GLOBAL REGULARITY OF SCHRÖDINGER MAPS

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Abstract. This paper summarizes the main results of the first two sections of Paul Smith's paper 'Conditional Global Regularity of Schrödinger maps: Sub-Threshold Dispersed Energy" [3]. After recalling some preliminimary facts about Sobolev spaces for functions mapping into $\mathbf{S}^2$, I will state Paul Smith's main theorem, namely a global well-posedness result of functions φ satisfying the Schrödinger map initial value problem, assuming an L^4 boundedness condition of the gradient $\partial_x \varphi$. Moreover, there is absence of weak turbulence. Then, I will present an outline of the proof, and finally I will discuss the two main tools he uses in the proof of the theorem: the caloric gauge, as well as frequency localization.

1. Introduction

In this paper, we consider the following equation, where $\varphi : \mathbb{R}^2 \to \mathbb{S}^2$:

$$\begin{cases} \partial_t \varphi = & \varphi \times \Delta \varphi \\ \varphi(x,0) = & \varphi_0(x) \end{cases} \tag{1.1}$$

In this case, the following energy is conserved:

$$E(\varphi(t)) := \frac{1}{2} \int_{\mathbb{R}^2} \left| \partial_x \varphi(t) \right|^2 dx \tag{1.2}$$

Moreover, both theF equation (1.1) and the energy (1.2) are invariant under the scaling

$$\varphi(x,t) \to \varphi(\lambda x, \lambda^2 t) \tag{1.3}$$

where $\lambda > 0$, and that's why we call (1.1) **energy-critical**.

So far, if the energy is small, the equation (1.1) basically has been solved: Namely, given sufficiently low-energy initial data, we have global wellposedness and absence of weak turbulence [1].

Hence, we need to turn our attention to what happens when the energy is large. So far, we have *local* well-posedness and an instability result near solitons. Moreover, there is a threshold $E_{crit} = 4\pi$, called the **critical energy** such that for $E < E_{crit}$, there exist no non-constant finite-energy harmonic maps with energy E. We call the range of energies $[0, E_{crit})$ as **sub-threshold**.

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Therefore, the goal is to show that we have *global* well-posedness, and this is what Paul Smith's paper is trying to demonstrate. More precisely, we will show that, assuming the gradient $\partial_x \varphi$ is bounded in L^4 and the initial data φ_0 is smooth enough, we have a unique, smooth, global solution φ. Along the way, we will try to define a new kind of frame which solves our problem: the **caloric gauge**.

Historical note: The caloric gauge was first discovered by Terence Tao in the study of wave maps [6]

2. Set-up

2.1. Sobolev Spaces H_Q.

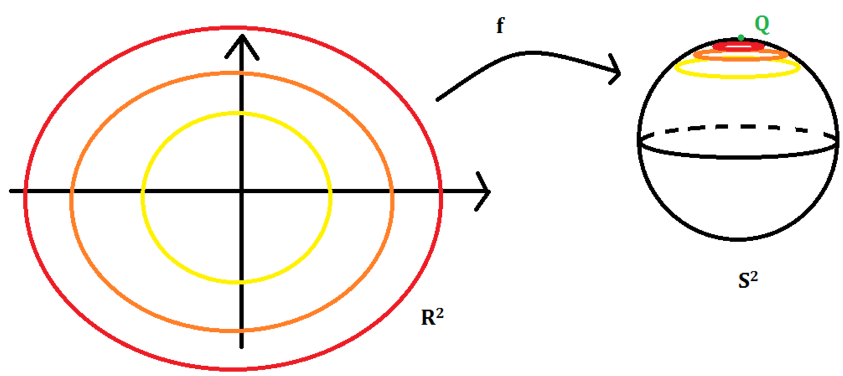
Before we can state the main result, we need to define Sobolev spaces H_Q appropriate for functions mapping from $\mathbb{R}^2$ into $\mathbb{S}^2$. In particular, we need to define an analog of 'decaying to 0 at ∞'

Definition. *Given $\sigma \in [0, \infty)$, and $Q \in \mathbb{S}^2$, let:*

$$H_Q^\sigma := \left\{ f : \mathbb{R}^2 \longrightarrow \mathbb{S}^2 \mid f - Q \in H^\sigma \right\}$$

In other words, functions in H_Q^σ decay to Q as $|x| \to \infty$, as illustrated by the following picture:

Maps/HQ.png



We also define:

$$H_Q^\infty = \bigcap_{\sigma \in \mathbb{N}} H_Q^\sigma$$

This definition can be also extended to spacetime Sobolev spaces:

First of all, define $H^{\sigma,\rho}(T)$ to be the Sobolev space of complex-valued functions on $\mathbb{R}^2 \times (-T,T)$ with the norm:

$$\|f\|_{H^{\sigma,\rho}(T)} := \sup_{t\in(-T,T)} \sum_{\rho'=0}^{\rho} \left\| \partial_t^{\rho'} f(.,t) \right\|_{H^\sigma}$$

Now define:

$$H_Q^{\sigma,\rho} := \left\{ f : \mathbb{R}^2 \times (-T,T) \longrightarrow \mathbb{S}^2 \mid f - Q \in H^{\sigma,\rho}(T) \right\}$$

and we define $H_Q^{\infty,\infty}$ analogously to H_Q^∞.

2.2. **Conservation laws for H_Q.** There are two important conservation laws related to the Sobolev spaces H_Q:

If $\varphi \in C((T_1,T_2) \to H_Q^\infty)$ solves the equation (1.1) on (T_1,T_2), then the following two quantities are conserved:

$$\int_{\mathbb{R}^2} |\varphi(t) - Q|^2 \, dx \qquad \text{and} \qquad \int_{\mathbb{R}^2} |\partial \varphi(t)|^2 \, dx$$

Which implies that the Sobolev norms H_Q^0 and H_Q^1, as well as the energy (1.2), are conserved. Moreover, we have time-reversibility.

3. Local Theory

As said before, the local theory is well-understood. More precisely, we have the following theorem, whose proof can be found for example in [4]:

Theorem (Local existence and uniqueness)**.** *If $\varphi_0 \in H_Q^\infty$ for some $Q \in \mathbb{S}^2$, then there exists a time $T = T(\|\varphi_0\|_{H_Q^{25}}) > 0$ for which there exists a unique solution $\varphi \in C((-T,T) \to H_Q^\infty$ of the equation* (1.1)*.*

This gives a short-time existence-uniqueness theorem as well as a blow-up criterion for solutions.

4. Global theory

The global theory, on the other hand, is not as well-understood. If the energy is small, then we have the following global regularity result, due to Bejeranu, Ionescu, Kenig, and Tataru [1]:

Theorem (Global Regularity)**.** *Let $Q \in \mathbb{S}^2$. Then there exists an $\epsilon_0 > 0$ such that for any $\varphi_0 \in H_Q^\infty$ with $\|\partial_x \varphi_0\|_{L_x^2} \leq \epsilon_0$, there is a unique solution $\varphi \in C(\mathbb{R} \to H_Q^\infty)$ of the initial value problem* (1.1)*. Moreover, for any $T \in [0,\infty)$ and $\sigma \in \mathbb{N}$,*

$$\sup_{t\in(-T,T)} \|\varphi(t)\|_{H^\sigma_Q} \leq C = C(\sigma, T, \|\varphi_0\|_{H^\sigma_Q})$$

Furthermore, given $\sigma_1 \in \mathbb{N}$, there exists a positive $\epsilon_1 = \epsilon_1(\sigma_1) \leq \epsilon_0$ such that the uniform bounds:

$$\sup_{t\in\mathbb{R}} \|\varphi(t)\|_{H^\sigma_Q} \leq C_\sigma \|\varphi_0\|_{H^\sigma_Q}$$

hold for all $1 \leq \sigma \leq \sigma_1$, provided $\|\partial_x \varphi_0\|_{L^2_x} \leq \epsilon_1$

The main idea is that the spaces H^σ_Q take into account a local smoothing effect, which controls the worst term in the nonlinearity. Also, the proof uses an important tool called the caloric gauge, which we'll define in section 7.

5. Main goal of Paul Smith's paper

The main goal of Paul Smith's paper is to prove the following theorem below. Before we can state it, recall that P_k denotes the standard Littlewood-Paley Fourier multiplier restricted to frequencies ~ 2^k.

Theorem (Main Theorem)**.** *Let $T > 0$ and $Q \in \mathbb{S}^2$. Let $\epsilon_0 > 0$ and let $\varphi \in H^{\infty,\infty}_Q$ be a solution of the Schrödinger map system (1.1) whose initial data φ_0 has energy $E_0 := E(\varphi_0) < E_{crit}$ and satisfies the energy dispersion condition:*

$$\sup_{k\in\mathbb{N}} \|P_k \partial_x \varphi_0\|_{L^2_x} \leq \epsilon_0 \tag{5.1}$$

Let $I \supset (-T, T)$ denote the maximal time interval for which there exists a smooth (necessarily unique) extension of φ satisfying (1.1). Suppose a priori that:

$$\sum_{k\in\mathbb{Z}} \|P_k \partial_x \varphi\|^2_{L^4_{t,x}(I\times\mathbb{R}^2)} \leq \epsilon_0^2 \tag{5.2}$$

Then, for $\epsilon_0 > 0$ sufficiently small,

$$\sup_{t\in(-T,T)} \|\varphi(t)\|_{H^\sigma_Q} \leq C = C(\sigma, T, \|\varphi_0\|_{H^\sigma_Q}) \tag{5.3}$$

for all $\sigma \in \mathbb{N}$.

Additionally, $I = \mathbb{R}$, so that, in particular, φ admits a unique smooth global extension; this extension satisfies $\varphi \in C(\mathbb{R} \to H^\infty_Q)$. Moreover, for any $\sigma_1 \in \mathbb{N}$, there exists a positive $\epsilon_1 = \epsilon_1(\sigma_1) \leq \epsilon_0$ such that:

$$\|\varphi\|_{L^\infty_t H^\sigma_Q(\mathbb{R}\times\mathbb{R}^2)} \leq C_\sigma \|\varphi\|_{H^\sigma_Q(\mathbb{R}^2)} \tag{5.4}$$

holds for all $0 \leq \sigma \leq \sigma_1$ provided (5.1) and (5.2) hold with ϵ_1 in place of ϵ_0.

Note that (5.1) is automatically satisfied if the energy is small. In that special case case we don't even need the L^4 bound.

Also note that (5.4) says that the solution does not exhibit weak turbulence.

6. Outline of the proof of the Main Theorem

6.1. **Step 1: Basic set-up and gauge selection.** By interpolation and mass/energy conservation, we only need to consider $\sigma \geq 1$.

Therefore, instead of controlling $\|\varphi(t)\|_{\dot{H}^\sigma}$, we only need to control $\|\partial_x \varphi(t)\|_{\dot{H}^{\sigma-1}}$

Next, we consider the time-evolution $\partial_x \varphi$, which can, as it turns out, be written only in terms of terms of derivatives of φ.

Here comes the crucial fact: Instead of expressing the equations in terms of coordinates, we will select a gauge, more specifically the *caloric gauge*.

Then it turns out that Sobolev bounds for the *gauged* derivative map impies corresponding Sobolev bounds for the *ungauged* derivative map.

So we can write the gauged equations as:

$$(\partial_t - \Delta)\,\psi = \mathcal{N}$$

where ψ is $\partial_x \varphi$ expressed in the caloric gauge and $\mathcal{N}$ is a nonlinearity term which includes ψ and $\partial_x \psi$.

6.2. **Step 2: Function spaces and their interrelation.** In the energy-critical setting, energy estimates won't help us any more and hence need to look for other bounds to control the solution. Instead we will look for local smoothing estimates and (specialized) Strichartz estimates.

More importantly, we will control ψ in a suitable space using a **bootstrap argument**.

For the boostrap assumption, we start with the estimate:

$$\|P_k \psi\|_{G_k} \leq C \left(\|P_k \psi(t=0)\|_{L^2_x} + \|P_k \mathcal{N}\|_{N_k} \right)$$

where G_k and N_k are frequency-localized spaces for $P_k \psi$ resp. $P_k \mathcal{N}$.

Then we want to refine the estimate by showing:

$$\|P_k \psi\|_{G_k} \leq C \left(\|P_k \psi(t=0)\|_{L^2_x} + C\epsilon \|P_k \mathcal{N}\|_{N_k} \right)$$

from which we conclude:

$$\|P_k \psi\|_{G_k} \leq C \|P_k \psi(t=0)\|_{L^2_x} \tag{6.1}$$

This allows us to prove (5.3) and (6.1) by unwinding the gauge and frequency localization steps and concluding with a standard continuity argument.

Step 3: Controlling the nonlinearity. So far, we required small energy, and the heart of this step (and in fact of the whole paper) is to show how we can get the estimate (1.7) without the small energy assumption.

The main task will be to control the terms in $P_k\mathcal{N}$ with a derivative landing on a high-frequency component, written as $A_{lo}\partial\psi_{hi}$.

Then we will work directly with the covariant frequency-localized Schrödinger equation and estimate the remaining terms of $P_k\mathcal{N}$ and close the bootstrap argument. $\square$

7. The caloric gauge

In this section, we will switch from a coordinate-formulation of the Schrödinger map system to a derivative formulation of those equations. Then we will define the caloric gauge.

7.1. **Derivative equations.**

Let $\varphi : \mathbb{R}^2 \times (-T, T) \to \mathbb{S}^2$ be a smooth map. We let $\partial_1 = \partial_{x_1}$, $\partial_2 = \partial_{x_2}$, $\partial_3 = \partial_t$.

Since $T_{\varphi(x,t)}\mathbb{S}^2$ is $2-$dimensional, select a smooth orthonormal frame $(v(x,t), w(x,t))$, where v and w are chosen such that $v \times w = \varphi$. This is possible by smoothness and orientability.

Then introduce the **derivative fields** ψ_α by:

$$\psi_\alpha := v \cdot \partial_\alpha \varphi + iw \cdot \partial \alpha \varphi \tag{7.1}$$

so that:

$$\partial_\alpha \varphi = vRe(\psi_\alpha) + wIm(\psi_\alpha) \tag{7.2}$$

Note that all we're identifying $\mathbb{R}^2$ with $\mathbb{C}$ via $v \leftrightarrow 1, w \leftrightarrow i$.

Through this identification, the Riemannian connection on $\mathbb{S}^2$ pulls back to a covariant derivative on $\mathbb{C}$, which we denote by:

$$D_\alpha := \partial_\alpha + iA_\alpha$$

where the real-valued connection coefficients A_α are defined by:

$$A_\alpha := w \cdot \partial_\alpha v \tag{7.3}$$

so that:

$$\begin{aligned} \partial_\alpha v = & \ -\varphi Re(\psi_\alpha) + wA_\alpha \\ \partial_\alpha w = & \ -\varphi Im(\psi_\alpha) - vA_\alpha \end{aligned}$$

Because the Riemannian connection on $\mathbb{S}^2$ is torsion-free, we have the following nice relation:

$$(\partial_\beta + iA_\beta) = (\partial_\alpha + iA_\alpha)$$

that is:

$$D_\beta \psi_\alpha = D_\alpha \psi_\beta \tag{7.4}$$

Which tells us that we can switch α and β whenever D is next to ψ.

Now if we define:

$$q_{\beta\alpha} := \partial_\beta A_\alpha - \partial_\alpha A_\beta = Im\left(\psi_\beta \overline{\psi_\alpha}\right)$$

then the curvature of the connection D is given by:

$$[D_\beta, D_\alpha] := D_\beta D_\alpha - D_\alpha D_\beta = i q_{\beta\alpha} \tag{7.5}$$

7.1.1. *Gauge formulation.* So far all of the above works for every smooth $\varphi : \mathbb{R}^2 \times (-T,T) \to \mathbb{S}^2$. From now on, assume φ solves our original equation (1.1). We will see that ψ_α satisfies a specific equation (7.10), which will lead to the Gauge formulation of the system (1.1).

First, using (7.4), (7.5) and the fact that $v \times w = \varphi$, we get (see convention below):

$$\psi_t = i \sum_{\ell=1}^{2} D_\ell \psi_\ell = i D_\ell \psi_\ell \tag{7.6}$$

Conventions:

- Roman letters are used to index spatial variables
- Greek letters are used to index all variables (spatial and time)
- Repeated indices within the same subscript or occuring in juxtaposed terms are summed over

Now apply D_m to both sides of (7.6) and use (7.4) and (7.5) to get:

$$D_t \psi_m = i D_t D_\ell \psi_m + q_{\ell m} \psi_\ell$$

which is equivalent to the nonlinear Schrödinger equation:

$$\left(i\partial_t + \Delta\right) \psi_m = \mathcal{N}_m \tag{7.7}$$

where $\mathcal{N}_m$ is the nonlinearity defined by:

$$\mathcal{N}_m := -iA_\ell \partial_\ell \psi_m - i\partial_\ell \left(A_\ell \psi_m\right) + \left(A_t + A_x^2\right) \psi_m - i\psi_\ell Im\left(\overline{\psi_\ell} \psi_m\right)$$

Next, we decompose $\mathcal{N}_m$ as $\mathcal{N}_m = B_m + V_m$, where:

$$\begin{aligned} B_m :=& - i\partial_\ell \left(A_\ell \psi_m\right) - iA_\ell \partial_\ell \psi_m \\ =& - 2iA_\ell \partial_\ell \psi_m - i\left(\partial_\ell A_\ell\right) \psi_m \\ =& - 2i\partial_\ell \left(A_\ell \psi_m\right) + i\left(\partial_\ell A_\ell\right) \psi_m \end{aligned} \tag{7.8}$$

$$V_m := \left(A_t + A_x^2\right) \psi_m - i\psi_\ell Im\left(\overline{\psi_\ell} \psi_m\right) \tag{7.9}$$

Physically, what we're doing is 'essentially' separating the (semilinear) magnetic potential terms B_m and (quasilinear) electric potential term V_m.

In the low-energy case, the problem is essentially harmless. The most dangerous term comes from the magnetic nonlinearity B_m, especially where the high-frequency part of ψ_m is differentiated. But because of small energy, this nonlinearity is perturbative, and we can use a bootstrap argument to control this term.

However, if the energy is $< E_{crit}$, the situation is a little bit more complicated. For this, we have to use energy dispesion of the initial data, as well as L^4 smallness in the sense of (5.2). Then the eletric potential is again easy to control, and we need to handle the terms where the high frequency part of ψ_m is differentiated. But this time, our smallness assumptions are not good enough, and we need to resort to other techniques.

Finally, if we put everything together, we arrive at the gauge formulation of the Schrödinger map system:

$$\begin{cases} \partial_t \varphi = & iD_\ell \psi_\ell \\ D_\beta \psi_\alpha = & D_\alpha \psi_\beta \\ iq_{\alpha\beta} = & [D_\alpha, D_\beta] \end{cases} \tag{7.10}$$

However, this formulation depends on the choice of orthonormal frame (v, w). Changing a given choice of orthonormal frame induces a gauge transformation and can be represented as:

$$\psi_m \to e^{i\theta}\psi_m$$
$$A_m \to A_m + \partial_m \theta$$

The system (7.10) is invariant with respect to the above gauge transformations.

The advantage of choosing such a gauge is that the nonlinearity (7.7) becomes perturbative (under extra assumptions and some work) and hence becomes much easier to control.

7.2. **Introduction to the caloric gauge.** It turns out that the Coulomb gauge, which is the usual choice in the study of wave maps, is not well-suited for (1.1). See for instance [5]. Instead, we will use a better gauge, called the *caloric* gauge, which we'll define here.

While it is hard to prove existence, uniqueness, and decay properties of the caloric gauge, once those have been proven, the caloric gauge becomes a powerful tool.

Here is the existence theorem for the caloric gauge. For a proof, see [2]

Theorem (The caloric gauge)**.** *Let $T \in (0, \infty)$, $Q \in \mathbb{S}^2$, and let $\psi(x,t) \in H_Q^{\infty,\infty}(T)$ be such that* $\sup_{t\in(-T,T)} E(\varphi(t)) \leq E_{crit}$. *Then there exists a unique smooth extension $\varphi(s,x,t) \in C([0,\infty) \to H_Q^{\infty,\infty}(T))$ solving the covariant heat equation:*

$$\partial_s \varphi = \Delta\varphi + \varphi \cdot |\partial_x \varphi|^2 \tag{7.11}$$

and with $\varphi(0,x,t) = \varphi(x,t)$. Moreover, for any given choice of a (constant) orthonormal basis (v_∞, w_∞) of $T_Q\mathbb{S}^2$, there exist smooth functions $v, w : [0,\infty) \times \mathbb{R}^2 \times (-T,T) \to \mathbb{S}^2$ such that for each point (s,x,t) the set $\{v, w, \varphi\}$ naturally forms an orthonormal basis for $\mathbb{R}^3$, and the following gauge condition is satisfied:

$$w \cdot \partial_s v \equiv 0 \tag{7.12}$$

and for each $f = \varphi - Q, v - v_\infty, orw - w_\infty$, multiindex ρ, and $s \geq 0$:

$$|\partial_x^\rho f(s)| \leq C_\rho \langle s \rangle^{\frac{-(|\rho|+1)}{2}} \tag{7.13}$$

Note: (7.11) is also known as the (harmonic) **heat flow**, and (7.12) is called the **parallel transport condition**.

Note: In all the applications in this paper, $E(\varphi(t))$ is conserved, and we let $E_0 := E(\varphi_0)$, where φ_0 is the initial data.

More conventions:

- $\partial_0 = \partial_s$

- All Greek indices range over heat time, spatial variables, and time

Now we can define the various gauge components:

$$\begin{aligned}
\psi_\alpha &:= v \cdot \partial_\alpha \varphi + i \cdot \partial_\alpha \varphi \\
A_\alpha &:= w \cdot \partial_\alpha v \\
D_\alpha &:= \partial_\alpha + A_\alpha \\
q_{\alpha\beta} &:= \partial_\alpha A_\beta - \partial_\beta A_\alpha
\end{aligned}$$

Then we get the following results:

$$\partial_\alpha = vRe(\psi_\alpha) + wIm(\psi_\alpha) \qquad (\alpha = 0,1,2,3)$$

And (7.12) is equivalent to:

$$A_s \equiv 0 \tag{7.14}$$

And the heat flow (7.11) is equivalent to:

$$\psi_s = D_\ell \psi_\ell \tag{7.15}$$

And using the ‘switching’ rules (7.4) and (7.5), we may rewrite the D_m covariant derivative of (7.15) as:

$$\partial_s \psi_m = D_\ell D_\ell \psi_m + iIm\left(\psi_m \overline{\psi_\ell}\right) \psi_\ell$$

or, equivalently:

$$(\partial_s - \Delta)\psi_m = iA_\ell \partial_\ell \psi_m + i\partial_\ell (A_\ell \psi_m) - A_x^2 \psi_m + i\psi_\ell Im\left(\overline{\psi_\ell}\psi_m\right) \tag{7.16}$$

More generally, we can take the D_α covariant derivative to obtain:

$$(\partial_s - \Delta)\psi_\alpha = U_\alpha \tag{7.17}$$

where:

$$\begin{aligned} U_\alpha := &\; iA_\ell\partial_\ell\psi_\alpha + i\partial_\ell(A_\ell\psi_\alpha) - A_x^2\psi_\alpha + i\psi_\ell Im(\overline{\psi_\ell}\psi_\alpha) \\ = &\; 2iA_\ell\partial_\ell\psi_\alpha + i(\partial_\ell A_\ell)\psi_\alpha - A_x^2\psi_\alpha + i\psi_\ell Im(\overline{\psi_\ell}\psi_\alpha) \\ = &\; 2i\partial_\ell(A_\ell\psi_\alpha) - i(\partial_\ell A_\ell)\psi_\alpha - A_x^2\psi_\alpha + i\psi_\ell Im(\overline{\psi_\ell}\psi_\alpha) \end{aligned} \tag{7.18}$$

While (7.16) and (7.7) look similar, their analytic properties are very different!

Now using (7.5) and (7.14), we get the following nice relation between A_m and ψ_m:

$$\partial_s A_m = Im\left(\psi_s\overline{\psi_m}\right)$$

Then, using (7.13) and integrating back from ∞ to s, we get:

$$A_m(s) = -\int_s^\infty Im\left(\overline{\psi_m}\psi_s\right)(s')ds' \tag{7.19}$$

We also have some nice estimates on A_m which will prove to be useful later:

$$\|A_x\|_{L_t^\infty L_x^2} \leq C_{E_0} \tag{7.20}$$

$$\|\partial_x v\|_{L_t^\infty L_x^2} \leq C_{E_0} \tag{7.21}$$

Proof of (7.20), (7.21). For (7.20), see [2].

Because $|v| \equiv 1$, we have $v \cdot v \equiv 1$, so $v \cdot \partial_\alpha v \equiv 0$, Therefore, with respect to the orthonormal frame (v, w, φ), the vector $\partial_\alpha v$ admits the representation:

$$\partial_\alpha v = A_\alpha \cdot w - Re(\psi_\alpha) \cdot \varphi \tag{7.22}$$

Then using $|w| \equiv 1 \equiv |\varphi|, \|\psi_\alpha\|_{L_x^2} \equiv \|\partial_\alpha\varphi\|_{L_x^2}$, energy conservation, (7.20) and (7.21), the proof follows. □

8. Frequency localization

The last ingredient we need is frequency localization. First we will do this in general, and then apply the general results to our caloric gauge.

8.1. **Bernstein inequalities and frequency envelopes.** We will use standard Littlewood-Paley frequency localization notation such as $P_k f$ and $P_{\leq k} f$.

An important set of inequalities we will use are called the *Bernstein inequalities* for $\mathbb{R}^2$ and $\sigma \geq 0$ and $1 \leq p \leq q \leq \infty$:

$$
\begin{aligned}
\|P_{\leq k} |\partial_x|^\sigma f\|_{L^p_x(\mathbb{R}^2)} \leq & \; C_{p,\sigma} 2^{\sigma k} \|P_{\leq k} f\|_{L^p_x}(\mathbb{R}^2) \\
\left\| P_k |\partial_x|^{\pm\sigma} f \right\|_{L^p_x(\mathbb{R}^2)} \leq & \; C_{p,\sigma} 2^{\pm\sigma k} \|P_k f\|_{L^p_x}(\mathbb{R}^2) \\
\|P_{\leq k} f\|_{L^q_x(\mathbb{R}^2)} \leq & \; C_{p,q} 2^{2k\left(\frac{1}{p}-\frac{1}{q}\right)} \|P_{\leq k} f\|_{L^p_x}(\mathbb{R}^2) \\
\|P_k f\|_{L^q_x(\mathbb{R}^2)} \leq & \; C_{p,q} 2^{2k\left(\frac{1}{p}-\frac{1}{q}\right)} \|P_k f\|_{L^p_x}(\mathbb{R}^2)
\end{aligned}
$$

Now let's define the frequency envelope:

Definition (Frequency envelopes)**.** *A positive sequence $\{a_k\}_{k\in\mathbb{Z}}$ is a* ***frequency envelope*** *if it belongs to ℓ^2 and is slowly-varying, i.e:*

$$a_k \leq a_j 2^{\delta|k-j|}, \qquad j,k \in \mathbb{Z} \tag{8.1}$$

*A frequency envelope $\{a_k\}_{k\in\mathbb{Z}}$ is ϵ-**energy dispersed** if it satisfies the additional condition:*

$$\sup_{k\in\mathbb{Z}} a_k \leq \epsilon$$

Note that the frequency envelopes satisfy the following summation rules:

$$\sum_{k'\leq k} 2^{pk'} a_{k'} \leq \frac{C}{p-\delta} 2^{pk} a_k \qquad\qquad p > \delta \tag{8.2}$$

$$\sum_{k'\geq k} 2^{-pk'} a_{k'} \leq \frac{C}{p-\delta} 2^{-pk} a_k \qquad\qquad p > \delta \tag{8.3}$$

Usually we'll work with $p >> \delta$, so we will drop the factors $\frac{1}{p-\delta}$.

8.2. **Applications to the heat flow and the caloric gauge.** From now on, fix $\delta_1 > 0$.

Given initial data $\varphi_0 \in H^\infty_Q$, define for all $\sigma \geq 0$ and $k \in \mathbb{Z}$

$$c_k(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'} \partial_x \varphi_0\|_{L^2_x} \tag{8.4}$$

Let $c_k := c_k(0)$ for short. Then, for $\sigma \in [0, \sigma_1]$, we have:

$$
\begin{aligned}
\|\partial_x \varphi_0\|^2_{\dot{H}^\sigma_x} = & \; C \sum_{k\in\mathbb{Z}} c_k^2(\sigma) \\
\|P_k \partial_x \varphi_0\|_{L^2_x} \leq & \; c_k(\sigma) 2^{-\sigma k}
\end{aligned} \tag{8.5}
$$

Similarly, for the time-dependent version, for $\psi \in H^{\infty,\infty}_Q(T)$, define for all $\sigma \geq 0$ and $k \in \mathbb{Z}$:

$$\gamma_k(\sigma) := \sup_{k'\in\mathbb{Z}} 2^{-\delta|k-k'|} 2^{\sigma k'} \|P_{k'} \varphi\|_{L^\infty_t L^2_x} \tag{8.6}$$

and set $\gamma_k := \gamma_k(1)$

Then we have the following frequency-localized energy bounds:

Theorem (Frequency-localized energy bounds for the heat flow)**.** *Let $f = \varphi, v$, or w. Then for $\sigma \in [1, \sigma_1]$, we have the bound:*

$$\|P_k f(s)\|_{L_t^\infty L_x^2} \leq C 2^{-\sigma k} \gamma_k(\sigma)(1 + s2^{2k})^{-20} \tag{8.7}$$

And for any $\sigma, \rho \in \mathbb{N}$:

$$\sup_{k \in \mathbb{Z}} \sup_{s \in [0,\infty)} (1+s)^{\frac{\sigma}{2}} 2^{\sigma k} \left\| P_k \partial_t^\rho f(s) \right\|_{L_t^\infty L_x^2} < \infty \tag{8.8}$$

Theorem (Frequency-localized energy bounds for the caloric gauge)**.** *For $\sigma \in [0, \sigma_1 - 1]$, we have:*

$$\|P_k \psi_x(s)\|_{L_t^\infty L_x^2} + \|P_k A_m(s)\|_{L_t^\infty L_x^2} \leq C 2^{k - \sigma k} \gamma_k(\sigma)(1 + s2^{2k})^{-20} \tag{8.9}$$

Moreover, for any $\sigma \in \mathbb{N}$:

$$\sup_{k \in \mathbb{Z}} \sup_{s \in [0,\infty)} (1+s)^{\frac{\sigma}{2}} 2^{\sigma k - k} \left(\left\| P_k \left(\partial_t^\rho \psi_x(s) \right) \right\|_{L_t^\infty L_x^2} + \left\| P_k \left(\partial_t^\rho A_x(s) \right) \right\|_{L_t^\infty L_x^2} \right) < \infty \tag{8.10}$$

and

$$\sup_{k \in \mathbb{Z}} \sup_{s \in [0,\infty)} (1+s)^{\frac{\sigma}{2}} 2^{\sigma k} \left(\left\| P_k \left(\partial_t^\rho \psi_t(s) \right) \right\|_{L_t^\infty L_x^2} + \left\| P_k \left(\partial_t^\rho A_t(s) \right) \right\|_{L_t^\infty L_x^2} \right) < \infty \tag{8.11}$$

Note that an elementary consequence of the above theorem is:

Corollary. *For $\sigma \in [0, \sigma_1 - 1]$, we have:*

$$\|P_k \sigma_x(0, \cdot, 0)\|_{L_x^2} \leq C 2^{-\sigma k} c_k(\sigma) \tag{8.12}$$

9. What comes next?

Essentially, Paul Smith's paper continues following the outline of the proof given in section 6: First, he defines function spaces which control ψ_m in (7.7) and the nonlinearity $\mathcal{N}_m$. Those spaces are based on the classical Strichartz estimates and its variant, as well as bilinear estimates. Then, he goes onto with actually proving the Main Theorem, starting with the Local Existence and Uniqueness Theorem in section 3 and extensively using the caloric gauge and the frequency-localized energy estimates.

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