1. (a) Let \( f \) be a function and \( a, L \) real numbers. Define carefully: \( \lim_{x \to a} f(x) = L \) if and only if...

For every number \( \epsilon > 0 \) there is a number \( \delta > 0 \) such that for all \( x \) in \( \mathbb{R} \),

\[
0 < |x - a| < \delta \quad \text{then} \quad |f(x) - L| < \epsilon
\]

(b) Show directly from the definition that \( \lim_{x \to 3} 2x = 6 \)

Let \( f(x) = 2x \)

Part I: Finding \( \delta \)

1) \( |f(x) - 6| = |2x - 6| = |2(x - 3)| = 2|x - 3| \)
2) \( 2|x - 3| < \epsilon \) implies \( |x - 3| < \frac{\epsilon}{2} \)
3) Let \( \delta = \frac{\epsilon}{2} \)

Part II: Showing your \( \delta \) works

1) Let \( \epsilon > 0 \) be given. Let \( \delta = \frac{\epsilon}{2} \), and suppose \( 0 < |x - 3| < \delta \). Then \( |x - 3| < \frac{\epsilon}{2} \)
2) Then \( |f(x) - 6| = 2|x - 3| < 2 \cdot \frac{\epsilon}{2} = \epsilon \)
3) Hence, if \( 0 < |x - 3| < \delta \), then \( |f(x) - 6| < \epsilon \)
2. Let \( f(x) = \sqrt{x} \) for all \( x \). Prove directly from the definition of the derivative that \( f'(a) = \frac{1}{2\sqrt{a}} \)

Solution 1:

\[
\begin{align*}
\lim_{x \to a} f'(a) &= \lim_{x \to a} \frac{f(x) - f(a)}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \\
&= \lim_{x \to a} \frac{\sqrt{x} - \sqrt{a}}{x - a} \times \left( \frac{\sqrt{x} + \sqrt{a}}{\sqrt{x} + \sqrt{a}} \right) \\
&= \lim_{x \to a} \frac{(\sqrt{x} - \sqrt{a})(\sqrt{x} + \sqrt{a})}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= \lim_{x \to a} \frac{x - a}{(x - a)(\sqrt{x} + \sqrt{a})} \\
&= \lim_{x \to a} \frac{1}{\sqrt{x} + \sqrt{a}} \\
&= \frac{1}{2\sqrt{a}}
\end{align*}
\]

Solution 2:

\[
\begin{align*}
\lim_{h \to 0} f'(a) &= \lim_{h \to 0} \frac{f(a + h) - f(a)}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{a + h} - \sqrt{a}}{h} \\
&= \lim_{h \to 0} \frac{\sqrt{a + h} - \sqrt{a}}{h} \times \frac{\sqrt{a + h} + \sqrt{a}}{\sqrt{a + h} + \sqrt{a}} \\
&= \lim_{h \to 0} \frac{(\sqrt{a + h} - \sqrt{a})(\sqrt{a + h} + \sqrt{a})}{h(\sqrt{a + h} + \sqrt{a})} \\
&= \lim_{h \to 0} \frac{(\sqrt{a + h})^2 - (\sqrt{a})^2}{h(\sqrt{a + h} + \sqrt{a})} \\
&= \lim_{h \to 0} \frac{a + h - a}{h(\sqrt{a + h} + \sqrt{a})} \\
&= \lim_{h \to 0} \frac{h}{h(\sqrt{a + h} + \sqrt{a})} \\
&= \lim_{h \to 0} \frac{1}{\sqrt{a + h} + \sqrt{a}} \\
&= \frac{1}{2\sqrt{a}}
\end{align*}
\]
3. Let \( f(x) = \sqrt{3 - e^{2x}} \)
   (a) Explain why \( f \) is one-to-one

   **Solution 1:**

   \[
   f(x) = f(y) \\
   \sqrt{3 - e^{2x}} = \sqrt{3 - e^{2y}} \\
   3 - e^{2x} = 3 - e^{2y} \\
   e^{2x} = e^{2y} \\
   \ln(e^{2x}) = \ln(e^{2y}) \\
   2x = 2y \\
   x = y
   \]

   Hence \( f \) is one-to-one.

   **Solution 2:** \( e^{2x} \) is increasing, so \(-e^{2x} \) is decreasing, so \( 3 - e^{2x} \) is decreasing, and hence \( f(x) = \sqrt{3 - e^{2x}} \) is (strictly) decreasing (as a composition of an increasing and a decreasing function). And hence \( f \) is one-to-one.

   (b) What is the domain of \( f^{-1} \)?

   **Solution 1:** Using our formula for \( f^{-1} \) in question (c), we see that we need \( 3 - x^2 > 0 \) (we want the number under the \( \ln \) to be positive), so \( x^2 < 3 \), so \(-\sqrt{3} < x < \sqrt{3}\). **However**, notice that we also want \( x \geq 0 \) (because \( f(x) \geq 0 \)), so \( \text{Domain } f^{-1} = [0, \sqrt{3}) \).

   **Solution 2:**

   First, let’s find the domain of \( f \) (this will be useful in a second). The only thing we need is that \( 3 - e^{2x} \geq 0 \) (the number under the square root be nonnegative)

   \[
   \begin{align*}
   3 - e^{2x} &\geq 0 \\
   -e^{2x} &\geq -3 \\
   e^{2x} &\leq 3 \\
   2x &\leq \ln(3) \\
   x &\leq \frac{\ln(3)}{2}
   \end{align*}
   \]

   So the domain of \( f \) is \( (-\infty, \frac{\ln(3)}{2}] \).

   Now, because \( f \) is decreasing, and \( f \left( \frac{\ln(3)}{2} \right) = 0 \) and \( \lim_{x \to -\infty} f(x) = \sqrt{3}, \) we get that the range of \( f \) is \( [0, \sqrt{3}) \). But the range of \( f \) is the domain of \( f^{-1} \), so \( \text{Domain of } f^{-1} = [0, \sqrt{3}) \).
(c) Find a formula for $f^{-1}$

1) Let $y = \sqrt{3 - e^{2x}}$

2)

\[y = \sqrt{3 - e^{2x}}\]
\[y^2 = 3 - e^{2x}\]
\[y^2 - 3 = -e^{2x}\]
\[3 - y^2 - e^{2x}\]
\[e^{2x} = 3 - y^2\]
\[2x = \ln(3 - y^2)\]
\[x = \frac{\ln(3 - y^2)}{2}\]

3) $f^{-1}(x) = \frac{\ln(3 - x^2)}{2}$
4. **A- Problem 3 on page 127**

   (a) \([-4]\) (f not defined at \(-4\)), \([-2]\) (\(\lim_{x \to -2} f(x)\) does not exist), \([0]\) (ditto), \([2]\) (ditto)

   (b)
   - \(-4\): Neither
   - \(-2\): Continuous from the left
   - \(2\): Continuous from the right
   - \(4\): Continuous from the right

**B- Problem 37 on page 164**

   - \([-4]\): Graph has a ‘kink’ at \(-4\) (left-hand-side limits and right-hand-side limits not equal)
   - \([0]\): Not continuous at 0

**C- Problem 39 on page 164**

   - \([-1]\): Vertical tangent line at \(-1\)
   - \([4]\): Graph has a ‘kink’ at 4
5. Compute:

(a) \( \lim_{x \to 3^+} \frac{x^2 - 9}{x^2 + 2x - 3} \)

We have \( \lim_{x \to 3^+} x^2 - 9 = 0 \) by continuity of \( y = x^2 - 9 \) while \( \lim_{x \to 3^+} x^2 + 2x - 3 = 3^2 + 2(3) - 3 = 9 + 6 - 3 = 12 \) by continuity of \( y = x^2 + 2x - 3 \), and hence:

\[ \lim_{x \to 3^+} \frac{x^2 - 9}{x^2 + 2x - 3} = \frac{0}{12} = 0 \]

(b) \( \lim_{x \to 3} \frac{x^2 - 9}{x^2 - 2x - 3} \) = \( \lim_{x \to 3} \frac{(x - 3)(x + 3)}{(x - 3)(x + 1)} = \lim_{x \to 3} \frac{x + 3}{x + 1} = \frac{3 + 3}{3 + 1} = \frac{6}{4} = \frac{3}{2} \)

(c) \( \lim_{x \to \infty} \frac{\sqrt{x^2 - 9}}{2x + 5} \) = \( \lim_{x \to \infty} \frac{\sqrt{x^2} \left( 1 - \frac{9}{x^2} \right)}{2x + 5} \)

\[ = \lim_{x \to \infty} \frac{\sqrt{1 - \frac{9}{x^2}}}{2x + 5} \]

\[ = \lim_{x \to \infty} \frac{|x| \sqrt{1 - \frac{9}{x^2}}}{2x + 5} \]

\[ = \lim_{x \to \infty} \frac{x \left( \sqrt{1 - \frac{9}{x^2}} \right)}{2x + 5} \]

\[ = \lim_{x \to \infty} \frac{x \left( \sqrt{1 - \frac{9}{x^2}} \right)}{x \left( 2 + \frac{5}{x} \right)} \]

\[ = \lim_{x \to \infty} \frac{\sqrt{1 - \frac{9}{x^2}}}{2 + \frac{5}{x}} \]

\[ = \frac{\sqrt{1 - 0}}{2 + 0} \]

\[ = \frac{1}{2} \]

(d) First of all,

\[ \frac{\cot(2t)}{t} = \frac{\cos(2t)}{\sin(2t)} \frac{t}{t} = \frac{\cos(2t)}{t(\sin(2t))} \]

Now \( \lim_{t \to 0^+} \cos(2t) = \lim_{t \to 0^+} \cos(2t) = 1 \)

And \( \lim_{t \to 0^+} t(\sin(2t)) = 0^+ \) and \( \lim_{t \to 0^-} t(\sin(2t)) = 0^+ \) (in the second case, it’s because both \( t \) and \( \sin(2t) \) are negative!)
Hence:

\[
\lim_{t \to 0^+} \frac{\cot(2t)}{t} = \lim_{t \to 0^+} \frac{\cos(2t)}{t \sin(2t)} = \frac{1}{0^+} = \infty
\]

and

\[
\lim_{t \to 0^-} \frac{\cot(2t)}{t} = \lim_{t \to 0^-} \frac{\cos(2t)}{t \sin(2t)} = \frac{1}{0^+} = \infty
\]

And thus:

\[
\lim_{t \to 0} \frac{\cot(2t)}{t} = \infty
\]
6. Compute:

(a) \[
\frac{d}{dt} \left( t^{\frac{1}{2}} \sec(t) \right) = \frac{1}{2} t^{-\frac{1}{2}} \sec(t) + t^{\frac{1}{2}} \sec(t) \tan(t)
\]

(b) \[
\frac{d}{du} \left( \frac{u}{u^2 + 1} \right) = \frac{1 \cdot (u^2 + 1) - u \cdot (2u)}{(u^2 + 1)^2} = \frac{u^2 + 1 - 2u^2}{(u^2 + 1)^2} = \frac{1 - u^2}{(u^2 + 1)^2}
\]
7. Find an equation for the line which is normal to the curve consisting of all points 
\((x, y)\) satisfying \(y = \frac{xe^x}{x^2 + 1}\) at the point \((0, 0)\) on this curve.

The normal line goes through the point \((0, 0)\) and has slope \(\frac{-1}{y'(0)}\) (the negative reciprocal of the slope of the tangent line to the graph at 0).

Now:

\[
y' = \frac{(xe^x)' \cdot (x^2 + 1) - xe^x \cdot (2x)}{(x^2 + 1)^2} = \frac{(e^x + xe^x)(x^2 + 1) - 2xe^x}{(x^2 + 1)^2}
\]

Hence:

\[
y'(0) = \frac{(e^0 + 0e^0)(0^2 + 1) - 2(0)^2e^0}{((0)^2 + 1)^2} = \frac{(1)(1)}{(1)} = 1
\]

And hence the equation of the normal line is:

\[
y - 0 = -\frac{1}{1} (x - 0)
\]

That is: \(y = -x\)
8. Problem 50 on page 190

(a) 
\[ P'(2) = F'(2)G(2) + F(2)G'(2) = 0 \times 2 + 3 \times \frac{1}{2} = \frac{3}{2} \]

(b) 
\[ Q'(7) = \frac{F'(7)G(7) - F(7)G'(7)}{G(7)^2} = \frac{\frac{1}{2} \times 1 - 5 \times \left( -\frac{2}{3} \right)}{12} = \frac{1}{4} + \frac{10}{3} = \frac{43}{12} \]