MATH 1A - SOLUTION TO 3.3.51

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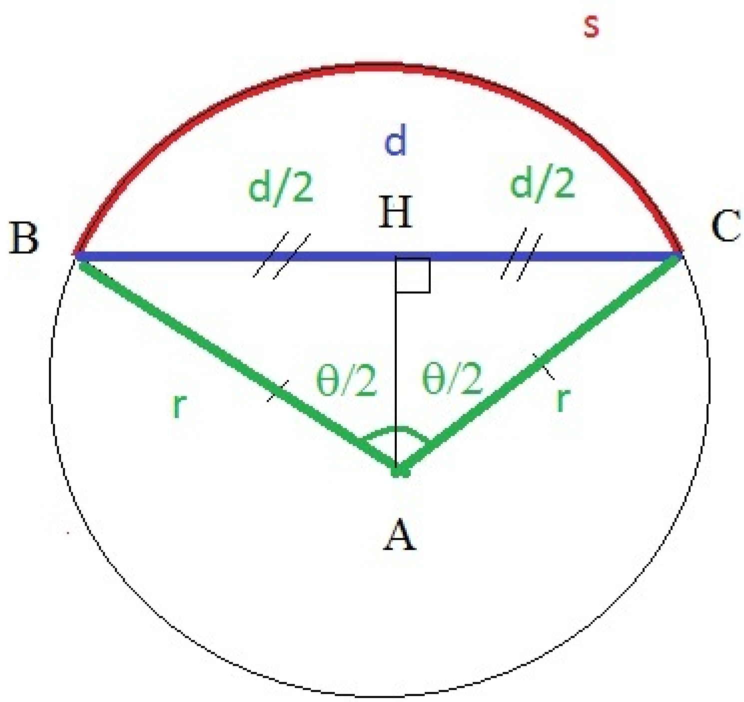
1A/Arclength.png



The problem is to calculate:

$$\lim_{\theta \to 0^+} = \frac{s}{d}$$

First of all, let's call the radius of the circle r. Now, let's divide this into three simple sub-problems:

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(1) **Calculate s**

But this is not very hard! Either you know about the formula for the length of an arc, or you can easily derive it! Namely, if an angle of 2π radians corresponds to a length of $2\pi r$ (i.e. the circumference of a circle), then an angle of θ radians corresponds to a length of s.

Now, because the length of an arc is proportional to the angle, we have the following equality:

$$\frac{s}{\theta} = \frac{2\pi r}{2\pi} = r$$

So $\boxed{s = \theta r}$

Notice that the use of radians makes this calculation particularly simple!

(2) **Calculate d**

This is a bit harder than the above step, but actually not that bad! Label the triangle in the figure ABC, and let H be the midpoint of BC. Then, the triangle AHB is right in A, and we can use our regular definition of sin to find a relationship between r and d, namely:

$$sin(\angle BAH) = \frac{BH}{AB}$$

But you can easily check that $\angle BAH = \frac{\theta}{2}$, that $BH = \frac{d}{2}$ (because H is the midpoint of BC), and that $AB = r$! Hence, we get:

$$sin(\frac{\theta}{2}) = \frac{\frac{d}{2}}{r}$$

That is, $\boxed{d = 2r sin(\frac{\theta}{2})}$.

(3) Compute the limit

The last thing we need to do is to calculate the required limit! But this is easy, since we know the values of s and d in terms of r:

$$\lim_{\theta \to 0^+} \frac{s}{d} = \lim_{\theta \to 0^+} \frac{r\theta}{2r sin(\frac{\theta}{2})} = \lim_{\theta \to 0^+} \frac{\theta}{2 sin(\frac{\theta}{2})} = \lim_{\theta \to 0^+} \frac{\frac{\theta}{2}}{sin(\frac{\theta}{2})}$$

Finally, let $t = \frac{\theta}{2}$ and notice that, as $\theta \to 0^+$, $t \to 0^+$, then, our limit becomes:

$$\lim_{\theta \to 0^+} \frac{s}{d} = \lim_{t \to 0^+} \frac{t}{sin(t)} = \lim_{t \to 0^+} \frac{1}{\frac{sin(t)}{t}} = \frac{\lim_{t \to 0^+} 1}{\lim_{t \to 0^+} \frac{sin(t)}{t}} = \frac{1}{1} = 1$$

And again, the next-to-last step is justified because both limits (numerator and denominator) exist and the limit of the denominator is nonzero!

Hence, we get:

$$\boxed{\lim_{\theta \to 0^+} \frac{s}{d} = 1}$$