# MATH 1A - HOW TO DERIVE THE FORMULA FOR THE DERIVATIVE OF ARCCOS(X)

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Here is one example of a theory question you might get on the exam:

$\boxed{\textbf{Problem: } \text{Show that the derivative of } y = \cos^{-1}(x) \text{ is } y' = \frac{-1}{\sqrt{1-x^2}}}$

## 1. Step 1: Finding $y'$

How would you start this problem? Well, we know that we are dealing with an inverse trig function, and the only thing to know about $\cos^{-1}(x)$ are the following two identities:

$$\cos(\cos^{-1}(x)) = x$$
$$\cos^{-1}(\cos(x)) = x$$

Now, which one of those two formulas would you use? We'll use the **first** one, for reasons that will become clearer later on (the reason is that we'd ultimately want to use the chain rule, and using the chain rule we'd like to 'fish' out the derivative for $\cos^{-1}(x)$, and this works really well when we use the first formula! You can try it using the second one, and you'll soon notice that you'll be having a hard time!).
For simplicity, let $y = cos^{-1}(x)$, so we ultimately want to find (abbreviated WTF) $y'$. Let's rewrite our first formula using what we **just** defined:

$$\cos(y) = x$$

See how we get an **implicit** formula for $y$! So, in order to calculate $y'$, let's use **implicit differentiation**! We get:

$$\begin{aligned}(\cos(y))' &= (x)' \\ y' \cdot (-\sin(y)) &= 1 \\ y' &= \frac{-1}{\sin(y)} \\ y' &= \frac{-1}{\sin(\cos^{-1}(x))}\end{aligned}$$

So $\boxed{y' = \frac{-1}{\sin(\cos^{-1}(x))}}$. This is a good formula, but we can do even better than that! We can actually write $\sin(\cos^{-1}(x))$ in a nicer form, and this is the point of **Step 2**.

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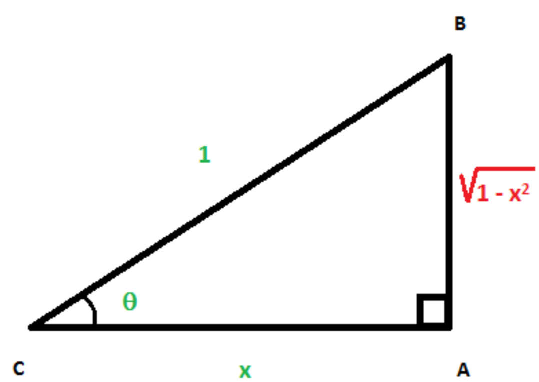
## 2. Step 2: Writing $\sin(\cos^{-1}(x))$ in a nicer form

Ideally, in order to solve the problem, we should get the identity: $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$, because then we'll get our desired formula $y' = \frac{-1}{\sqrt{1-x^2}}$, and we solved the problem! Now how the hell can we derive this identity (the left-hand-side and the right-hand-side don't seem to be related!!!). I'll give you two ways of doing this: the geometric way (which is longer, but easier), and the algebraic way (which is shorter, but slicker).

2.1. **Geometric way.** Let's think a little bit about $\sin(\cos^{-1}(x))$. First of all, since $\cos$ takes an angle and gives a number between -1 and 1, we should expect that $\cos^{-1}$ takes a number between -1 and 1, and gives an angle. It then makes sense to define $\theta = \cos^{-1}(x)$ to emphasize the fact that $cos^{-1}(x)$ is an **angle**! Since $\theta = cos^{-1}(x)$, this automatically means that $x = \cos(\theta)$ (by definition of the inverse function, or you can apply $\cos$ to both sides, and use the first formula at the beginning of the solution).

Here comes the 'geometric' part! Let's draw a triangle ABC that is right in A (see below), and let $\theta$ be the angle $< C$ (this method seems weird, but trust me, it works!) Now think about it: what *is* $\cos(\theta)$? From geometry, we know that $\cos(\theta) = \frac{AC}{BC}$. But we also know from above that $\cos(\theta) = x$, so let's choose values of AC and BC such that $\cos(\theta) = x$. For example, $AC = x$ and $BC = 1$ works, which we labeled in green in our figure below! (any values of AC and BC satisfying the above identity would work, here we chose the simplest one to make our algebra less messy!)




1A/Triangle.png

Now what do we **really** want to calculate? $\sin(\cos^{-1}(x))$! And this is equal to $\sin(\theta)$. Now, in our triangle, $\sin(\theta) = \frac{AB}{BC}$, so all we need to calculate is AB! But, from the Pythagorean theorem:

$$\begin{aligned}
BC^2 &= AB^2 + AC^2 \\
AB^2 &= BC^2 - AC^2 \\
AB^2 &= 1 - x^2 \\
AB &= \sqrt{1 - x^2}
\end{aligned}$$

And now we're done, because: $\sin(\cos^{-1}(x)) = \sin(\theta) = \frac{AB}{BC} = AB = \sqrt{1-x^2}$, and hence:

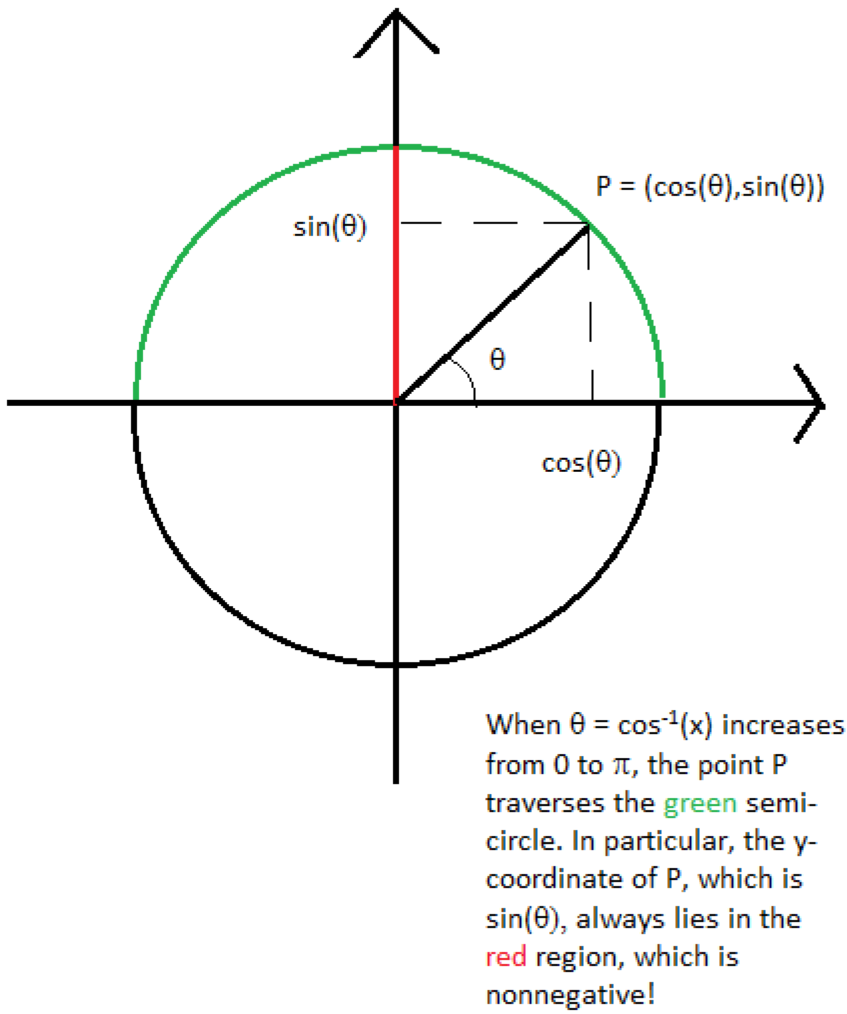
$$y' = \frac{-1}{\sin(\cos^{-1}(x))} = \frac{-1}{\sqrt{1-x^2}}$$

2.2. **Algebraic way.** The idea is: We want to calculate $\sin(\cos^{-1}(x))$, but we also have a nice identity $\cos(\cos^{-1}(x)) = x$, so we somehow want to combine both things! Now, we know an identity that relates $\sin$ and $\cos$, namely: $\boxed{\sin^2(x) + \cos^2(x) = 1}$, and we can use this identity to solve our problem, just by plugging in $\cos^{-1}(x)$ for $x$:

$$\begin{aligned}
\sin^2(\cos^{-1}(x)) + \cos^2(\cos^{-1}(x)) &= 1 \\
\sin^2(\cos^{-1}(x)) + x^2 &= 1 \\
\sin^2(\cos^{-1}(x)) &= 1 - x^2 \\
\sin(\cos^{-1}(x)) &= \pm\sqrt{1-x^2}
\end{aligned}$$

Now the question is: Which do we choose, $\sqrt{1-x^2}$, or $-\sqrt{1-x^2}$, and this requires some thinking!
The thing is: We defined $\cos^{-1}(x)$ to have range $[0, \pi]$ (see page 68 of your textbook for example), so, in particular, $\sin(\cos^{-1}(x))$ has range $[0, 1]$, and is in particular nonnegative (see picture below for more clarification), so the answer **has** to be $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$, because the other answer wouldn't make sense!

1A/Theta.png




Finally, we'll end our argument like in the geometric way! Since $\sin(\cos^{-1}(x)) = \sqrt{1-x^2}$, we get:

$$y' = \frac{-1}{\sin(\cos^{-1}(x))} = \frac{-1}{\sqrt{1-x^2}}$$