MATH 1A - SOLUTION TO 3.9.44

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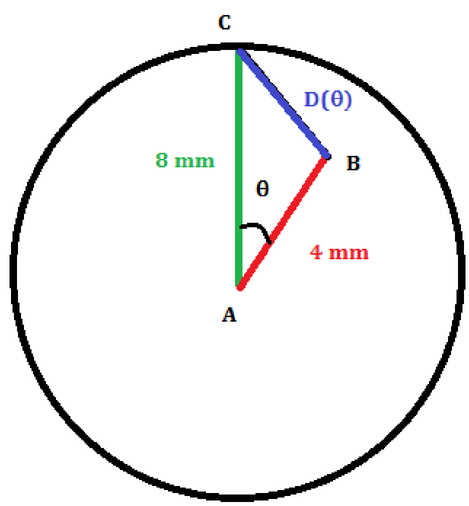
As is usual for related rates problems, let's draw a picture:



1A/Clock.png

Let θ be the angle between the hour hand and the minute hand. Now, what we want to calculate is $D'(\theta)$, where the $'$ indicates differentiation with respect to the time variable.

How can we relate $D(\theta)$ with what we know? This is easy! We know an angle θ and the lengths of AB and AC in the picture, so let's just use the **law of cosines**. We get:

$$BC^2 = AC^2 + AB^2 - 2 \cdot AC \cdot AB \cdot \cos(\theta)$$

That is:

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$$D(\theta)^2 = 8^2 + 4^2 - 2 \cdot 8 \cdot 4 \cdot \cos(\theta)$$

Which you can write as:

$$D(\theta)^2 = 80 - 64\cos(\theta)$$

Now differentiate with respect to time!
We get:

$$2D(\theta)D'(\theta) = 64\sin(\theta)\frac{d\theta}{dt}$$

And now, all we need to do is to plug in everything we know!

First of all $\theta = \frac{2\pi}{12} = \frac{\pi}{6}$ (basically, the whole circle corresponds to 2π, and so $\frac{1}{12}$ of the circle corresponds to $\frac{2\pi}{12}$).

In particular, $\boxed{\sin(\theta) = \sin(\frac{\pi}{6}) = \frac{1}{2}}$.

Now, for $d(\theta)$, we use the law of cosines again, this time with the value $\theta = \frac{\pi}{6}$:

$$D(\frac{\pi}{6})^2 = 80 - 64 \cdot \cos(\frac{pi}{6}) = 80 - 64\frac{\sqrt{3}}{2} = 80 - 32\sqrt{3}$$

So, taking square roots, we get $\boxed{d(\frac{\pi}{6}) = \sqrt{80 - 32\sqrt{3}} = 4\sqrt{5 - 2\sqrt{3}} \text{ mm}}$.

Finally, we need to compute $\frac{D\theta}{dt}$. But think about it! In 12 hours, $\theta = 2\pi$, so the speed of θ should be $\frac{2\pi}{12} = \frac{\pi}{6} = -\frac{11\pi}{6}$ rad/h (notice that θ is decreasing, so we wanted a negative answer!)

Finally, we have all our information to get our final answer:

$$\begin{aligned}
2D(\theta)D'(\theta) &= 64\sin(\theta)\frac{D\theta}{dt} \\
2 \cdot (4\sqrt{5 - 2\sqrt{3}}) \cdot D'\left(\frac{\pi}{6}\right) &= 64 \cdot \frac{1}{2} \cdot \frac{-11\pi}{6} \\
2 \cdot (4\sqrt{5 - 2\sqrt{3}}) \cdot D'\left(\frac{\pi}{6}\right) &= 32 \cdot \frac{-11\pi}{6} \\
D'\left(\frac{\pi}{6}\right) &= 16 \cdot \frac{\frac{-11\pi}{6}}{4\sqrt{5 - 2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &= 4 \cdot \frac{\frac{-11\pi}{6}}{\sqrt{5 - 2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &= -\frac{22\pi}{3\sqrt{5 - 2\sqrt{3}}} \\
D'\left(\frac{\pi}{6}\right) &\approx -18.55mm/h
\end{aligned}$$

So our final answer is $\boxed{D'\left(\frac{\pi}{6}\right) = -\frac{22\pi}{3\sqrt{5-2\sqrt{3}}} \text{ mm/h} \approx -18.55mm/h}$