Math 54 Cheat Sheet

Vector spaces

Subspace: If u and v are in W, then u + v are in W, and cu is in W.
Null(A): Solutions of Ax = 0. Row-reduce A.
Row(A): Space spanned by the rows of A. Row-reduce A and choose the columns of A that contain the pivots.
Col(A): Space spanned by columns of A. Row-reduce A and choose the columns of A that contain the pivots.
Rank(A) = Dim(Null(A)) + Dim(Col(A)) = number of pivots

Rank-Nilpotency theorem:
rank(A) + dim(Nul(A)) = n, where A is a matrix of rank r.

Linear transformation: T(u + v) = T(u) + T(v), T(cu) = cT(u), which c is a number.
T is one-to-one if T(u) = 0 ⇒ u = 0
T is onto if Col(T) = \mathbb{R}^n.

Linear dependence: \[ a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \Rightarrow a_1 = a_2 = \cdots = a_n = 0. \]

To show lin. ind. form the matrix of the vectors, and show that Null(A) = \{ 0 \}.

Linear dependence: \[ a_1v_1 + a_2v_2 + \cdots + a_nv_n = 0 \text{ for } a_1, a_2, \ldots, a_n \text{ all zero.} \]

Span: Set of linear combinations of \( v_1, \ldots, v_n \).

Basis B for V: A linearly independent set such that Span(B) = V.
To show sthg is a basis, show it is linearly independent and spans.

To find a basis from a collection of vectors, form the matrix A of the vectors, and find Col(A).
To find a basis for a vector space, take any element of that v.s. and express it as a linear combination of ‘simpler’ vectors. Then show those vectors form a basis.

Dimension: Number of elements in a basis.
To find dim, find a basis and find num. els.

Theorem: If V has a basis of vectors, then every basis of V must have n vectors.

Basis theorem: If V is an n-dim v.s., then any lin. ind. set with n elements is a basis, and any set of n els. which spans V is a basis.

Matrix of a lin. transf T with respect to bases B and C:
For every vector v in B, evaluate T(v), and express T(v) as a linear combination of vectors in C. Put the coefficients in a column vector, and then form the matrix of the column vectors you found!

Coordinates: To find [x]_C, express x in terms of the vectors in B.

Invertible matrix theorem: If A is invertible, then A is row-equivalent to I, A has n pivots, T(x) = Ax is one-to-one and onto, Ax = b has a unique solution for every b, A^T is invertible, det(A) \neq 0, the columns of A form a basis for \( \mathbb{R}^n \).

Rank(A) = n

Change of basis: [x]_C = P_{C \to B} [x]_B \text{ (think of C as the new, cool basis)}.

Basis for a vector space, take any element of \[ v_1, \ldots, v_n \] is a basis.

Diagonalization

Diagonalizable: A is diagonalizable if A = PDP^{-1} for some diagonal D and invertible P.

Theorem: A is diagonalizable \implies A has n linearly independent eigenvectors.

If A has n distinct eigenvalues, THEN A is diagonalizable, but the opposite is not always true!!!

Notes: A can be diagonalizable even if it’s not invertible (Ex: A = \[ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \]). Not all matrices are diagonalizable (Ex: \[ \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix} \]).

Consequence: A = PDP^{-1} \implies A^n = PDP^{n-1}.

How to diagonalize: To find the eigenvalues, calculate det(A - \lambda I), and find the roots of that.
To find the eigenvectors, for each \lambda find a basis for Null(A - \lambda I), which you do by row-reducing.

Rational roots theorem: If p(\lambda) = 0 has a rational root \( \frac{r}{s} \), then a divides the constant term of p, and b divides the leading coefficient.

Use this to guess zeros of p. Once you have a zero that works, use long division! Then A = PDP^{-1}, where D is diagonal matrix of eigenvalues.

Orthogonality

u, v are orthogonal if u \cdot v = 0.
\|u\| = \sqrt{u \cdot u} {u_1, \ldots, u_n} is orthogonal if u_i \cdot u_j = 0 if i \neq j.

Orthogonal if u_1 \cdot u_1 = 1 \implies \text{ orthogonal to every } w \in W.

If {u_1, \ldots, u_n} is an orthogonal basis, then:
y = c_1u_1 + \cdots + c_nu_n \implies c_j = \langle y, u_j \rangle / \langle u_j, u_j \rangle

Orthogonal matrix Q has orthonormal columns!
Consequence: QTQ = I, QQ^T = \text{Orthogonal projection on } \text{Col}(Q).

Orthogonal projection: If {u_1, \ldots, u_k} is a basis for W, then orthogonal projection of y on W is:
y = \left( \frac{y \cdot u_1}{\langle u_1, u_1 \rangle} {u_1, \ldots, u_k} \right) u_1 + \cdots + \left( \frac{y \cdot u_k}{\langle u_k, u_k \rangle} {u_k} \right) u_k

y - \hat{y} is orthogonal to \hat{y}, shortest distance btw y and W is \| y - \hat{y} \|.

Gram-Schmidt: Start with B = {u_1, \ldots, u_n}. Let:

v_1 = u_1
v_2 = u_2 - \left( \frac{v_2 \cdot u_1}{v_1 \cdot v_1} \right) v_1
v_3 = u_3 - \left( \frac{v_3 \cdot u_1}{v_1 \cdot v_1} \right) v_1 - \left( \frac{v_3 \cdot u_2}{v_2 \cdot v_2} \right) v_2

Then {v_1, \ldots, v_n} is an orthogonal basis for Span(B), and if w_1 = \frac{y}{\langle y, y \rangle}, then {w_1, \ldots, w_n} is an orthonormal basis for Span(B).

QR-factorization: To find Q, apply G-S to columns of A. Then R = QT A

Least-squares: To solve Ax = b in the least squares-way, solve \hat{A}T \hat{x} = \hat{A}T b.

Least squares solution makes \| A\hat{x} - b \| smallest.
\hat{x} = R^{-1} Q^T b, where A = QR.
Inner product spaces $f \cdot g = \int_a^b f(t)g(t)dt$. G-S applies with this inner product as well.

Cauchy-Schwarz: $|u \cdot v| \leq ||u|| ||v||$

Triangle inequality: $||u + v|| \leq ||u|| + ||v||$

Symmetric matrices ($A = A^T$)

Has $n$ real eigenvalues, always diagonalizable, orthogonally diagonalizable ($A = PDP^T$, $P$ is an orthogonal matrix, equivalent to symmetry!)

**Theorem:** If $A$ is symmetric, then any two eigenvectors from different eigenspaces are orthogonal. How to orthogonally diagonalize: First diagonalize, then apply G-S on each eigenspace and normalize. Then $P$ = matrix of (orthonormal) eigenvectors, $D$ = matrix of eigenvalues.

**Spectral decomposition:** $A = \sum_{i=1}^{n} \lambda_i e_i e_i^T$

**Undetermined coefficients:** where $\lambda_i$ is a real eigenvalue.

**Useful formulas:**

- $Ae^{\lambda t} = \sum_{i=1}^{n} \lambda_i e_i e_i^T$ for $n$ real eigenvalues.
- $\lambda_i$ is a solution if $det(A - \lambda I) = 0$.
- $e^{\lambda t}$, where $\lambda$ is a complex eigenvalue:

**Second-order and Higher-order differential equations**

Homogeneous solutions: Auxiliary equation: Replace equation by polynomial, so $y''$ becomes $r^2$ etc. Then find the zeros (use the rational roots theorem and long division, see the ‘Diagonalization-section’). ‘Simple zeros’ give you $e^{rt}$. Repeated zeros (multiplicity $m$) give you $Ae^{rt} + Bte^{rt} + \cdots + Zte^{mr-1}r^{m-1}$. Complex zeros $r = a + bi$ give you $Ae^{ar}e^{bit} + Be^{ar}e^{-bit}$.

**Undetermined coefficients:** $y(t) = y_0(t) + y_p(t)$, where $y_0$ solves the hom. eqn. (equation = 0), and $y_p$ is a particular solution. To find $y_p$:

- If the inhom. term is $Ct^{m}e^{rt}$, then:
  \[ y_p = t^{m}(A_m t^{m-1} + A_1 e^{rt}) \]
  where $m = 0$ if $r$ is a root of $A_m x^m + \cdots + A_1 = 0$.
- If the inhom. term is $Ct^{m}e^{rt} \sin(\beta t)$, then:
  \[ y_p = t^{m}(A_m t^{m-1} + A_1 \cos(\beta t) + B_1 t + 1)e^{rt} \sin(\beta t) \]
  where $s = m$, if $a + bi$ is also a root of $A_m x^m + \cdots + A_1 = 0$.
- If the inhom. term is $Ct^{m}e^{rt} \cos(\beta t)$, then:
  \[ y_p = t^{m}(A_m t^{m-1} + A_1 \cos(\beta t) + B_1 t + 1)e^{rt} \cos(\beta t) \]
  where $s = m$, if $a + bi$ is also a root of $A_m x^m + \cdots + A_1 = 0$.

**Systems of differential equations**

To solve $x' = Ax$:

- $x(t) = Ae^{\lambda t}v_1 + Be^{\lambda_2 t}v_2 + \cdots + e^{\lambda_n t}v_n$ ($\lambda_i$ are your eigenvalues, $v_i$ are your eigenvectors).

**Fundamental matrix:** Matrix whose columns are the solutions, without the constants (the columns are solutions and linearly independent).

- Complex eigenvalues: $A = (A - \lambda I)u = v$ (one solution is enough).
- Then $x(t) = Ae^{\lambda t}v + B(t)e^{\lambda t}v + e^{\lambda t}u$.

Undetermined coefficients First find hom. solution. Then for $\mu_\lambda$, just like regular undetermined coefficients, except that instead of guessing $\mu_\lambda u = ae^{\lambda t} + b \cos(t)$, you guess $ae^{\lambda t} + b \cos(t)$, where $a = \frac{a_1}{\omega^2 - \lambda^2}$ is a vector. Then plug into $x' = Ax + f$ and solve for $a$ etc.

**Variation of parameters** First hom. solution $x_h(t) = Ax_1(t) + Bx_2(t)$.

- Then:
  $x_p(t) = v_1(t)x_1(t) + v_2(t)x_2(t)$
  Then:
  $W(t) = \left[ \begin{array}{c} v_1(t) \\ v_2(t) \end{array} \right] = e^{At}$, where $\hat{x}(t) = \left[ \begin{array}{c} x_1(t) \\ x_2(t) \end{array} \right]$.

- Finally:
  $x(t) = x_h + x_p$.

**Matrix exponential** $e^{At} = \sum_{n=0}^{\infty} \frac{A^n t^n}{n!}$. To calculate $e^{At}$, diagonalize:

- $A = PDP^{-1} \Rightarrow e^{At} = Pe^{Dt}P^{-1}$, where $e^{Dt}$ is a diagonal matrix with diag. entries $e^{\lambda t}$.
- Or $A$ only has one eigenvalue $\lambda$ with multiplicity $m$, use $e^{At} = e^{\lambda t} \sum_{n=0}^{m-1} \frac{(A - \lambda I)^n t^n}{n!}$.

- Solution of $x' = Ax$ is then $x(t) = e^{At}c$, where $c$ is a constant vector.
Coupled mass-spring system

Case $N = 2$
Equation: $x'' = Ax, \ A = \begin{bmatrix} -2 & 1 \\ 1 & -2 \end{bmatrix}$
Proper frequencies: Eigenvalues of $A$ are:
$\lambda = -1, -3$, then proper frequencies $\pm i, \pm \sqrt{3}i$ (± square roots of eigenvalues)
Proper modes: $v_1 = \begin{bmatrix} \sin \left( \frac{\pi}{2} \right) \\ \sin \left( \frac{2\pi}{3} \right) \end{bmatrix}$
$v_2 = \begin{bmatrix} \sin \left( \frac{2\pi}{3} \right) \\ -\sin \left( \frac{\pi}{2} \right) \end{bmatrix}$

Partial differential equations

Full Fourier series: $f$ defined on $(-T, T)$:
$f(t) = \sum_{n=1}^{\infty} \left( a_n \cos \left( \frac{2\pi nt}{T} \right) + b_n \sin \left( \frac{2\pi nt}{T} \right) \right)$, where:
$a_0 = \frac{1}{T} \int_{-T}^{T} f(t) \, dt$
$a_n = \frac{1}{T} \int_{-T}^{T} f(t) \, \cos \left( \frac{2\pi nt}{T} \right) \, dt$
b$0 = \frac{1}{T} \int_{-T}^{T} f(t) \, \sin \left( \frac{2\pi nt}{T} \right) \, dt$

Cosine series: $f$ defined on $(0, T)$:
$f(t) = \sum_{n=1}^{\infty} a_n \cos \left( \frac{2\pi nt}{T} \right)$, where:
a$0 = \frac{2}{T} \int_{0}^{T} f(t) \, \cos \left( \frac{2\pi nt}{T} \right) \, dt$
a$0 = \frac{2}{T} \int_{0}^{T} f(t) \, \sin \left( \frac{2\pi nt}{T} \right) \, dt$

Sine series: $f$ defined on $(0, T)$:
$f(t) = \sum_{n=1}^{\infty} b_n \sin \left( \frac{2\pi nt}{T} \right)$, where:
b$0 = \frac{2}{T} \int_{0}^{T} f(t) \, \cos \left( \frac{2\pi nt}{T} \right) \, dt$
b$m = \frac{2}{T} \int_{0}^{T} f(t) \, \sin \left( \frac{2\pi nt}{T} \right) \, dt$
Tabular integration: (BIP: $\int f'g = fg - \int fg'$) To integrate $f(t)g(t) \, dt$ where $f$ is a polynomial, make a table whose first row is $f(t)$ and $g(t)$. Then differentiate $f$ as many times until you get 0, and antidifferentiate as many times until it aligns with the 0 for $f$. Then multiply the diagonal terms and do + first term second term etc.

Orthogonality formulas:
$\int_{-T}^{T} \cos \left( \frac{2\pi mt}{T} \right) \, dx = 0$
$\int_{-T}^{T} \cos \left( \frac{2\pi nt}{T} \right) \, dx = 0$ if $m \neq n$
$\int_{-T}^{T} \sin \left( \frac{2\pi nt}{T} \right) \, dx = 0$ if $m \neq n$

Convergence: Fourier series $F$ goes to $f(x)$ if $F$ is continuous at $x$, and if $f$ has a jump at $x$, $F$ goes to the average of the jumps. Finally, at the endpoints, $F$ goes to the average of the left/right endpoints.

Heat/Wave equations:

Step 1: Suppose $u(x, t) = X(x)T(t)$, plug this into PDE, and group $X$-terms and $T$-terms. Then $X''(x) \overline{X(x)} = \lambda \overline{X}$, then proper frequencies

$\pm 2i \sin \left( \frac{\pi m}{N+1} \right)$
k = 1, 2, ··· $N$
equation and $-\infty < x < \infty$: $u(x, t) = \frac{1}{2} (f(x + \alpha t) + f(x - \alpha t)) + \frac{1}{2\alpha} \int_{x - \alpha t}^{x + \alpha t} g(s)ds$, where

$$u_{tt} = \alpha^2 u_{xx}, u(x, 0) = f(x), \frac{\partial u}{\partial t} u(x, 0) = g(x).$$

The integral just means ‘antidifferentiate and plug in’.

Laplace equation:
Same as for Heat/Wave, but $T(t)$ becomes $Y(t)$, and we get $Y''(y) = -\lambda Y(y)$. Also, instead of writing $Y(y) = A_m e^{\omega y} + B_m e^{-\omega y}$, write $Y(y) = A_m \cosh(\omega y) + B_m \sinh(\omega y)$. Remember $\cosh(0) = 1, \sinh(0) = 0$. 