Math 110 Homework 2 Solutions

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Chapter 1:

8.) Prove that the intersection of an arbitrary collection of subspaces of V is a subspace of V.

Solution: Let $(U_i)_{i\in I}$ be an arbitrary collection of subspaces of V (I is just an index set used to keep track of the different subspaces). We define the intersection of all elements of the collection as follows:

$$\bigcap_{i\in I} U_i = \{x : x \in U_i \text{ for all } i \in I\}.$$

We now prove it is a subspace of V.

First, we note that as each U_i is a subspace, we have $0 \in U_i$ for all $i \in I$. Therefore, $0 \in \bigcap_{i\in I} U_i$.

Let x and y be in $\bigcap_{i\in I} U_i$, i.e. x and y are in U_i for each $i \in I$. As each U_i is a subspace, they are all closed under addition. Therefore, $x + y \in U_i$ for every $i \in I$. This implies $x + y \in \bigcap_{i\in I} U_i$ so $\bigcap_{i\in I} U_i$ is closed under addition.

Let $c \in \mathbb{F}$ and $x \in \bigcap_{i\in I} U_i$. Then as $x \in U_i$ for each i and the U_i are all subspaces, we have $c \cdot x \in U_i$ for all i as each U_i is closed under scalar multiplication. Therefore $c \cdot x \in \bigcap_{i\in I} U_i$ so $\bigcap_{i\in I} U_i$ is closed under multiplication and is thus a subset of V.

14.) Suppose U is the subspace of $\mathcal{P}(\mathbb{F})$ consisting of all polynomials p of the form

$$p(z) = az^2 + bz^5.$$

Find a subspace, W of $\mathcal{P}(\mathbb{F})$ such that $\mathcal{P}(\mathbb{F}) = U \oplus W$.

Solution: Set $W = \{p \in \mathcal{P}(\mathbb{F}) : p(z) = \sum_{k=0}^{n} a_k z^k \text{ and } a_2 = 0 = a_5\}$. We claim W is a subspace of $\mathcal{P}(\mathbb{F})$ and $\mathcal{P}(\mathbb{F}) = U \oplus W$.

To see that W is a subspace, we note that the zero polynomial has all coefficients zero, in particular a_2 and a_5 so $0 \in W$. Given p and $q \in W$, there exists and n such that $p(z) = \sum_{k=0}^{n} a_k z^k$ and $q(z) = \sum_{k=0}^{n} b_k z^k$ with a_2, a_5, b_2, and b_5 are all zero. The polynomial $p+q$ is $(p+q)(z) = \sum_{k=0}^{n}(a_k + b_k)z^k$. We see that $a_2 + b_2 = 0 + 0 = 0$ and $a_5 + b_5 = 0 + 0 = 0$ which implies $p + q \in W$ so W is closed under addition. Finally, if $c \in \mathbb{F}$ and p is as in the previous sentence, then $cp(z) = \sum_{k=0}^{n} ca_k z^k$ and $ca_2 = c \cdot 0 = 0$ and $ca_5 = c \cdot 0 = 0$ so W is closed under scalar multiplication and is a subspace.

To show that $\mathcal{P} = U \oplus W$, we invoke proposition 1.9. This means we have to show $\mathcal{P}(\mathbb{F}) = U + W$ and $U \cap W = \{0\}$. Let $f \in \mathcal{P}(\mathbb{F})$. Then for some m, we write

$$f(z) = \sum_{k=0}^{m} c_k z^k = (c_2 z^2 + c_5 z^5) + \left(\sum_{k=0}^{1} c_k z^k + \sum_{k=3}^{4} c_k z^k + \sum_{k=6}^{m} c_k z^k \right) \in U + W$$

which shows $U + W = \mathcal{P}(\mathbb{F})$. Finally, if $g(z) = \sum_{k=0}^{j} d_k z^k$ is in $U \cap W$, then $d_k = 0$ for all $k \neq 2$ or 5 since $g \in U$, and $d_2 = 0 = d_5$ since $g \in W$. Therefore, $d_k = 0$ for all k and we conclude that $U \cap W = \{0\}$. This means $\mathcal{P}(\mathbb{F}) = U \oplus W$ as desired.

15.) Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \text{ and } V = U_2 + W$$

then $U_1 = U_2$.

Solution: This is false. For an example, we take $V = \mathbb{F}^2$, $U_1 = \{(x, 0) : x \in \mathbb{F}\}$, $U_2 = \{(z, z) : z \in \mathbb{F}\}$ and $W = \{(0, y) : y \in \mathbb{F}\}$. From the textbook, these are all subspaces of V. We first note that $V = U_1 \oplus W$. Indeed, if $v = (a_1, a_2) \in U_1 \cap W$, then $a_1 = 0$ since $v \in W$ and $a_2 = 0$ since $v \in U_1$. Therefore, $U_1 \cap W = \{0\}$. Furthermore, any vector $(x, y) \in \mathbb{F}^2$ can be written as $(x, 0) + (0, y) \in U_1 + W$ so that $U_1 + W = V$. Therefore $V = U_1 \oplus W$ by proposition 1.9.

We also claim $V = U_2 \oplus W$. Let $v = (a_1, a_2) \in U_2 \oplus W$. Then $a_1 = 0$ since $v \in W$ and $a_1 = a_2$ since $v \in U_2$. This implies $a_2 = 0$ hence $v = 0$ and $U_2 \cap W = \{0\}$. Also, note that any vector, $(x, y) \in W$ can be written as

$$(x, y) = (x, x) + (0, y - x) \in U_2 + W$$

so it follows that $U_2 + W = V$. Therefore, $V = U_2 \oplus W$ by proposition 1.9. Clearly, $U_2 \neq U_1$ so we are finished.

Chapter 2:

1.) Prove that if $(v_1, \cdots, v_n)$ spans V, then so does the list

$$(v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

Solution: Let $v \in V$. Since $(v_1, \cdots, v_n)$ spans V, we know that there exist scalars $a_1, \cdots, a_n \in \mathbb{F}$ such that

$$a_1 v_1 + a_2 v_2 + \cdots + a_n v_n = v.$$

We seek coefficients b_i such that:

$$b_1(v_1 - v_2) + b_2(v_2 - v_3) + \cdots + b_{n-1}(v_{n-1} - v_n) + b_n v_n = v = a_1 v_1 + a_2 v_2 + \cdots + a_n v_n.$$

Using the distributive property, the left hand side can be rewritten as

$$b_1 v_1 + (b_2 - b_1) v_2 + \cdots + (b_{n-1} - b_{n-2}) v_{n-1} + (b_n - b_{n-1}) v_n$$

so comparing this with the expression in the a_i gives us the following equations:

$$b_1 = a_1$$
$$b_k - b_{k-1} = a_k \text{ for } 2 \leq k \leq n.$$

We claim that $b_k = \sum_{i=1}^{k} a_i$ for all k between 1 and n. Indeed, the result is clearly true for b_1. We now assume $2 \leq j < n$ and $b_j = \sum_{i=1}^{j} a_i$. Then:

$$b_{j+1} = b_j + a_{j+1} = \sum_{i=1}^{j} a_i + a_{j+1} = \sum_{i=1}^{j+1} a_i$$

so the formula is proved by induction. This proves that:

$$v = a_1(v_1 - v_2) + (a_1 + a_2)(v_2 - v_3) + \cdots + \left(\sum_{k=1}^{n} a_k \right) v_n.$$

so $v \in \text{span}(v_1, \cdots, v_n)$. Therefore, the list $(v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n)$ spans V.

2.) Prove that if $(v_1, \cdots, v_n)$ is linearly independent in V, then so is the list

$$(v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n)$$

obtained by subtracting from each vector (except the last one) the following vector.

Solution: Consider the following equation:

$$a_1(v_1 - v_2) + a_2(v_2 - v_3) + \cdots + a_n v_n = 0$$

for each $a_i \in \mathbb{F}$. To show linear independence, we must show that each coefficient in the above equation must be zero. The equation can be rewritten as

$$a_1 v_1 + (a_2 - a_1)v_2 + (a_3 - a_2)v_3 + \cdots (a_n - a_{n-1})v_n = 0.$$

Since $(v_1, \cdots, v_k)$ are linearly independent, we see that

$$a_1 = 0$$
$$a_k - a_{k-1} = 0 \text{ for } 2 \leq k \leq n$$

We claim that this implies that $a_k = 0$ for k between 1 and n. We again proceed by induction. Clearly the result is true for $k = 1$. Assume that the result is true for some j such that $2 \leq j < n$. Then $a_{j+1} - a_j = 0$, implying $a_{j+1} = a_j = 0$, completing the proof. Therefore the list $(v_1 - v_2, v_2 - v_3, \cdots, v_{n-1} - v_n, v_n)$ is linearly independent.

3.) Suppose $(v_1, ..., v_n)$ is linearly independent in V, and $w \in V$. Prove that if $(v_1 + w, \cdots, v_n + w)$ is linearly dependent, then $w \in \text{span}(v_1, ..., v_n)$.

Solution: Since $(v_1 + w, \cdots, v_n + w)$ is linearly dependent, we can find scalars $a_1, \cdots, a_n$ not all 0 such that

$$a_1(v_1 + w) + \cdots + a_n(v_n + w) = 0.$$

Rearranging this equation, we get:

$$a_1 v_1 + \cdots + a_n v_n = -(a_1 + \cdots + a_n)w.$$

We claim that $a_1 + \cdots + a_n \neq 0$. Indeed if this were not the case, then we would have $a_1 v_1 + \cdots + a_n v_n = -0w = 0$. Since the a_i are not all zero, this contradicts the linear independence of $\{v_1, \cdots, v_n\}$. Therefore, we can divide by $(a_1 + \cdots + a_n)$, producing:

$$w = -\frac{a_1}{a_1 + \cdots + a_n} v_1 - \cdots - \frac{a_n}{a_1 + \cdots + a_n} v_n,$$

implying $w \in \text{span}(v_1, ..., v_n)$.