EXTRA PROBLEMS FOR HOMEWORK 9

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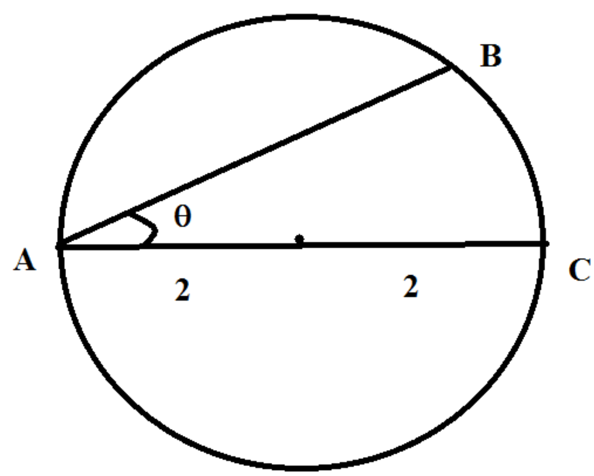
Problems

There are a total of 4 extra problems to do.

Problem 4.5.71: Show that the curve $y = x - \tan^{-1}(x)$ has two slant asymptotes $y = x + \frac{\pi}{2}$ and $y = x - \frac{\pi}{2}$. Use this fact to help sketch the curve.

Problem 4.7.48: A woman at a point A on the shore of a circular lake with radius 2 mi wants to arrive at the point C diametrically opposite A on the other side of the lake in the shortest possible time (see the figure). She can walk at the rate of 4 mph and row a boat at 2 mph. How should she proceed?

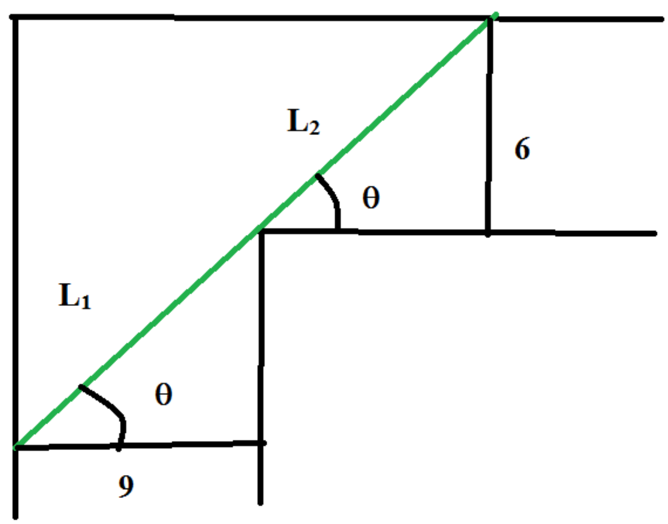
1A/Math 1A - Fall 2013/Homeworks/Lake.png



Date: Friday, November 8th, 2013.

Problem 4.7.70: A steel pipe is being carried down a hallway 9 ft wide. At the end of the hall there is a right-angled turn into a narrower hallway 6 ft wide. What is the length of the longest pipe that can be carried horizontally around the corner?

1A/Math 1A - Fall 2013/Homeworks/Pipe.png



Problem 4.9.77: A car is traveling at 100 km/h when the driver sees an accident 80 m ahead and slams on the brakes. What constant decceleration is required to stop the car in time to avoid a pileup?

Solutions

Solution to 4.5.71. :

At ∞:

Suppose the slant asymptote is $y = ax + b$, then:

$$a = \lim_{x \to \infty} \frac{x - \tan^{-1}(x)}{x} = \lim_{x \to \infty} 1 - \frac{\tan^{-1}(x)}{x} = 1 - \frac{\frac{\pi}{2}}{\infty} = 1$$

$$b = \lim_{x \to \infty} x - \tan^{-1}(x) - x = \lim_{x \to \infty} -\tan^{-1}(x) = -\frac{\pi}{2}$$

Hence $x - \tan^{-1}(x)$ has a slant asymptote of $y = x - \frac{\pi}{2}$ at ∞

At $-\infty$:

Suppose the slant asymptote is $y = ax + b$, then:

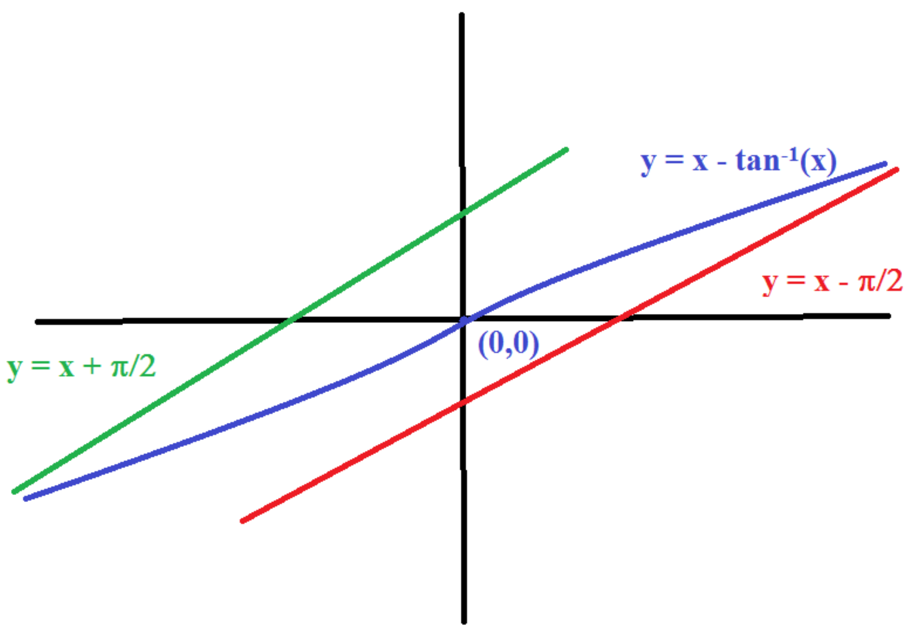
$$a = \lim_{x \to -\infty} \frac{x - \tan^{-1}(x)}{x} = \lim_{x \to -\infty} 1 - \frac{\tan^{-1}(x)}{x} = 1 - \frac{\frac{\pi}{2}}{-\infty} = 1$$

$$b = \lim_{x \to -\infty} x - \tan^{-1}(x) - x = \lim_{x \to -\infty} -\tan^{-1}(x) = -\left(-\frac{\pi}{2}\right) = \frac{\pi}{2}$$

Hence $x - \tan^{-1}(x)$ has a slant asymptote of $y = x + \frac{\pi}{2}$ at $-\infty$

- D $Dom = \mathbb{R}$
- I $y-$intercept: $f(0) = 0$, $x-$intercept: 0 (there are no others, because f is increasing; see Increasing/Decreasing section)
- S No symmetries
- A No vertical asymptotes (f is defined everywhere), Slant Asymptotes $y = x - \frac{\pi}{2}$ at ∞, $y = x + \frac{\pi}{2}$ at $-\infty$; No H.A. because there are already two S.A.
- I $f'(x) = 1 - \frac{1}{1+x^2} = \frac{x^2}{1+x^2} \geq 0$, so f is increasing everywhere; No local max/min
- C $f''(x) = \frac{2x(1+x^2) - x^2(2x)}{(1+x^2)^2} = \frac{2x}{(1+x^2)^2}$, so f is concave down on $(-\infty, 0)$ and concave up on $(0, \infty)$. Inflection point $(0, f(0)) = (0, 0)$

1A/Math 1A - Fall 2013/Homeworks/x - arctan(x).png



Solution to 4.7.48:

- Let t_{AB} be the time spent rowing from A to B and t_{BC} be the time spent walking from B to C
- By the formula time $= \frac{\text{distance}}{\text{velocity}}$, we have:

$$t_{AB} = \frac{AB}{2} = \frac{\cos(\theta)AC}{2} = \frac{4\cos(\theta)}{2} = 2\cos(\theta)$$

$$t_{BC} = \frac{BC}{4} = \frac{2 \times \angle BOC}{4} = \frac{2 \times 2\theta}{4} = \theta$$

(here O is the origin; it is a geometric fact that $\angle BOC = 2\angle BAC$)
- Let $f(\theta) = 2\cos(\theta) + \theta$
- Constraint: $0 \leq \theta \leq \frac{\pi}{2}$ (see the picture!)
- $f'(\theta) = -2\sin(\theta) + 1 = 0 \Leftrightarrow \sin(\theta) = \frac{1}{2} \Leftrightarrow \theta = \frac{\pi}{3}$
- $f(0) = 2$, $f\left(\frac{\pi}{2}\right) = \frac{\pi}{2}$ and $f\left(\frac{\pi}{3}\right) = 2 \times \frac{\sqrt{3}}{2} + \frac{\pi}{3} = \sqrt{3} + \frac{\pi}{3}$.

  By the **closed interval method**, $\theta = \frac{\pi}{2}$ is an absolute minimizer.
- Therefore, she should just walk! (which makes sense because she walks much faster than she rows!)

Solution to 4.7.70:

- We want to minimize $L_1 + L_2$
- $\cos(\theta) = \frac{L_1}{9}$, so $L_1 = \frac{9}{\cos(\theta)}$, $\sin(\theta) = \frac{L_2}{6}$, so $L_2 = \frac{6}{\sin(\theta)}$
- Let $f(\theta) = \frac{9}{\cos(\theta)} + \frac{6}{\sin(\theta)}$
- Constraint: $0 < \theta < \frac{\pi}{2}$ (Notice that at 0 and $\frac{\pi}{2}$, we can't carry the pipe horizontally around the corner; it would break at that corner)
- $f'(\theta) = \frac{9\sin(\theta)}{\cos^2(\theta)} + \frac{-6\cos(\theta)}{\sin^2(\theta)} = \frac{9\sin^3(\theta) - 6\cos^3(\theta)}{\cos^2(\theta)\sin^2(\theta)} = 0$

  $\Leftrightarrow 9\sin^3(\theta) - 6\cos^3(\theta) = 0 \Leftrightarrow \left(\frac{\sin(\theta)}{\cos(\theta)}\right)^3 = \frac{6}{9} = \frac{2}{3} \Leftrightarrow \tan^3(\theta) = \frac{2}{3} \Leftrightarrow \theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$
- By FDTAEV, $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ is the absolute minimizer of f
- Answer: $\frac{9}{\cos(\theta)} + \frac{6}{\sin(\theta)}$, where $\theta = \tan^{-1}\left(\sqrt[3]{\frac{2}{3}}\right)$ (if you want to, you can simplify this using the triangle method: $\frac{1}{\cos(\tan^{-1}(x))} = \sqrt{1+x^2}$ and $\frac{1}{\sin(\tan^{-1}(x))} = \frac{\sqrt{1+x^2}}{x}$, but I think this is enough torture for now :)

Solution to 4.9.76: Suppose the acceleration of the car is $a(t) = A$. Then $v(t) = At + B$ and $s(t) = \frac{A}{2}t^2 + Bt + C$.

However, at $t = 0$, the car is moving at 100 km/h, so $v(0) = 100$, so $B = 100$, hence $v(t) = At + 100$ and $s(t) = \frac{A}{2}t^2 + 100t + C$.

Moreover, at $t = 0$, the car is at its initial position 0, so $s(0) = 0$, so $C = 0$, hence $s(t) = \frac{A}{2}t^2 + 100t$

Now let t^* be the time needed to real the pile-up.

We want the car to have 0 velocity at t^*, hence $v(t^*) = 0$, hence $At^* + 100 = 0$, so $At^* = -100$

Moreover, we want $s(t^*) = 80\text{m} = 0.08$ km, so $\frac{A}{2}(t^*)^2 + 100t^* = 0.08$, but using the fact that $At^* = -100$, this just becomes: $\frac{-100t^*}{2} + 100t^* = 0.08$, so $50t^* = 0.08$, so $t^* = \frac{1}{625}$.

Therefore $A = -\frac{100}{t^*} = -100 \times 625 = -62500\ km/h^2$, so the answer is $\boxed{62500\ km/h^2}$.