \( T(u + v) = T(u) + T(v) \), \( T(cu) = cT(u) \), where \( c \) is a number.

Linear independence: \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) \( \Rightarrow a_1 = a_2 = \cdots = a_n = 0 \).

To show Linear indep. form the matrix of the vectors, and show that \( \text{Nul}(A) = \{0\} \).
Linear dependence: \( a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \) for \( a_1, a_2, \ldots, a_n \) not all zero.
Span: Set of linear combinations of \( v_1, v_2, \ldots, v_n \).

Basis \( B \) for \( V \): A linearly independent set such that \( \text{Span}(B) = V \).
To show \( B \) is a basis, show it is linearly independent and spans.
To find a basis from a collection of vectors, form the matrix \( A \) of the vectors, and find \( \text{Col}(A) \).
To find a basis for a vector space, take any element of that v.s. and express it as a linear combination of ‘simpler’ vectors. Then show those vectors form a basis.

Dimension: Number of elements in a basis.
To find dim, find a basis and find number of elements.

Orthogonality
\( \|u\| = \sqrt{u \cdot u} \)
\( u_1 \cdot u_2 = 0 \) if \( i \neq j \), orthonormal if \( u_1 \cdot u_1 = 1 \).

Orthogonal matrix Q has orthonormal columns!

Consequence: \( QT = Q^T \)
Orthogonal projection on \( C(\mathbf{Q}) \).

Orthogonal projection: \( \text{Proj}_W(y) = \left( \frac{y \cdot u_1}{u_1 \cdot u_1} \right) u_1 + \cdots + \left( \frac{y \cdot u_n}{u_n \cdot u_n} \right) u_n \)
\( y - \text{Proj}_W(y) \) is orthogonal to \( y \), shortest distance between \( y \) and \( W \).

Invertible matrix theorem: If \( A \) is invertible, then \( A \) is row-equivalent to \( I \). A has \( n \) pivots, \( T(x) = Ax \) is one-to-one onto, \( A\) is \( b \) if and only if \( b \) is in \( \text{Col}(A) \).

Invertible matrix theorem and inverse:

Diagonalization:

Diagonalizability: \( A \) is diagonalizable if \( A = PDP^{-1} \) for some diagonal \( D \) and invertible \( P \).

A and \( B \) are similar if \( A = PBP^{-1} \) for \( P \) invertible.

Theorem: \( A \) is diagonalizable \( \iff A \) has \( n \) linearly independent eigenvectors.

Notes: A can be diagonalizable even if it’s not invertible (Ex: \( A = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \)).

Consequence: \( A = PDP^{-1} \Rightarrow A^m = PDP^{-1} \)

Least squares solution makes \( \|Ax - b\| \) smallest.

Symmetric matrices \( A = A^T \)

Has \( n \) real eigenvalues, always diagonalizable, orthogonal diagonalization \((A = PDP^T, \quad P \) is an orthogonal matrix, equivalent to symmetry!).

Theorem: If \( A \) is symmetric, then any two eigenvectors from different eigenspaces are orthogonal.

To orthogonal diagonalize: First diagonalize, then apply G-S on each eigenspace and normalize.

Spectral decomposition: \( \lambda_1 u_1 u_1^T + \lambda_2 u_2 u_2^T + \cdots + \lambda_n u_n u_n^T \)

Second-order and Higher-order differential equations
Homogeneous solutions:
Auxiliary equation: Replace equation by polynomial, so \( y'' + \cdots \) becomes \( r^n \) etc. Then find the zeros (use the ratioal roots theorem and long division, see the “Diagonalization” section).
Simple zeros “give you \( e^{rt} \), Repeated zeros (multiplicity \( m \)) give you \( Ae^{rt} + Bte^{rt} + \cdots + Zt^{m-1}e^{rt} \). Complex zeros \( r = a + bi \) give you \( Ae^{at}\cos(bt) + Be^{at}\sin(bt) \).
Undetermined coefficients: \( y(t) = y_p(t) + y_h(t) \), where \( y_h(t) \) solves the hom. eqn. (equation = 0), and \( y_p(t) \) is a particular solution.

If the inhom. term is \( Ct^m \), then \( y_p(t) = \frac{t^m}{(m+1)!} \). If \( r \) is a root of aux with multiplicity \( m \), then \( s = m \), and if \( r \) is not a root, then \( s = 0 \).

If the inhom. term is \( C_1e^{st} \sin(bt) \), then \( y_p(t) = C_1e^{st} \). Complex forms: To find the matrix, put the \( x^2 \)-coefficients on the diagonal, and evenly distribute the other terms. For example, if \( a_1x^2 - \cdots \), then \( 2 \) becomes 2 and \( 3 \) becomes 3.

Eigenfunction: \( y(t) = \cos(at) + \sin(bt) \).
Orthogonal always goes with \( \cos \) and \( \sin \) and \( \text{versa} \). Only have to look at \( a + bi \) as one entity.

Variation of parameters: First, make sure the leading coefficient (usually the coeff. of \( y'' \)) is 1. Then \( y_p(t) = y_h(t) + y_r(t) \), where \( y_r(t) \) and \( y_r(t) \) are your particular solutions. Then \( y_1y_2 y_3 = 0 \). Invert the matrix and solve for \( v_1' \) and \( v_2' \), and integrate to get \( v_1 \) and \( v_2 \), finally use: \( y_p(t) = v_1(t)y_1(t) + v_2(t)y_2(t) \).
Least-squares: Solve \( Ax = b \) in the least squares-way, solve \( A^TA = A^TB \).
If 

First find homogeneous solution. Then for

\[ f(x) = g_1(t) + g_2(t) \] (2 functions),

\[ W(t) = \left[ \begin{array}{c} f(t) \\ f'(t) \\ g(t) \\ g'(t) \end{array} \right] \] (3 functions). Then pick a point \( t_0 \) where \( \det(W(t_0)) \) is easy to evaluate. If \( \det \neq 0 \), then \( f, g, h \) are linearly independent! Try to look for simplifications before you differentiate.

Fundamental solution set: If \( f, g, h \) are solutions and linearly independent, then the general solution is

\[ y(t) = c_1 f(t) + c_2 g(t) + c_3 h(t) \]

Largest interval of existence: First make sure the leading coefficient equals 1. Then look at the domain of each term. For each domain, consider the part of the interval which contains the initial condition. Finally, intersect the intervals and change any brackets to parentheses.

Harmonic oscillator: \( my'' + by' + ky = 0 \) (\( m = \text{inertia} \), \( b = \text{damping} \), \( k = \text{stiffness} \))

**Systems of differential equations**

To solve \( \mathbf{x}' = A\mathbf{x} \): \( \mathbf{x}(t) = e^{At} \mathbf{v}_1 + Be^{\lambda_1 t} \mathbf{v}_2 + e^{\lambda_2 t} \mathbf{v}_3 \) (\( \lambda_i \) are eigenvalues, \( \mathbf{v}_i \) are eigenvectors)

**Fundamental matrix:** Matrix whose columns are the solutions, without the constants (the columns are solutions and linearly independent)

Complex eigenvalues \( A = \alpha + i\beta, \) and \( \mathbf{v} = \alpha + i\beta \mathbf{v} \) then:

\[ \mathbf{x}(t) = (e^{\alpha t} \cos(\beta t))\mathbf{v}_1 + e^{\alpha t} \mathbf{v}_2 + (e^{\alpha t} \sin(\beta t))\mathbf{v}_3 \]

Notes: You only need to consider one complex eigenvalue. For real eigenvalues, use the formula above. Also, \( a_{i+bi} = \frac{1}{2}(a+bi) \)

Fundamental eigenvectors If you only find one eigenvector \( \mathbf{v} \) (even though there are supposed to be 2), then solve the following equation for \( u: (A - \lambda I)u = v \) (one solution is enough). Then:

\[ \mathbf{x}(t) = e^{\lambda t} \mathbf{v} + B(e^{\lambda t} + e^{-\lambda t}) \]

Undetermined coefficients First find hom. solution. Then for \( x_{p} \), just like regular undetermined coefficients, except that instead of guessing \( x_{p}(t) = ae^{\lambda t} \mathbf{v}_1 + be^{\lambda t} \mathbf{v}_2 + ce^{k t} \mathbf{v}_3 \), you guess \(ae + b \cos(\omega t)\), where \( a = \frac{a_1}{a_2} \) is a vector. Then plug into \( \mathbf{x}' = A\mathbf{x} + \mathbf{f} \) and solve for \( a, b, c \)

Variation of parameters First hom. solution \( \mathbf{x}_h(t) = e^{\lambda t} \mathbf{v}_1 + Be^{\lambda t} \mathbf{v}_2 + e^{\lambda t} \mathbf{v}_3 \). Then:

\[ \mathbf{x}_p(t) = \mathbf{v}_1(t) \mathbf{x}_h(t) + \mathbf{v}_2(t) \mathbf{x}_h(t) + \mathbf{v}_3(t) \mathbf{x}_h(t) \]

Partial differential equations

Full solution; \( f \) defined on \([-T, T] \),

\[ f(x) = \sum_{m=0}^{\infty} a_m \cos \left( \frac{m\pi}{L} x \right) \]

Proper modes:

\[ f(x) = \sum_{m=0}^{\infty} \sin \left( \frac{m\pi}{L} x \right) \]

Proper modes:

\[ f(x) = \sum_{m=0}^{\infty} \alpha_m \cos \left( \frac{m\pi}{L} x \right) \]

\[ b_m = \frac{1}{L} \int_{0}^{L} f(x) \sin \left( \frac{m\pi}{L} x \right) \, dx \]

Cosine series: \( f \) defined on \((0, T) \), \( f(x) = \sum_{m=0}^{\infty} a_m \cos \left( \frac{m\pi}{L} x \right) \)

where:

\[ a_0 = \frac{1}{T} \int_{0}^{T} f(x) \, dx \]

\[ a_m = \frac{1}{T} \int_{0}^{T} f(x) \cos \left( \frac{m\pi}{L} x \right) \, dx \]

Sine series: \( f \) defined on \((0, T) \), \( f(x) = \sum_{m=0}^{\infty} b_m \sin \left( \frac{m\pi}{L} x \right) \),

where:

\[ b_0 = 0 \]

\[ b_m = \frac{1}{T} \int_{0}^{T} f(x) \sin \left( \frac{m\pi}{L} x \right) \, dx \]

Tabular integration (IBP): \( \int g(x) \sin \left( \frac{m\pi}{L} x \right) \, dx \)