MATH 1A - SOLUTION TO 4.8.38

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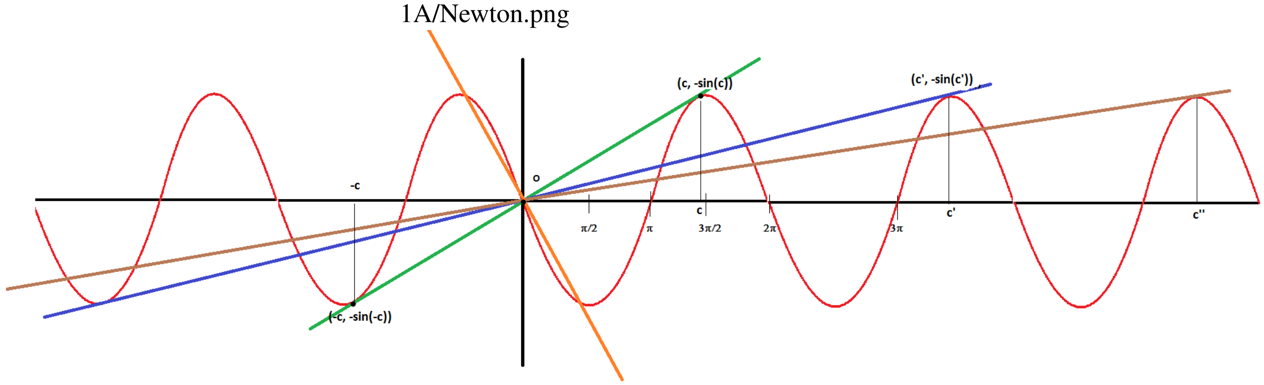
As usual, a good picture is the key to solving the problem:

1A/Newton.png



Note: This picture is very complete. It is meant to illustrate all the points I am making below.

Here, we are given a lot of information, so let's try to tackle this problem one step at a time!

First, let's calculate the equation of *any* tangent line to the graph of $f(x) = -sin(x)$ that goes through $(0,0)$. Later, we will be worrying about finding the one that has the largest slope.

By definition of the derivative, the tangent line to the graph of f at c has slope $f'(c) = -\cos(c)$, so any such tangent line that **also** goes through $(0,0)$ has equation: $y - 0 = -\cos(c)(x-0)$, i.e. $y = -\cos(c)x$. Finally, we know that the tangent line goes through $(c, -\sin(c))$ (i.e. goes through the graph of f at c), so we get: $-\sin(c) = -\cos(c) \cdot c$, i.e. $\tan(c) = c$.

So any tangent line at c with the above properties must solve $\tan(c) = c$, i.e. $\tan(c) - c = 0$.

So what we really need to do is to approximate the zero of the function $g(x) = \tan(x) - x$. Now this looks like a Newton's method problem! But remember, that for Newton's method, we need to find a good initial guess, and **here** is where we use the information

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that the tangent line must have largest slope!

First of all, notice that $f(x) = -\sin(x)$ is odd, so the graph is symmetric about the origin! If you look at the picture above, you'll see that the tangent line to the graph at c is the same as the tangent line at $-c$. This means we can restrict ourselves to the right-hand-side of the picture, i.e. $c \geq 0$! (if you don't understand this argument, don't worry, it's just a simplification)

And if you look at the picture again, you'll notice that if your initial guess is between 0 and $\frac{\pi}{2}$, your successive approximations will got to 0. And you don't want that because the slope of the tangent line at 0 is -1 (which is not the greatest slope). The same problem arises with the initial guess between $\frac{3\pi}{2}$ and 2π (the approximations go to 2π)

Finally, notice that when c gets larger and larger, the tangent line at c has smaller and smaller slope (see picture: The brown line has a smaller slope than the blue line, which has a smaller slope than the green line), so you'd like your initial guess not to be too large. In particular, we don't want the initial guess to be larger than $\frac{3\pi}{2}$!

From this analysis, we conclude that any initial guess between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ is good enough! (also look at the picture above: the green tangent line seems to be the winner here)

For example, with $x_0 = 4.5$ (or you could try $x_0 = \pi$), we get the successive approximations (remember that you are applying Newton's method to $g(x) = \tan(x) - x$, **NOT** $f(x)$!):

$$\begin{aligned} x_0 &= 4.5 \\ x_1 &= 4.49361390 \\ x_2 &= 4.49340966 \\ x_3 &= 4.49340946 \\ x_4 &= 4.49340946 \end{aligned}$$

And so, our approximation is: $\boxed{c \approx 4.49340946}$. And hence the largest **slope** is approximately equal to $\boxed{-\cos(4.49340946) \approx 0.2172336}$ (because $f'(c) = -\cos(c)$).

To summarize:

- Draw a picture
- Derive the function that you want to apply Newton's method to (i.e. $g(x) = \tan(x) - x$)
- Argue that your intial approximation must be between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (**YOU NEED TO JUSTIFY THIS PART!**, maybe not as precise as I did, but there needs to be some justification
- Apply Newton's method to g with initial approximation between $\frac{\pi}{2}$ and $\frac{3\pi}{2}$ (4 would work)