4.5.11.

D : \( \mathbb{R} - \{ \pm 3 \} \)

I : No \( x \)-intercepts, \( y \)-intercept: \( y = -\frac{1}{9} \)

S : \( f \) is even

A : Horizontal Asymptote \( y = 0 \) (at \( \pm \infty \)), Vertical Asymptotes \( x = \pm 3 \)

I : \( f'(x) = -\frac{2x}{(x^2-9)^2} \); \( f \) is increasing on \(( -\infty, -3 ) \cup ( -3, 0 )\) and decreasing on \(( 0, 3 ) \cup ( 3, \infty ) \). Local maximum of \( -\frac{1}{9} \) at 0.

C : \( f''(x) = 6 \frac{x^2+3}{(x^2-9)^3} \); \( f \) is concave up on \(( -\infty, -3 ) \cup ( 3, \infty )\) and concave down on \(( -3, 3 )\); No inflection points

1A/Homeworks/hw10graph1.png

4.5.31.

Note: First of all, \( f \) is periodic of period \( 2\pi \), so we’re only focusing on \([0, 2\pi]\).

D : \( \mathbb{R} \)

I : \( x \)-intercepts: \( x = 0, x = 2\pi \) (basically you should get \( \sin(x) = 3 \), which is impossible), \( y \)-intercept: \( y = 0 \)

S : Again, \( f \) is periodic of period \( 2\pi \). Also, \( f \) is odd.

A : No asymptotes

Date: Monday, April 11th, 2011.
\[ f'(x) = 3 \cos(x) - 3 \cos(x) \sin(x) = 3 \cos(x)(1 - \sin^2(x)) = 3 \cos^3(x); \]
Increasing on \((0, \frac{\pi}{2}) \cup (\frac{3\pi}{2}, 2\pi)\); Decreasing on \((\frac{\pi}{2}, \frac{3\pi}{2})\). Local maximum of 2 at \(x = \frac{\pi}{2}\). Local minimum of \(-1\) at \(x = \frac{3\pi}{2}\).

\[ f''(x) = -9 \sin(x) \cos^2(x); \]
Concave down on \((0, \pi)\) and Concave up on \((\pi, 2\pi)\). Inflection point \((\pi, 0)\)

4.5.41.

D : \(\mathbb{R}\)
I : No \(x\)-intercepts, \(y\)-intercept: \(y = \frac{1}{2}\)
S : No symmetries
A : Horizontal Asymptotes: \(y = 0\) (at \(-\infty\)), \(y = 1\) (at \(\infty\))
I : \( f'(x) = \frac{e^{-x}}{(1+e^{-x})^2} > 0 \), so \(f\) is decreasing on \(\mathbb{R}\)
C : \( f''(x) = \frac{e^{-x} - 1}{(e^{x}+1)^3} \) (multiply numerator and denominator by \((e^x)^3\) after simplifying), so \(f\) is concave down on \((\infty, 0)\) and concave up on \((0, \infty)\). Inflection point at \((0, \frac{1}{2})\)
Note: First of all, \( f \) is periodic of period \( 2\pi \), so from now on we may assume that \( x \in [0, 2\pi] \).

D: We want \( \sin(x) > 0 \), so the domain is \((0, \pi)\).

I: No \( y \)-intercepts, \( x \)-intercepts: Want \( \ln(\sin(x)) = 0 \), so \( \sin(x) = 1 \), so \( x = \frac{\pi}{2} \).

S: Again, \( f \) is periodic of period \( 2\pi \).

A: No horizontal/slant asymptotes, but \( \lim_{x \to 0^+} \ln(\sin(x)) = \ln(0^+) = -\infty \), so \( x = 0 \) is a vertical asymptote. Also \( \lim_{x \to \pi^-} \ln(\sin(x)) = -\infty \), so \( x = \pi \) is also a vertical asymptote.

I: \( f''(x) = \frac{\cos(x)}{\sin(x)} = \cot(x) \), then \( f'(x) = 0 \leftrightarrow x = \frac{\pi}{2} \), and using a sign table, we can see that \( f \) is increasing on \((0, \frac{\pi}{2})\) and decreasing on \((\frac{\pi}{2}, \pi)\).

Moreover, \( f(\frac{\pi}{2}) = \ln(1) = 0 \) is a local maximum of \( f \).

C: \( f''(x) = -\csc^2(x) < 0 \), so \( f \) is concave down on \((0, \pi)\).
1A/Homeworks/hw10graph.png
4.7.3.
- Want to minimize $x + y$
- But $xy = 100$, so $y = \frac{100}{x}$, so $x + y = x + \frac{100}{x}$
- Let $f(x) = x + \frac{100}{x}$
- $x > 0$ ($x$ is positive)
- $f'(x) = 0 \iff \frac{1}{x} - \frac{100}{x^2} = 0 \iff x^2 = 100 \iff x = 10$
- By FDTAEV, $x = 100$ is the absolute minimum of $f$
- Answer: $x = 100, y = \frac{100}{100} = 1$

4.7.11. The picture is as follows:

![Fence.png](1A/Practice Exams/Fence.png)

- Want to minimize $3w + 4l$.
- But $2lw = 1.5$, so $l = \frac{0.75}{w}$, so $3w + 4l = 3w + \frac{3}{w}$
- Let $f(w) = 3w + \frac{3}{w}$
- $w > 0$
- $f'(w) = 0 \iff 3 - \frac{3}{w^2} = 0 \iff w^2 = 1 \iff w = 1$
- By FDTAEV, $w = 1$ is the absolute minimum of $f$
- Answer: $w = 1, 2l = 1.5$

4.7.19.
- We have $D = \sqrt{(x - 1)^2 + y^2}$, so $D^2 = (x - 1)^2 + y^2$
- But $y^2 = 4 - 4x^2$, so $D^2 = (x - 1)^2 + 4 - 4x^2$
- Let $f(x) = (x - 1)^2 + 4 - 4x^2$
- No constraints
- $f'(x) = 2(x - 1) - 8x = -6x - 2 = 0 \iff x = -\frac{1}{3}$
- By the FDTAEV, $x = -\frac{1}{3}$ is the maximizer of $f$
- Since $y^2 = 4 - 4x^2$, we get $y^2 = 4 - \frac{4}{9} = \frac{32}{9}$, so $y = \pm \sqrt{\frac{32}{9}} = \pm \frac{4\sqrt{2}}{3}$
4.7.21. Picture:

\[ x^2 + y^2 = r \]

- We have \( A = xy \), but the trick here again is to maximize \( A^2 = x^2y^2 \) (thanks for Huiling Pan for this suggestion!)
- But \( x^2 + y^2 = r^2 \), so \( y^2 = r^2 - x^2 \), so \( A^2 = x^2(r^2 - x^2) = x^2y^2 - x^4 \)
- Let \( f(x) = x^2y^2 - x^4 \)
- Constraint \( 0 \leq x \leq r \) (look at the picture)
- \( f'(x) = 2xy^2 - 4x^3 = 0 \iff x = 0 \) or \( x = \frac{r}{\sqrt{2}} \)
- By the closed interval method, \( x = \frac{r}{\sqrt{2}} \) is a maximizer of \( f \) (basically \( f(0) = f(r) = 0 \)
- Answer: \( x = \frac{r}{\sqrt{2}}, y = \sqrt{r^2 - \frac{x^2}{2}} = \frac{r}{\sqrt{2}} \)

4.7.30.

- Let \( w \) be the width of the rectangle, and \( h \) the height of the rectangle.
- We have \( A = wh + \pi(\frac{w}{2})^2 = wh + \frac{\pi}{4}w^2 \), but \( w + 2h + 2\pi \frac{w}{2} = 30 \), so \( 2h + \pi w + w = 30 \), so \( h = \frac{30 - (\pi + 1)w}{2} \). Hence \( A = w\left(\frac{30 - (\pi + 1)w}{2}\right) + \frac{\pi}{4}w^2 \)
- Let \( f(w) = w\left(\frac{30 - (\pi + 1)w}{2}\right) + \frac{\pi}{4}w^2 \)
- Constraint: \( w > 0 \)
- \( f'(w) = 15 - \frac{(\pi + 2)}{2}w = 0 \iff w = \frac{30}{\pi + 2} \) (there’s a big cancellation going on!)
- By FDTEAEV, \( w = \frac{30}{\pi + 2} \) is the maximizer of \( f \)
- Answer: \( w = \frac{30}{\pi + 2}, h = \frac{15}{\pi + 2} \)
4.7.53. (a) \( c'(x) = \frac{C'(x)x - C(x)}{x^2} \). When \( c \) is at its minimum, \( c'(x) = 0 \), so \( C'(x)x - C(x) = 0 \), so \( C'(x) = \frac{C(x)}{x} = c(x) \), so \( C'(x) = c(x) \), i.e. marginal cost equals the average cost!

4.7.63. (thank you Brianna Grado-White for the solution to this problem!)

The picture is as follows:

Here, \( h_1 \) and \( h_2 \) and \( L \) are fixed, but \( x \) varies.
Now the total time taken is \( t = t_1 + t_2 = \frac{d_1}{v_1} + \frac{d_2}{v_2} \).

Now, by the Pythagorean theorem: \( d_1 = \sqrt{x^2 + h_1^2} \) and \( d_2 = \sqrt{(L - x)^2 + h_2^2} \), so we get:

\[
t(x) = \frac{\sqrt{x^2 + h_1^2}}{v_1} + \frac{\sqrt{(L - x)^2 + h_2^2}}{v_2}
\]

And

\[
t'(x) = \frac{x}{v_1 \sqrt{x^2 + h_1^2}} + \frac{x - L}{v_2 \sqrt{(L - x)^2 + h_2^2}} = \frac{x}{v_1 d_1} + \frac{x - L}{v_2 d_2}
\]

Setting \( t'(x) = 0 \) and cross-multiplying, we get:

\[
v_1 d_1 (L - x) = v_2 d_2 x
\]

So, by definition of \( \sin(\theta_1) \) and \( \sin(\theta_2) \), we get:

\[
\frac{v_1}{v_2} = \frac{d_2 x}{(L - x) d_1} = \frac{\frac{d_2 x}{d_1}}{\frac{L - x}{d_2}} = \frac{\sin(\theta_1)}{\sin(\theta_2)}
\]
Section 4.9: Antiderivatives

4.9.7. \( F(x) = 5 \frac{x^3}{x} - 4x^2 + C \)

4.9.24. \( f'(x) = 2x + \frac{1}{4}x^4 + \frac{1}{4}x^7 + A \), so \( f(x) = x^2 + \frac{1}{20}x^5 + \frac{1}{80}x^8 + Ax + B \)

4.9.33. \( f(x) = -2 \sin(t) + \tan(t) + C \), but \( 4 = f\left(\frac{\pi}{3}\right) = -\sqrt{3} + \sqrt{3} + C = C \), so \( f(x) = -2 \sin(t) + \tan(t) + 4 \)

4.9.33. If \( f''(\theta) = \sin(\theta) + \cos(\theta) \), then \( f'(\theta) = -\cos(\theta) + \sin(\theta) + C \).

Hence \( f'(\theta) = -\sin(\theta) - \cos(\theta) + 5\theta + C' \).

\( f(0) = 3 \), so \(-0 + 0 + C' = 3 \), so \( C' = 3 \).

Hence \( f'(\theta) = -\sin(\theta) - \cos(\theta) + 5\theta + 4 \)

4.9.61. \( a(t) = 10 \sin(t) + 3 \cos(t) \), so \( v(t) = -10 \cos(t) + 3 \sin(t) + A \), so \( s(t) = -10 \sin(t) - 3 \cos(t) + At + B \)

Now, \( s(0) = 0 \), but \( s(0) = -10(0) - 3(1) + A(0) + B \), so \(-3 + B = 0 \), so \( B = 3 \)

So \( s(t) = -10 \sin(t) - 3 \cos(t) + At + 3 \)

Moreover, \( s(2\pi) = 12 \), but \( s(2\pi) = -10(0) - 3(1) + A(2\pi) + 3 = A(2\pi) \), so \( A(2\pi) = 12 \), so \( A = \frac{12}{2\pi} = \frac{6}{\pi} \)

So altogether, you get: \( s(t) = -10 \sin(t) - 3 \cos(t) + \frac{6}{\pi}t + 3 \)

4.9.74. First of all, the acceleration of the car is \( a(t) = -16 \), so \( v(t) = -16t + C \).

We want to find \( v(0) = C \), so once we find \( C \), we’re done!

Let \( t^* \) be the time when the car comes to a stop.
Then \( v(t^*) = 0 \), so \(-16t^* + C = 0 \), so \( C = 16t^* \). So once we find \( t^* \), we’re done!

Now we know that \( s(t^*) - s(0) = 200 \), but \( s(t) = -8t^2 + C + C' \), so \( 200 = -8(t^*)^2 + C + C' + 0 - C(0) - C' = -8(t^*)^2 + 16t^* \), so \( 8(t^*)^2 = 200 \), so \((t^*)^2 = 25 \) so \( t^* = 5 \) (assuming time is positive)

Whence \( v(0) = C = 16t^* = 80 \)