Final Exam — Review — Problems

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Note: In all the problems below, \( V \) is a finite-dimensional inner-product space (except in problems 1 and 7(a)-(d), where \( V \) is just a finite-dimensional vector space)

Problem 1:
Let \( U \) and \( W \) be subspaces of a vector space \( V \), with \( \dim(U) \geq \dim(W) \). Show that there exists \( T \in \mathcal{L}(V) \) such that \( T(U) = W \).

Problem 2:
Suppose \( T \in \mathcal{L}(V) \) satisfies \( \langle T(e_i), e_j \rangle = 0 \) if \( i \neq j \) and 1 otherwise (for all \( i \) and \( j \)). Calculate \( M(T) \).

Problem 3:
Let \( T \) and \( S \) be self-adjoint operators on \( V \) such that \( TS = ST \). Show that there exists an orthonormal basis of \( V \) whose elements are eigenvectors of both \( S \) and \( T \) (that is, \( S \) and \( T \) are simultaneously diagonalizable)

Problem 4:
In the following \( V^* \) denotes the set of all linear functionals on \( V \), and given \( v \), \( \phi_v \in V^* \) denotes the functional \( \phi_v(u) = \langle u, v \rangle \).

Define \( \Phi : V \rightarrow V^* \) by: \( \Phi(v) = \phi_v \)

Show that \( \Phi \) is an isomorphism of vector spaces!

\(^1\)that is, the set of linear transformations from \( V \) to \( \mathbb{F} \)
Problem 5:
Let $U$ be a subspace of $V$, and $P$ be the orthogonal projection on $U$. Let $J : U \to V$ denote the inclusion map, that is, $J(u) = u$. Show that $J^* = P$.

Problem 6:
Let $V$ be an inner-product space and $W$ be any vector space, and $T \in \mathcal{L}(V, W)$. Given $w \in W$, define $S_w = \{v \in V \mid T(v) = w\}$ (the set of vectors in $V$ that map to $W$). Show that the smallest element $\hat{w}$ of $S_w$ (if it exists) is orthogonal to any vector $\text{Nul}(T)$.

Problem 7: TRUE/FALSE EXTRAVAGANZA!!!

(a) If $U, W, Z$ are subspaces of $V$, and $\dim(V) = \dim(U) + \dim(W) + \dim(Z)$, then $V = U \oplus W \oplus Z$.

(b) If $W$ is a fixed subspace of $V$, then $\{T \in \mathcal{L}(V) \mid W$ is a $T$-invariant subspace of $V\}$ is a subspace of $\mathcal{L}(V)$.

(c) If $T, S \in \mathcal{L}(V)$, and $S$ is invertible, then $T$ and $STS^{-1}$ have the same eigenvalues, including multiplicities.

(d) If $V = \mathbb{R}^2$ and $T^2 = T$, then there is a basis of $V$ consisting of eigenvectors of $T$.

(e) If $T = S^*S$ for $S \in \mathcal{L}(V)$, then all the eigenvalues of $T$ are nonnegative.

(f) If $\mathbb{F} = \mathbb{C}$, and $T$ is normal and nilpotent, then $T = 0$.

(g) If $\mathbb{F} = \mathbb{C}$, and $\|Tx\| = \|x\|$ for all $x$, then there is a basis of $V$ consisting of eigenvectors of $T$.

\footnote{By this we mean that if $u$ is any other vector in $S_w$, then $\|\hat{w}\| \leq \|u\|$}