MATH 1A - SOLUTION TO 5.1.26

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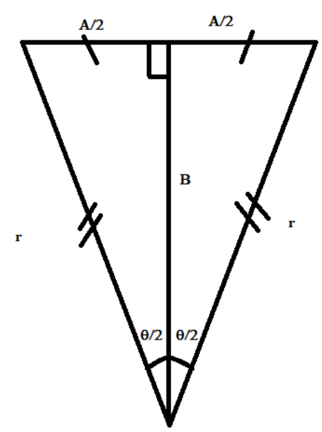
(a) We are given that the polygon is made out of n congruent triangles, so $A_n = n \cdot T$, where T is the area of each triangle. So all we need to find is T.

Here again, a picture tells a thousand words, so by drawing the picture of such a triangle, we can figure out its area:



1A/Polygon.png

Using the picture, you'll notice that:

$$T = \frac{1}{2} \cdot A \cdot B = \frac{A}{2} B$$

And we can divide the triangle into two right triangles, and hence use trigonometry to calculate $\frac{A}{2}$ and B! Here, $\theta = \frac{2\pi}{n}$, the central angle!
We get:

$$\cos\left(\frac{\theta}{2}\right) = \frac{B}{r}$$
$$B = r \cdot \cos\left(\frac{\theta}{2}\right)$$

Date: Wednesday, November 17th, 2010.

$$\sin\left(\frac{\theta}{2}\right) = \frac{\frac{A}{2}}{r}$$
$$\frac{A}{2} = r \cdot \sin\left(\frac{\theta}{2}\right)$$

And so, we get:

$$T = \frac{A}{2} \cdot B = r \cdot \sin\left(\frac{\theta}{2}\right) \cdot r \cdot \cos\left(\frac{\theta}{2}\right) = r^2 \cdot \sin\left(\frac{\theta}{2}\right)\cos\left(\frac{\theta}{2}\right) = r^2 \frac{1}{2}\sin\left(2 \cdot \frac{\theta}{2}\right) = \frac{1}{2}r^2\sin(\theta) = \frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right)$$

Here, we used the fact that, in general, $2\sin(x)\cos(x) = \sin(2x)$, so $\sin(x)\cos(x) = \frac{1}{2}\sin(2x)$.

And so, we have:

$$A_n = n \cdot T = n \cdot \frac{1}{2}r^2\sin\left(\frac{2\pi}{n}\right) = \frac{1}{2}nr^2\sin\left(\frac{2\pi}{n}\right)$$

(b) Actually, the hint tells us that we don't even have to use l'Hopital's rule, but rather the rule that:

$$\lim_{x\to 0}\frac{\sin(x)}{x} = 1$$

We have that:

$$\begin{aligned}
\lim_{n\to\infty} A_n = &\lim_{n\to\infty}\frac{1}{2}r^2 n\sin\left(\frac{2\pi}{n}\right)\\
=&\frac{1}{2}r^2\lim_{n\to\infty} n\sin\left(\frac{2\pi}{n}\right)\\
=&\frac{1}{2}r^2\lim_{n\to\infty}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{1}{n}}\\
=&\frac{1}{2}r^2\lim_{n\to\infty}\frac{2\pi\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}\\
=&2\pi\cdot\frac{1}{2}r^2\lim_{n\to\infty}\frac{\sin\left(\frac{2\pi}{n}\right)}{\frac{2\pi}{n}}\\
=&\pi r^2\lim_{x\to 0}\frac{\sin(x)}{x} \qquad \left(x = \frac{2\pi}{n}\right)\\
=&\pi r^2(1)\\
=&\pi r^2
\end{aligned}$$

The basic idea is that, if we have an indeterminate form of the form "$0 \cdot \infty$", we rewrite ∞ as $\frac{1}{0}$, or we rewrite 0 as $\frac{1}{\infty}$. Here, for example, we wrote $n = \frac{1}{\frac{1}{n}}$ in order to apply the hint in the problem!

And hooray, you just proved that the formula for the area of a circle of radius r is πr^2. But actually, you didn't, because trigonometry, which you used in (a), relies heavily on this formula!