Problem: A number $a$ is called a fixed point of a function $f$ if $f(a) = a$ (for example, 1 is a fixed point of $f(x) = x^2$). Prove that if $f'(x) \neq 1$ for all $x$, then $f$ has at most one fixed point.

Solution:

This is again a proof by contradiction!

Suppose $f$ has (at least) two fixed points $a$ and $b$.

Then, by definition of a fixed point, $f(a) = a$, and $f(b) = b$.

However, by the Mean Value Theorem, there is a $c$ in $(a, b)$ such that:

$$\frac{f(b) - f(a)}{b - a} = f'(c)$$

Now using the fact that $f(b) = b$ and $f(a) = a$, we get:

$$\frac{b - a}{b - a} = f'(c)$$

So

$$1 = f'(c)$$

That is, $f'(c) = 1$. However, by assumption, $f'(x) \neq 1$ for all $x$, so in particular setting $x = c$ gives $f'(c) \neq 1$.

but since $f'(c) = 1$, we get $1 \neq 1$, which is a contradiction!

Hence $f$ has at most one fixed point!