MATH 1A - SOLUTION TO 3.5.69

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Let's denote the point of intersection between the ellipse and the tangent line by (a, b).

Then, using implicit differentiation, we can show that the slope of the tangent line is

$$-\frac{a}{4b}$$

Now, let K be the altitude of the lamp, our goal is to find K.

Notice that the same tangent line goes through the points $(-5, 0)$ and $(3, K)$, so by the slope formula, we have:

$$\text{Slope} = \frac{K - 0}{3 - (-5)} = \frac{K}{8}$$

In particular, since the slope is also equal to $-\frac{a}{4b}$, we have:

$$\frac{K}{8} = -\frac{a}{4b}$$

So

$$K = -8\frac{a}{4b} = -\frac{2a}{b}$$

So all we really need to do to solve this problem is to find $-\frac{2a}{b}$!

Now we also know that the tangent line goes through the points $(-5, 0)$ and (a, b), so its slope is $\frac{b-0}{a-(-5)} = \frac{b}{a+5}$, but again we know that its slope is also $-\frac{a}{4b}$, and so we get:

$$\frac{b}{a+5} = -\frac{a}{4b}$$

So cross-multiplying, we have $4b^2 = -(a)(a + 5)$, that is $a^2 + 4b^2 = -5a$.

HOWEVER, We also know that (a, b) is on the ellipse, so it satisfies the equation of the ellipse, and so $a^2 + 4b^2 = 5$, whence we get $-5a = 5$, and so $a = -1$.

And plugging $a = -1$ into $a^2 + 4(b)^2 = 5$ and assuming $b > 0$, we get $b = 1$, and so $K = -\frac{2a}{b} = \frac{2}{1} = 2$, and we're done!

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