MATH 1A - SOLUTION TO 3.5.40

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First of all, by implicit differentiation:

$$
\begin{aligned}
\frac{2x}{a^2} + \frac{2yy'}{b^2} =& 0 \\
y'\left(\frac{2y}{b^2}\right) =& -\frac{2x}{a^2} \\
y' =& -\frac{b^2}{a^2}\frac{2x}{2y} \\
y' =& -\frac{b^2}{a^2}\frac{x}{y}
\end{aligned}
$$

It follows that the tangent line to the ellipse at (x_0, y_0) has slope $-\frac{b^2}{a^2}\frac{x_0}{y_0}$, and since it goes through (x_0, y_0), its equation is:

$$y - y_0 = \left(-\frac{b^2}{a^2}\frac{x_0}{y_0}\right)(x - x_0)$$

And the rest of the problem is just a little algebra!

First of all, by multiplying both sides by $a^2 y_0$, we get:

$$(y - y_0)(a^2 y_0) = -b^2 x_0 (x - x_0)$$

Expanding out, we get:

$$ya^2 y_0 - a^2 (y_0)^2 = -b^2 x_0 x + b^2 (x_0)^2$$

Now rearranging, we have:

$$ya^2 y_0 + b^2 x_0 x = a^2 (y_0)^2 + b^2 (x_0)^2$$

Now dividing both sides by a^2, we get:

$$yy_0 + \frac{b^2}{a^2} x_0 x = (y_0)^2 + \frac{b^2}{a^2}(x_0)^2$$

And dividing both sides by b^2, we get:

$$\frac{yy_0}{b^2} + \frac{x_0 x}{a^2} = \frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2}$$

But now, since (x_0, y_0) is on the ellipse, $\frac{(y_0)^2}{b^2} + \frac{(x_0)^2}{a^2} = 1$, so we get:

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$$\frac{yy_0}{b^2} + \frac{x_0 x}{a^2} = 1$$

Whence,

$$\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$$

Which is what we want!