

VARIATION OF LAGRANGIAN SKELETONS AND VARIATION OF GIT QUOTIENTS

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ABSTRACT. Let \mathbb{C}^* acts on \mathbb{C}^N with weight (a_1, \dots, a_N) , and let $\eta = \sum a_i$. It is well-known that there are two possible GIT quotients stacks \mathcal{X}_\pm , whose dg-derived categories $\text{Coh}(X_+)$ and $\text{Coh}(X_-)$ are related: if $\eta = 0$, we have $\text{Coh}(X_+) \simeq \text{Coh}(X_-)$; if $\eta > 0$, we have $\text{Coh}(X_+) = \langle T_\eta, \dots, T_1, \text{Coh}(X_-) \rangle$ and similarly for $\eta < 0$. It is also well-known that, using homological mirror symmetry, one can get mirror skeleton $\Lambda_\pm \subset T^*T^{N-1}$ such that $\text{Sh}^w(T^{N-1}, \Lambda_\pm) \simeq \text{Coh}(X_\pm)$. In this paper, we construct an interpolation family of skeleton $\{\Lambda_t\}_{t \in [0,1]}$, such that if $\eta = 0$, then variation Λ_t is non-characteristic ($\text{Sh}^w(T^{N-1}, \Lambda_t)$ invariant), and if $\eta \neq 0$ then there are $|\eta|$ many discontinuities in the family $\text{Sh}^w(T^{N-1}, \Lambda_t)$. The construction is based on ‘window subcategories’ from variation of GIT quotients, and is closely related to the Floer theoretic approach by [Ker17].

0.1. Motivation. The homological mirror symmetry for toric varieties has been understood from different point of views, from Abouzaid’s Lagrangian section [Abo09, Abo06], through Fang-Liu-Treumann-Zaslow’s treatment through constructible sheaves [FLTZ11, FLTZ12], to wrapped Fukaya categories or wrapped microlocal sheaves [Kuw16].

There is an interesting refinement of homological mirror symmetry of toric variety, that not only matches the entire category of $\text{Coh}(X_\Sigma)$ to certain wrapped Fukaya category, but also matches certain semi-orthogonal decomposition of them. On the B-side (coherent side), it involves using a run of minimal model program on X_Σ to get a sequence of birational morphisms

$$X_\Sigma = X_0 \dashrightarrow X_1 \dashrightarrow \dots \dashrightarrow X_n$$

as done by Kawamata [Kaw06], or Ballard-Favero-Katzarkov [BFK19]. On the A-side, suppose we are given a Landau-Ginzburg model $W : (\mathbb{C}^*)^n \rightarrow \mathbb{C}$, one may tropicalize W (by making a choice), and decompose the critical values of W into different sets, each lies approximately on a concentric circle [DKK16]. Conjecturally, the choices of the minimal model run on the B-side matches the choices of the tropicalization of W on the A-side, and the semi-orthogonal decompositions of $\text{Coh}(X_\Sigma)$ matches that with the $\text{Fuk}((\mathbb{C}^*)^n, W)$, and there has been progress towards it [BDF⁺15, Ker17].

Here we are proposing a third player in the picture, the constructible sheaf category. Given a minimal model run $X_0 \dashrightarrow \dots \dashrightarrow X_n$, we can use coherent-constructible correspondence [Bon06, FLTZ11, Kuw16] to get a sequence of Lagrangian skeleta $\Lambda_0, \dots, \Lambda_n$ in T^*T^n , however it is not clear (as far as the author know) how to build functor between $\text{Sh}(T^n, \Lambda_i)$ and $\text{Sh}(T^n, \Lambda_{i+1})$ except in the special case that $\Lambda_i \subset \Lambda_{i+1}$ or X_i and X_{i+1} is related by blow-up. The more general flip / flop between X_i needs to be explained using Lagrangian skeletons.

Our final goal is to build functors between $\text{Sh}(T^n, \Lambda_i)$ and $\text{Sh}(T^n, \Lambda_{i+1})$ using a continuous family of skeletons interpolating from Λ_i to Λ_{i+1} . These family of skeletons are slices of a skeleton of one-dimension higher, called ‘window skeleton’¹ inspired by window subcategories in VGIT setting [Seg11, BFK19, HL15]. In this paper, we study a baby example of \mathbb{C}^* acting on \mathbb{C}^N with two possible quotients. This models a birational transformation where the transition center is a point.

0.2. Setup . Let \mathbb{C}^* act on \mathbb{C}^N with weight vector $a = (a_1, \dots, a_N)$, that is

$$\mathbb{C}^* \times \mathbb{C}^N \rightarrow \mathbb{C}^N, \quad (t, z_1, \dots, z_N) \mapsto (t^{a_1} z_1, \dots, t^{a_N} z_N).$$

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Denote

$$[N] = \{1, \dots, N\}, \quad [N]_{\pm} = \{i \in [N] : \pm a_i > 0\},$$

and let $\eta_{\pm} = \sum_{i \in [N]_{\pm}} |a_i|$. We assume from the beginning that

$$a_i \neq 0, \quad \gcd(\{|a_i|\}) = 1, \quad [N]_{\pm} \neq \emptyset, \quad \eta_+ \geq \eta_-.$$

Then $\eta := \eta_+ - \eta_- \geq 0$.

There are two possible GIT quotient stacks:

$$\mathcal{X}_{\pm} = [(\mathbb{C}^N)_{\pm}^{ss} / \mathbb{C}^*], \quad \text{where } (\mathbb{C}^N)_{\pm}^{ss} := \mathbb{C}^N \setminus \{z \in \mathbb{C}^N \mid z_i = 0 \text{ if } i \in [N]_{\pm}\}$$

They are open substacks of $\mathcal{X} = [\mathbb{C}^N / \mathbb{C}^*]$. We denote the inclusions as $\iota_{\pm} : \mathcal{X}_{\pm} \hookrightarrow \mathcal{X}$.

From Coherent-Constructible-Correspondence (CCC) for toric variety (or DM stack) [FLTZ11, Kuw16], we have equivalences of dg categories

$$\text{Coh}(\mathcal{X}_{\pm}) \simeq \text{Sh}^w(\mathbb{T}^{N-1}, \Lambda_{\pm}), \quad (0.1)$$

where \mathbb{T}^{N-1} is the $(N-1)$ dimensional real torus, $\Lambda_{\pm} \subset T^*\mathbb{T}^{N-1}$ are conical Lagrangian skeletons, and $\text{Sh}^w(\mathbb{T}^{N-1}, \Lambda_{\pm})$ denote the categories wrapped constructible sheaves on \mathbb{T}^{N-1} with singular support in Λ_{\pm} .

The dg category of coherent sheaves on the two quotients are related by a semi-orthogonal decomposition [BFK19, HL15],

$$\text{Coh}(\mathcal{X}_+) = \underbrace{\langle \text{Vect}, \dots, \text{Vect} \rangle}_{\eta \text{ times}}, \text{Coh}(\mathcal{X}_-) \quad (0.2)$$

hence we have corresponding decomposition from CCC,

$$\text{Sh}^w(\mathbb{T}^{N-1}, \Lambda_+) = \underbrace{\langle \text{Vect}, \dots, \text{Vect} \rangle}_{\eta \text{ times}}, \text{Sh}^w(\mathbb{T}^{N-1}, \Lambda_-). \quad (0.3)$$

The goal of this paper is to understand the above semi-orthogonal decomposition using Lagrangian isotopies from Λ_- to Λ_+ .

0.3. Window Skeleton and Results. One way to obtain the semi-orthogonal decomposition on the B-side is through window subcategory of $\text{Coh}(\mathcal{X})$.

Definition 0.1 (Window). *A window is a set of consecutive integers.*

Let W be a window. We define the **B-model window subcategory** to be the full triangulated subcategories

$$\mathcal{B}_W = \langle \{\mathcal{L}_i, i \in W\} \rangle \subset \text{Coh}(\mathcal{X}),$$

where \mathcal{L}_k is the \mathbb{C}^* -equivariant line bundle over \mathbb{C}^N with weight k .

Theorem 0.2 ([BFK19, HL15]). *Let W be a window. Let $\iota_{\pm} : \mathcal{X}_{\pm} \hookrightarrow \mathcal{X}$ be inclusions of open stacks.*

- (1) *If $|W| = \eta_+$, then we have equivalence of categories $\iota_+^*|_{\mathcal{B}_W} : \mathcal{B}_W \xrightarrow{\sim} \text{Coh}(\mathcal{X}_+)$.*
- (2) *If $|W| = \eta_-$, then we have equivalence of categories $\iota_-^*|_{\mathcal{B}_W} : \mathcal{B}_W \xrightarrow{\sim} \text{Coh}(\mathcal{X}_-)$.*

If we fix $a \in \mathbb{Z}$, and consider windows

$$W_+ = \{a, a+1, \dots, a+\eta_+-1\}, \quad W_- = \{a+\eta, \dots, a+\eta_+-1\} \subset W_+$$

then we have windows of sizes η_+ and η_- respectively. The inclusion of window subcategories $\mathcal{B}_{W_-} \hookrightarrow \mathcal{B}_{W_+}$ induces an embedding

$$(\iota_+^*) \circ (\iota_-^*)^{-1} : \text{Coh}(\mathcal{X}_-) \hookrightarrow \text{Coh}(\mathcal{X}_+).$$

The η copies of Vect corresponds to structure sheaf of the unstable loci $(\mathbb{C}^N)_-^{us}$ in $\text{Coh}([\mathbb{C}^N / \mathbb{C}^*])$ twisted by some character of \mathbb{C}^* . The details are reviewed in section 7.1.

Next we consider the A -side. From CCC, we have

$$\tau : \text{Coh}(\mathcal{X}) \xrightarrow{\sim} \text{Sh}^w(\mathcal{Y}, \Lambda_{\mathcal{Y}}), \quad \tau_{\pm} : \text{Coh}(\mathcal{X}_{\pm}) \xrightarrow{\sim} \text{Sh}^w(\mathcal{Y}, \Lambda_{\mathcal{Y}, \pm}).$$

where $\mathcal{Y} \simeq \mathbb{T}^{N-1} \times \mathbb{R}$ and $\Lambda_{\mathcal{Y}, \pm} \subset \Lambda_{\mathcal{Y}} \subset T^*\mathcal{Y}$ are a conical Lagrangian of dimension N . The Lagrangian skeletons $\Lambda_{\pm} \subset T^*\mathbb{T}^{N-1}$ in (0.1) arises as restriction of $\Lambda_{\mathcal{Y}, \pm}$ over $\mathbb{T}^{N-1} \times \{t\}$ for $t \in \mathbb{Z}$.

Thus, it is natural to define the window categories on the A -side as

$$\mathcal{A}_W = \tau(\mathcal{B}_W) = \langle \{\tau \mathcal{L}_i, i \in W\} \rangle \subset \text{Sh}^w(\mathcal{Y}, \Lambda_{\mathcal{Y}}).$$

The question is, how to characterize \mathcal{A}_W geometrically.

We define the **window skeleton** Λ_W as the minimal sub-skeleton of $\Lambda_{\mathcal{Y}}$ that contains the CCC image of \mathcal{B}_W :

$$\Lambda_W := \bigcup_{i \in W} \text{SS}(\tau(\mathcal{L}_i)) \subset \Lambda_{\mathcal{Y}}. \quad (0.4)$$

Thus, we have natural fully faithful embedding of subcategories

$$\iota_W : \mathcal{A}_W \hookrightarrow \text{Sh}^w(\mathcal{Y}, \Lambda_W).$$

It is not always true that the embedding is an equivalence. However, we have

Theorem 0.3. *If $|W| \geq \eta_-$, then the fully faithful embedding ι_W is an equivalence of categories.*

Example 0.4. *Here is an example where ι_W fails to be an equivalence. Consider \mathbb{C}^* acts on \mathbb{C}^3 with weights $(1, 1, -2)$, and let $W = \{0\}$ a window of size 1. The category \mathcal{B}_W is equivalent to $\text{Perf}(\mathbb{C}^2/\mathbb{Z}_2)$, where $\mathbb{C}^2/\mathbb{Z}_2$ is the singular affine variety $\text{Spec}(\mathbb{C}[x, y, z]/(xy - z^2))$. Then $\text{Sh}^w(\mathcal{Y}, \Lambda_W) \simeq \text{Coh}(\mathbb{C}^2/\mathbb{Z}_2)$. Clearly, we have a discrepancy here. \triangle*

Next, we consider the torus fibration

$$\mu : \mathcal{Y} \rightarrow \mathbb{R}, \quad \mathcal{Y}_t = \mu^{-1}(t) \simeq \mathbb{T}^{N-1}.$$

Then we have a family of categories labelled by $t \in \mathbb{R}$

$$\mathcal{A}_W(t) = \text{Sh}^w(\mathcal{Y}_t, \Lambda_t), \quad \Lambda_t := \Lambda_W|_{\mathcal{Y}_t}.$$

The slices of Λ_W over \mathcal{Y}_t to the left and right of the window has nice relation with skeleton of GIT quotients.

Theorem 0.5 (Theorem 3.1). *Let W be a window.*

- (1) *If $|W| \geq \eta_-$, then over $\mu^{-1}(\max(-W), +\infty)$, we have $\Lambda_W = \Lambda_{\mathcal{Y}, +}$.*
- (2) *If $|W| \geq \eta_+$, then over $\mu^{-1}(-\infty, \min(-W))$, we have $\Lambda_W = \Lambda_{\mathcal{Y}, -}$.*

Theorem 0.6. *Let W be a window, and $|W| = \eta_+$.*

- (a) *If $\eta = 0$, then $\mathcal{A}_{W, t}$ are constant (up to equivalence) for all $t \in \mathbb{R}$.*
- (b) *If $\eta > 0$, then there is a set \mathcal{J} of size η ,*

$$\mathcal{J} = \{a, a + 1, \dots, a + \eta - 1\}, \quad a = \min(-W),$$

such that $\mathcal{A}_{W, t}$ is locally constant (up to equivalence) away from \mathcal{J} . Moreover,

- (1) *If $t > \max(\mathcal{J})$, then*

$$\mathcal{A}_W(t) \simeq \text{Sh}^w(\mathbb{T}^{N-1}, \Lambda_+).$$

- (2) *If $t < \min(\mathcal{J})$, then*

$$\mathcal{A}_W(t) \simeq \text{Sh}^w(\mathbb{T}^{N-1}, \Lambda_-).$$

- (3) *If $t \in \mathcal{J}$, then for any small positive $\epsilon > 0$, we have semi-orthogonal decomposition*

$$\mathcal{A}_W(t + \epsilon) = \langle \text{Vect}, \mathcal{A}_W(t - \epsilon) \rangle.$$

0.4. Sketch of the Proof. Our main tool is the notion of variation of Lagrangian skeleton, which we now recall.

Let M be a real analytic compact manifold (we may relax the ‘compact’ condition in practice), B a smooth manifold and $\pi : M \times B \rightarrow B$. A **variation of Lagrangian skeleton** over B is a Lagrangian $\Lambda_B \subset T^*(M \times B)$ containing the zero section, such that ²

$$\Lambda_B^\infty \cap T_{M \times \{b\}}^\infty(M \times B) = \emptyset, \quad \forall b \in B.$$

For $b \in B$, denote $M_b = \pi^{-1}(b)$ and $\Lambda_b = \Lambda_B|_{M_b}$. We are interested in when $Sh^\diamond(M_b, \Lambda_b)$ will be invariant.

We study this problem locally in $M \times B$. We consider the sheaf of categories $Sh_{\Lambda_B}^\diamond$ on $M \times B$, and define its singular support $SS(Sh_{\Lambda_B}^\diamond)$. We also define the push-forward of $\pi_* Sh_{\Lambda_B}^\diamond$, then

$$SS(\pi_* Sh_{\Lambda_B}^\diamond) \subset \pi_*(SS(Sh_{\Lambda_B}^\diamond))$$

where push-forward Lagrangian skeleton is by Lagrangian correspondence.

In general, given a skeleton $\Lambda \subset T^*Z$, we are interested in how to compute $SS(Sh_\Lambda^\diamond)$ from Λ . Let $V \subset U \subset Z$ be nested open sets, the restriction functor

$$\rho_{U,V} : Sh_\Lambda^\diamond(U) \rightarrow Sh_\Lambda^\diamond(V)$$

is an equivalence if and only if $\rho_{U,V}$ is fully-faithful and its left-adjoint $\rho_{U,V}^L$, i.e., the co-restriction functor is fully-faithful. Hence we defined two other singular supports $SS_{Hom}(\Lambda)$ and $SS_{Hom}^L(\Lambda)$ that measures the failure of fully-faithfulness. We have

$$SS(Sh_\Lambda^\diamond) = SS_{Hom}^L(\Lambda) \cup SS_{Hom}(\Lambda).$$

In our case, $M \times B = \mathbb{T}^{n-1} \times \mathbb{R}$, the variation of skeleton is $\Lambda_W \subset T^*(\mathbb{T}^{n-1} \times \mathbb{R})$. Since the computation is local, we work in the universal cover $\tilde{\Lambda}_W \subset T^*\mathbb{R}^N$. The computation of $SS_{Hom}(\Lambda_W)$ boils down to an estimation of the singular support of the hom-sheaf. A useful trick in the estimation is the following theorem.

Theorem 0.7 (Theorem 2.15 later). *Let F, G be constructible sheaves over M and $\{G_t\}_{t \in (0, \epsilon)}$ a positive isotopy of G . Then*

$$SS(\mathcal{H}om(F, G)) \subset \limsup_{t \rightarrow 0^+} SS(\mathcal{H}om(F, G_t)).$$

In particular, if $SS^\infty(F) \cap SS^\infty(G_t) = \emptyset$ for $t \in (0, \epsilon)$ then

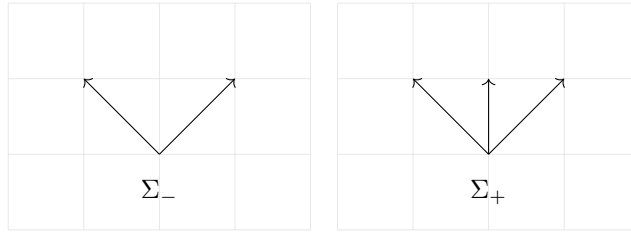
$$SS(\mathcal{H}om(F, G)) \subset \limsup_{t \rightarrow 0^+} SS(F)^a + SS(G_t).$$

The computation of $SS_{Hom}^L(\Lambda_W)$, in our case, is done by a straight-forward, if tedious, study on the skeleton and the generators of the sheaf category locally, i.e, the microlocal skyscrapers sheaves. The main result is that, if the window size is large enough, more precisely $|W| \geq \eta_+$, then $SS_{Hom}^L(\Lambda)$ is the zero-section.

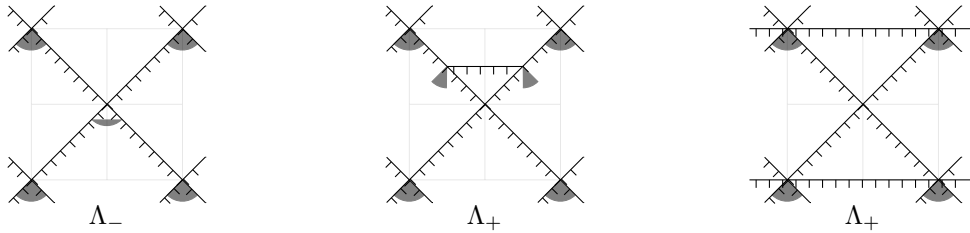
0.5. Examples. We present three examples of variation of Lagrangians. The first two are examples for non-characteristic deformation, i.e. $\eta = 0$, for resolution of $\mathbb{C}^2/2$ and the Atiyah flop; the last one is \mathbb{C}^* acting on \mathbb{C}^2 with weight $(3, -1)$, so that we can demonstrate the shape of the window skeleton and the behavior of the ‘thimble’.

²Here we use notation that $T^\infty M = (T^*M \setminus T_M^*M)/\mathbb{R}_+$ is the contact cosphere bundle at infinity, and $\Lambda^\infty = (\Lambda \setminus T_M^*M)/\mathbb{R}_+$ is the (singular) Legendrian boundary of Λ .

Example 0.8. Consider \mathbb{C}^* acts on \mathbb{C}^3 with weight $(1, 1, -2)$. The two GIT quotient stacks are $\mathcal{X}_- = [\mathbb{C}^2/\mathbb{Z}_2]$ and $\mathcal{X}_+ = \mathcal{O}_{\mathbb{P}^1}(-2)$. The corresponding toric fans are shown below.



From this, we can build the skeletons Λ_- and Λ_+ on \mathbb{T}^2 . The following picture shows an open neighborhood of a fundamental domain of the unwrapping of \mathbb{T}^2 . The hairs lines are the projection image of $\Lambda^\infty \subset T^\infty\mathbb{T}^2$, under the 'front projection' map $\pi : T^\infty\mathbb{T}^2 \rightarrow \mathbb{T}^2$. The small arcs indicate that the pre-image of $\pi|_{\Lambda^\infty}$ over the corresponding intersection point is a small arc.

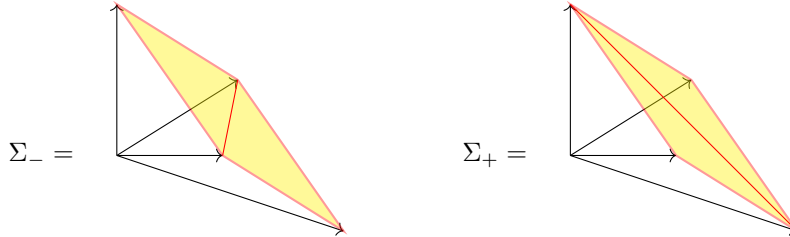


An interpolation of skeletons, generated by the slices of window skeleton is of the form. △

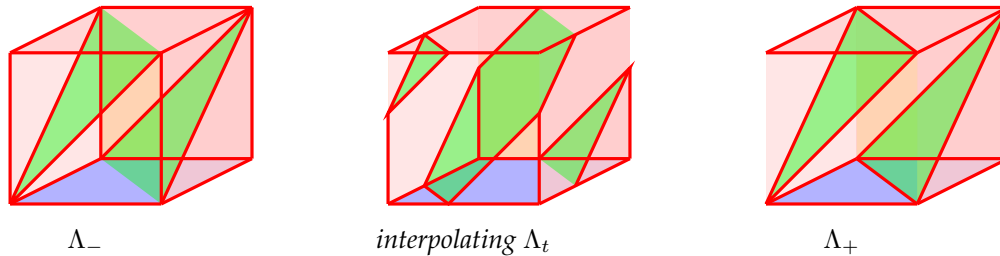
Example 0.9. Atiyah Flop. Consider \mathbb{C}^* acts on \mathbb{C}^4 with weight $(1, 1, -1, -1)$. The toric fan Σ_- and Σ_+ both have rays generated by

$$\{v_1 = (0, 0, 1), v_2 = (0, 1, 0), v_3 = (0, 0, -1), v_4 = (1, 1, -1)\}.$$

Σ_- and Σ_+ differ at the triangulation of the (non-simplicial) 3-dimensional cone generated by the four rays. In particular Σ_- contains $\text{cone}(v_2, v_3)$, Σ_+ contains $\text{cone}(v_1, v_4)$.

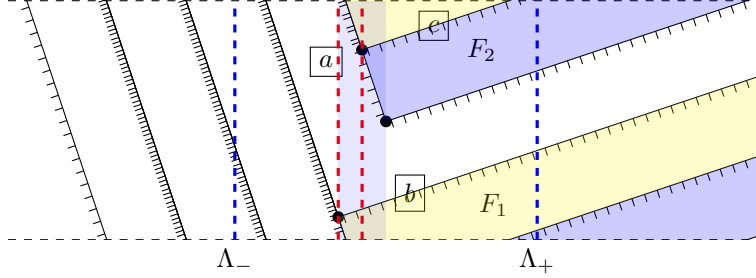


The corresponding Lagrangian skeletons Λ_\pm and the interpolation is shown as following. The red thick line corresponds the image where $\pi|_{\Lambda^\infty}$ is not finite.



△

Example 0.10. Consider \mathbb{C}^* acting on \mathbb{C}^2 with weight $(3, -1)$. The window skeleton is shown as below, living over $S^1 \times \mathbb{R}$ (drawn as $\mathbb{R} \times [0, 1]$ with top and bottom edge identified).



The window skeleton is the union of three skeleton $\Lambda(0), \Lambda(1), \Lambda(2)$, whose vertices are marked in black nodes. The window region is marked in shadow. Take a vertical slice on the right of the window region, we get the skeleton Λ_+ for $[\mathbb{C}/\mathbb{Z}_3]$; and the vertical slice on the left of the window region gives skeleton Λ_- for \mathbb{C} .



As the slice pass through the window region, from $t > 2$ to $t < 0$, the skeleton Λ_t under goes two jumps at $t = 0$ and $t = 1$, which are marked in red line. The change of skeleton is easy to understand, the three half rays in Λ_+ gets merged to two rays, then to one ray.

Next, we show how to identify the semi-orthogonal decomposition on the B-side

$$\text{Coh}([\mathbb{C}/\mathbb{Z}_3]) = \langle \text{Vect}_2, \text{Vect}_1, \text{Coh}(\mathbb{C}) \rangle.$$

First, we claim that

$$\text{Sh}^w(\mathbb{T} \times \mathbb{R}, \Lambda) \simeq \text{Sh}^w(\mathbb{T}, \Lambda_+) \simeq \text{Coh}([\mathbb{C}/\mathbb{Z}_3]).$$


$\text{Coh}(\mathbb{C})$ corresponds to the full triangulated subcategory of $\text{Sh}^w(\mathbb{T} \times \mathbb{R}, \Lambda)$ generated by sheaf F_0 co-representing the stalk at point a ; Vect_1 generated by the microlocal skyscraper sheaf³ F_1 at point b (support marked in yellow); Vect_2 the microlocal skyscraper sheaf F_2 at point c (support marked in blue). We can easily verify that $\text{Hom}(F_i, F_j) = 0$, if $i < j$, and also the generation condition. Hence F_2, F_1, F_0 forms an exceptional collection and induces the semi-orthogonal condition.

△

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0.7. Notation. We summarize all the notations used in the paper for the ease of reference.

1. $[N], [N]_{\pm}$ are defined in section 0.2. If $I \subset [N]$, we let $I_{\pm} = I \cap [N]_{\pm}$.
2. $\mathbb{T} = \mathbb{R}/\mathbb{Z}$. If N is a lattice, we let $N_A := N \otimes_{\mathbb{Z}} A$ for abelian groups $A = \mathbb{R}, \mathbb{C}, \mathbb{C}^*, \mathbb{T}$.
3. Let \mathbb{Z}^N denote the character lattice of $(\mathbb{C}^*)^N$, with basis $e_i, i \in [N]$. Let $(\mathbb{Z}^N)^{\vee}$ denote dual lattice, with dual basis $e_i^{\vee}, i \in [N]$.

³To be precise, we should say 'a' microlocal skyscraper sheaf, since there are choices of grading. In practice, we fix an initial microstalk sheaf , then apply geodesic flow stopped by the desired skeleton.

4. For $I \subset [N]$, we define closed cones

$$\kappa_I = \text{cone}(e_i, i \in I) \subset \mathbb{R}^N, \quad \sigma_I = \text{cone}(e_i^\vee, i \in I) \subset (\mathbb{R}^N)^\vee.$$

We also define standard vectors

$$e_I = \sum_{i \in I} e_i, \quad e_I^\vee = \sum_{i \in I} e_i^\vee.$$

5. There are a dual pair of short exact sequences (SES)

$$\begin{aligned} 0 \rightarrow \mathbb{Z} \xrightarrow{a} (\mathbb{Z}^N)^\vee \rightarrow \mathbf{N} \rightarrow 0 \\ 0 \rightarrow \mathbf{M} \rightarrow \mathbb{Z}^N \xrightarrow{\mu_Z} \mathbb{Z} \rightarrow 0. \end{aligned}$$

where $\mu_Z(x) = \sum_i x_i a_i$. We tensor the second SES by \mathbb{R} and quotient by \mathbf{M} , to get

$$0 \rightarrow \frac{\mathbf{M}_{\mathbb{R}}}{\mathbf{M}} \rightarrow \frac{\mathbb{R}^N}{\mathbf{M}} \xrightarrow{\mu} \mathbb{R} \rightarrow 0.$$

We identify $\mathbf{M}_{\mathbb{R}}/\mathbf{M}$ with $\mathbf{M}_{\mathbb{T}}$, then we get a torus fibration

$$\mu : \mathcal{Y} \rightarrow \mathbb{R}, \quad \mathcal{Y} := \frac{\mathbb{R}^N}{\mathbf{M}}, \quad \mathcal{Y}_t := \mu^{-1}(t).$$

6. If $Z \subset \mathbb{R}^N$ is a locally closed subset, then let \mathbb{C}_Z denote the constant sheaf with non-zero stalk \mathbb{C} only on Z .

7. If $v \in \mathbb{R}^N$, we denote

$$Q_v = \mathbb{C}_{v+(\mathbb{R}_{>0})^N}$$

the constant sheaf supported on the shifted open quadrant.

8. If $v \in \mathbb{Z}^N$, $I \subset [N]$, we denote

$$\mathcal{P}_I(v) := \underline{Q}_v \rightarrow \bigoplus_{J \subset I, |J|=1} Q_{v-e_J} \rightarrow \bigoplus_{J \subset I, |J|=2} Q_{v-e_J} \rightarrow \cdots \rightarrow Q_{v-e_I},$$

where Q_v is sitting at degree 0⁴ and the maps are induced by natural inclusion of open sets.

9. We say \mathbb{C}^* acts on \mathbb{C} **with weight** k if $\mathbb{C}^* \times \mathbb{C} \rightarrow \mathbb{C}$ is given by $(t, z) \mapsto t^k z$. Let \mathcal{L} be a \mathbb{C}^* -equivariant line bundle on \mathbb{C}^N . We say \mathcal{L} is of weight k , if \mathbb{C}^* acts on the fiber over fixed point $\mathcal{L}|_0$ with weight k . We denote the weight- k line bundle by \mathcal{L}_k .

10. If \mathbb{C}^* acts on \mathbb{C} with weight 1, then \mathbb{C}^* acts on its dual \mathbb{C}^\vee with weight -1 . Thus the space of function on \mathbb{C} has \mathbb{C}^* -action with non-positive weights

$$\mathcal{O}_{\mathbb{C}} = \text{Sym}^*(\mathbb{C}^\vee) = \text{Sym}^*(\mathbb{C} \cdot z) = \mathbb{C}[z], \quad \text{weight}(z) = -1.$$

If \mathcal{L}_k is the weight k line bundle on \mathbb{C} , then

$$\text{Hom}(\mathcal{L}_0, \mathcal{L}_k) = \Gamma(\mathcal{L}_k) \simeq \mathbb{C}[z] \cdot e_k, \quad \text{weight}(z) = -1, \text{weight}(e_k) = k.$$

We define the \mathbb{C}^* -equivariant map to be the weight-0 subspace above

$$\text{Hom}_{\mathbb{C}^*}(\mathcal{L}_0, \mathcal{L}_k) = \Gamma(\mathcal{L}_k)_0 = \begin{cases} \mathbb{C} \cdot z^k & k \geq 0 \\ 0 & k < 0 \end{cases}.$$

11. If Z is a real analytic manifold. We let $\pi_Z : T^*Z \rightarrow Z$ be its cotangent bundle, $\dot{T}^*Z = T^*Z \setminus T_Z^*Z$ the punctured bundle, $T^\infty Z = T^*Z/\mathbb{R}_+$. If $\Lambda \subset T^*Z$ is a conical Lagrangian, we let $\Lambda^\infty = (\Lambda \cap \dot{T}^*Z)/\mathbb{R}_+$ be its Legendrian boundary. If $S \subset Z$ is a submanifold, we let $T_S^\infty Z = (T_S^*Z)^\infty$.

If $S \subset Z$ is an embedded submanifold, $\Lambda \subset T^*Z$ a conical Lagrangian, we define the **restriction of Λ to S as**

$$\Lambda|_S = q(\Lambda \cap T^*Z|_S) \subset T^*S$$

⁴We always use wavy underline to indicate the degree 0 component.

where q is defined as

$$0 \rightarrow T_S^*Z \rightarrow TZ|_S \xrightarrow{q} T^*S.$$

Let $f : Z \rightarrow Y$ be a smooth submersion map, $Z_y = f^{-1}(y)$ for $y \in Y$, $\Lambda \subset T^*Z$ a conical Lagrangian. We say Λ is **non-characteristic with respect to f** , if

$$\Lambda^\infty \cap T_{Z_y}^\infty = \emptyset.$$

We denote $\Lambda_y = \Lambda|_{Z_y} \subset T^*Z_y$.

1. BACKGROUND

1.1. Wrapped Microlocal Sheaves. We review the basic definitions and properties of wrapped microlocal sheaves, as introduced by Nadler in [Nad16] to mimic the definition of wrapped Fukaya categories. There is nothing original in this subsection, we only give an abridged account of loc.cit, section 1.1.

We fix $k = \mathbb{C}$, or any other characteristic zero field.

1.1.1. Motivation. First we introduce the notion of Lagrangian skeleton of a Weinstein manifold or Weinstein pair. Since these heuristics are only for motivations, we will not give the precise definitions but refer the reader to the survey [Eli17] and references in there.

Let $(W, \omega = d\lambda)$ be an exact symplectic manifold, with Liouville one-form λ and (retracting) Liouville vector field X_λ defined by $\iota_{X_\lambda}\omega = -\lambda$. Roughly speaking, if the Liouville vector field 'points inward' near infinity and is gradient-like, then W is a Weinstein manifold. The skeleton $\text{Skel}(W)$ (or core) of a Weinstein manifold W is the union of the unstable manifolds for X_λ . Hence $\text{Skel}(W)$ is a deformation retract of W that is also compact and isotropic. We will only consider the case where the skeletons are Lagrangian.

A variant of Weinstein manifold is **Weinstein pair** (W, H) , where H can be thought of as a Weinstein hypersurface in the contact boundary $\partial^\infty W$ of W at infinity. Then $\text{Skel}(W, H)$ is the union of $\text{Skel}(W)$ and the flow-out of $\text{Skel}(H)$ under the retracting flow X_λ . We assume $\text{Skel}(W, H)$ is again a Lagrangian in W .

One may define the **(partially) wrapped Fukaya category** $\mathcal{W}(W, H)$ associated to the Weinstein pair (W, H) , where the objects are Lagrangian submanifolds, and morphisms $\text{Hom}(L_1, L_2)$ are computed by wrapping L_1 positively near infinity.

Kontsevich outlined an approach to compute the category $\mathcal{W}(W, H)$ as global section of a cosheaf of categories on the skeleton $L = \text{Skel}(W, H)$. This idea is realized first in the sheaf-theoretic approach in [Nad16, She17] as cosheaf of dg categories μsh_L^w of **wrapped microlocal sheaf**, then as wrapped Fukaya category of Liouville sectors in [GPS19, GPS18].

1.1.2. Three categories of microlocal sheaves. Let $\Lambda \subset T^*Z$ be a closed conical Lagrangian in a cotangent bundle of a real analytic manifold Z . To a conical open subset $\Omega \subset T^*Z$, we associate the following three dg categories.

- (1) $\mu Sh_\Lambda^\diamond(\Omega)$: the category of **large** microlocal sheaves. It is a cocomplete dg category consists of microlocal sheaves with microstalks being arbitrary k -modules supported on Λ .
- (2) $\mu Sh_\Lambda^w(\Omega)$: the category of **wrapped** microlocal sheaves. By definition, this is the category of compact objects in $\mu Sh_\Lambda^\diamond(\Omega)$.

$$\mu Sh_\Lambda^w(\Omega) = (\mu Sh_\Lambda^\diamond(\Omega))_c.$$

- (3) $\mu Sh_\Lambda(\Omega)$: the category of **traditional** microlocal sheaves. This is a full dg subcategory of $\mu Sh_\Lambda^\diamond(\Omega)$, consisting of objects whose microstalks are perfect k -modules (i.e. finite dimensional k -vector spaces) supported along Λ .

The wrapped microlocal sheaves $\mu Sh_\Lambda^w(\Omega)$ can recover the other two. The large microlocal sheaves is its ind-completion:

$$\mu Sh_\Lambda^\diamond(\Omega) = \text{Ind}(\mu Sh_\Lambda^w(\Omega)).$$

The traditional microlocal sheaves is the category of exact functors on it valued in perfect modules:

$$\mu Sh_\Lambda(\Omega) = \text{Fun}^{ex}(\mu Sh_\Lambda^w(\Omega)^{op}, \text{Perf}_k).$$

There is a distinguished set of wrapped microlocal sheaves, called *microlocal skyscrapers*, that co-represents the microstalk functors associated to smooth strata of $\Lambda \cap \Omega$. The collection of microlocal skyscrapers forms a compact generators of $\mu Sh_\Lambda^\diamond(\Omega)$, which split-generate $\mu Sh_\Lambda^w(\Omega)$ [Nad16][Lemma 1.4].

If $\Omega_1 \subset \Omega_2$ are two conic open subsets in T^*Z , we have the restriction functor

$$\rho : \mu Sh_\Lambda^\diamond(\Omega_2) \rightarrow \mu Sh_\Lambda^\diamond(\Omega_1).$$

ρ has a left-adjoint ρ^L that preserves compact objects, we have then

$$\rho^L : \mu Sh_\Lambda^\diamond(\Omega_1) \rightarrow \mu Sh_\Lambda^\diamond(\Omega_2), \quad \rho^w : \mu Sh_\Lambda^w(\Omega_1) \rightarrow \mu Sh_\Lambda^w(\Omega_2).$$

If $\Lambda_1 \subset \Lambda_2$ are two closed conic Lagrangians, then we have a fully-faithful embedding

$$i : \mu Sh_{\Lambda_1}^\diamond(\Omega) \rightarrow \mu Sh_{\Lambda_2}^\diamond(\Omega).$$

i also has a left-adjoint that preserves compact objects, we have

$$i^L : \mu Sh_{\Lambda_2}^L(\Omega) \rightarrow \mu Sh_{\Lambda_1}^L(\Omega), \quad i^w : \mu Sh_{\Lambda_2}^w(\Omega) \rightarrow \mu Sh_{\Lambda_1}^w(\Omega).$$

The realization of microlocal sheaves are described in [Nad16][Section 3]. Here we only use corresponding three versions for constructible sheaves, Sh_Λ^\diamond , Sh_Λ^w and Sh_Λ . Given an open set $U \subset Z$, we have $Sh_\Lambda^\diamond(U) = \mu Sh_\Lambda^\diamond(\pi^{-1}(U))$, where $\pi : T^*Z \rightarrow Z$ is the projection. All the statement about functors and left-adjoints applies here.

1.1.3. Stalk Probe Sheaves. If $U \subset Z$ is an open set, and $p \in U$, then the stalk functor $\phi_p : Sh_\Lambda^\diamond(U) \rightarrow \text{Mod}_k$ is co-representable by an object $\mathcal{F}_p \in Sh_\Lambda^w(U)$, i.e. $\text{Hom}(\mathcal{F}_p, \mathcal{G}) \simeq G_p$ for all $\mathcal{G} \in Sh_\Lambda^\diamond(U)$. We call \mathcal{F}_p the **stalk probe sheaf** for point p in $Sh_\Lambda^\diamond(U)$. This is an alias for Nadler's microlocal skyscraper sheaf in the special case that the microstalk functor is a stalk functor.

In general, it is hard to explicitly describe the stalk probe sheaves. In this paper, we employ two methods to find the descriptions:

- (1) **Geometric wrapping.** We start from a sheaf $P_0 = \mathbb{C}_B$ supported on a small open ball around p , then we consider **positive isotopy** P_t of P_0 , such that

$$SS^\infty(P_t) \cap \Lambda^\infty = \emptyset, \text{ and } , SS^\infty(P_1) \subset \Lambda^\infty.$$

Or more generally, an arbitrary isotopy P_t such that there exists a positive isotopy of Λ , satisfying $SS^\infty(P_1) \subset \Lambda^\infty$ and the **left non-characteristic (LNC)** condition

$$SS^\infty(P_t) \cap \Lambda_s^\infty = \emptyset, \forall t \in [0, 1], s \in (0, \epsilon).$$

- (2) **Algebraic Constraint.** If the stalk functor that we are interested in can be resolved using other stalks functors with known probe sheaves, then the probe sheaf that we are interested enjoys the same resolution. This is useful if we have know a closed embedding of skeleton $\Lambda_1 \subset \Lambda_2$, and we know the probes for Λ_2 and want to find probes for Λ_1 .

1.2. Coherent-Constructible Correspondence. Let $\Sigma \subset \mathbb{R}^n$ be a toric fan, X_Σ be the corresponding toric variety (or smooth DM toric stack). The Coherent-Constructible Correspondence (CCC) [FLTZ11, Kuw16] is an equivalence of dg categories

$$\mathrm{Coh}(X_\Sigma) \simeq \mathrm{Sh}^w(\mathbb{T}^n, \Lambda_\Sigma)$$

where $\Lambda_\Sigma \subset T^*\mathbb{T}^n$ is a Lagrangian skeleton that only depends on Σ . We say Λ_Σ is the **mirror** to X_Σ .

Example 1.1. (1) The mirror of \mathbb{C}^* is the zero-section $\Lambda = \mathbb{T} = S^1 \subset T^*\mathbb{T}$. The structure sheaf $\mathcal{O}_{\mathbb{C}^*}$ is sent to $p_!\mathbb{C}_{\mathbb{R}}$, where $p: \mathbb{R} \rightarrow \mathbb{T}$ is the quotient map.

(2) The mirror of \mathbb{C}^* is the union of zero section with the half cotangent fiber over 0, $\{(x=0, \xi < 0) \in T^*\mathbb{T}\}$. The structure sheaf $\mathcal{O}_{\mathbb{C}}$ is sent to (up to a shift in cohomological degree) $p_!\mathbb{C}_{(0,\infty)}$ (or equivalently $p_!\mathbb{C}_{(k,\infty)}$ for any $k \in \mathbb{Z}$), and the structure sheaf \mathcal{O}_0 for $0 \in \mathbb{C}$ is sent to $p_!\mathbb{C}_{(0,1]}$. Indeed

$$\mathrm{Hom}_{\mathbb{T}}(p_!\mathbb{C}_{(0,1]}, p_!\mathbb{C}_{(0,1]}) = \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}_{(0,1]}, p^!\mathbb{C}_{(0,1]}) = \bigoplus_{k \in \mathbb{Z}} \mathrm{Hom}_{\mathbb{R}}(\mathbb{C}_{(0,1]}, \mathbb{C}_{(k,k+1]}) = \mathbb{C} \oplus \mathbb{C}[-1]$$

where the contribution for $\mathbb{C}[-1]$ is from summand $k = 1$.

(3) The mirror of \mathbb{P}^1 is the union of zero section with the cotangent fiber $T_0^*\mathbb{T}$. The structure sheaf $\mathcal{O}_{\mathbb{P}^1}$ is sent to $\mathbb{C}_{\{0\}}$, $\mathcal{O}_{\mathbb{P}^1}(1)$ to $p_!\mathbb{C}_{(0,1]}[1]$ and $\mathcal{O}_{\mathbb{P}^1}(-1)$ to $p_!\mathbb{C}_{[0,1]}$. Here the choice of the cohomological degree is fixed by the requirement that tensor product on $\mathrm{Coh}(\mathbb{P}^1)$ is sent to convolution product on $\mathrm{Sh}(\mathbb{T}, \Lambda)$. \triangle

Now we establish some notations for toric geometry and describe the mirror skeleton Λ_Σ for a simplicial stacky fan Σ following [FLTZ11].

Let N be a rank n lattice, $N \simeq \mathbb{Z}^n$ after we fix a basis of N . Let $M = \mathrm{Hom}_{\mathbb{Z}}(N, \mathbb{Z})$ be the dual lattice. We use $N_{\mathbb{R}}$ for $N \otimes_{\mathbb{Z}} \mathbb{R}$, and similarly for N replaced by M , \mathbb{R} replaced by \mathbb{T}, \mathbb{C}^* .

We recall the following definitions.

- (1) A convex polyhedral cone (abbreviated as ‘cone’ later) $\sigma \subset N_{\mathbb{R}}$ is a set of the form $\sigma = \mathrm{cone}(S) = \{\sum_{u \in S} \lambda_u u \mid \lambda_u \geq 0\}$, where the cone generator $S \subset N_{\mathbb{R}}$ is a finite subset. A cone σ is *rational* if there is a generator S for σ such that $S \subset N$. A cone is *strongly convex* if it does not contain any non-trivial linear subspace of $N_{\mathbb{R}}$.
- (2) A cone $\sigma \subset N_{\mathbb{R}}$ is *simplicial* if σ is generated by linearly independent vectors in $N_{\mathbb{R}}$. It is *smooth* if σ is generated by a subset of \mathbb{Z} -basis of N . Let $\langle -, - \rangle$ denote the pairing $M \times N \rightarrow \mathbb{Z}$, and the induced other pairings, e.g $M_{\mathbb{T}} \times N \rightarrow \mathbb{T}$.
- (3) Let σ be a cone, we define the dual (closed) cone σ^\vee as

$$\sigma^\vee := \{x \in M_{\mathbb{R}} \mid \langle x, y \rangle \geq 0, \forall y \in \sigma\}.$$

We also define $\sigma^\perp = \{x \in M \mid \langle x, y \rangle = 0, \forall y \in \sigma\} \subset M_{\mathbb{R}}$, and σ° (resp. $(\sigma^\vee)^\circ$) as the relative interior of σ (resp. σ^\vee).

- (4) A face of a cone σ is the subset $H_m \cap \sigma$ for some $m \in \sigma^\vee$ and $H_m = m^\perp$. We use $\sigma(r)$ to denote the collection of r -dimensional faces of σ .
- (5) A fan Σ in $N_{\mathbb{R}}$ is a finite collection of strongly convex rational polyhedral cones $\sigma \subset N_{\mathbb{R}}$, such that (a) if $\sigma \in \Sigma$ then any face of σ is in Σ , and (b) if σ_1, σ_2 are cones in Σ then $\sigma_1 \cap \sigma_2$ is a face in both σ_1 and σ_2 . We use $\Sigma(r)$ to denote the collection of r -dimensional cones in Σ .
- (6) A fan Σ in $N_{\mathbb{R}}$ is *complete*, if its support $|\Sigma| := \cup_{\sigma \in \Sigma} \sigma$ is the entire $N_{\mathbb{R}}$. A fan Σ is *simplicial* (resp. *smooth*), if each maximal cone $\sigma \in \Sigma(n)$ is simplicial (resp. smooth).

Definition 1.2. A smooth stacky fan $(\Sigma, \{v_\rho\})$ is a simplicial fan Σ together with a choice of non-zero lattice vector $v_\rho \in \rho$ for each ray $\rho \in \Sigma(1)$. We sometimes abbreviate $(\Sigma, \{v_\rho\})$ as Σ .

Let Σ be a smooth stacky fan. We recall the definition of the (non-equivariant) FLTZ skeleton

$$\Lambda_\Sigma := \bigcup_{\sigma \in \Sigma} \Lambda_\sigma, \quad \text{where} \quad \Lambda_\sigma := M_{\mathbb{T}}^\sigma \times (-\sigma), \quad M_{\mathbb{T}}^\sigma := \{\theta \in M_{\mathbb{T}} \mid \langle \theta, v_\rho \rangle = 0, \forall \rho \in \sigma(1)\}.$$

Thanks to σ being simplicial, if σ is k -dimensional, then $M_{\mathbb{T}}^\sigma$ is $(n - k)$ -dimensional.

Let $p : M_{\mathbb{R}} \rightarrow M_{\mathbb{T}}$ be the quotient map, which is a covering map with fiber M . Then p induces a covering map on cotangent bundle, which we also denoted as $p : T^*M_{\mathbb{R}} \rightarrow T^*M_{\mathbb{T}}$. Then we define the equivariant FLTZ skeleton $\tilde{\Lambda}_{\Sigma}$ as

$$\tilde{\Lambda}_{\Sigma} = p^{-1}(\Lambda_{\Sigma})$$

The smooth DM toric stack for Σ is denoted by \mathcal{X}_{Σ} . Then non-equivariant and equivariant CCC says

$$\tau : Coh(\mathcal{X}_{\Sigma}) \simeq Sh^w(M_{\mathbb{T}}, \Lambda_{\Sigma}), \quad \text{and } \tau_{eq} : Coh_{(\mathbb{C}^*)^n}(\mathcal{X}_{\Sigma}) \simeq Sh^w(M_{\mathbb{R}}, \tilde{\Lambda}_{\Sigma}).$$

2. VARIATION OF LAGRANGIAN SKELETONS

Let M be a real analytic manifold, $\Lambda \subset T^*M$ a conical Lagrangian. We assume Λ always contains the zero-section $T^*_M M$, hence Λ is determined by its Legendrian boundary at infinity $\Lambda^{\infty} \subset T^{\infty}M$.

In this section, we are going to consider a family of Lagrangians $\{\Lambda_t\}_{t \in \mathbb{R}}$ and the corresponding wrapped (constructible) sheaves supported on it $Sh^w(M, \Lambda_t)$. We are interested in how $Sh^w(M, \Lambda_t)$ varies as Λ_t moves. If $Sh^w(M, \Lambda_t)$ remains invariant, we say the variation of Λ_t is **non-characteristic**, following Nadler [Nad15]. For example, Nadler has shown that there exists non-characteristic deformation from arbitrary Legendrian singularities into certain particular nice ones called arboreal singularities.

It is convenient to put the family of skeletons into one big skeleton $\Lambda_{\mathbb{R}} \subset M \times \mathbb{R}$, such that the restriction of $\Lambda_{\mathbb{R}}$ to the slice $M \times \{t\}$ equals Λ_t , and consider the categories $Sh_{\Lambda_{\mathbb{R}}}^w(M \times J)$ for $J \subset \mathbb{R}$ open. This brings us to consider the following more general problem, that given $\Lambda \subset T^*Z$ a skeleton, how will $Sh_{\Lambda}^w(U)$, or equivalently $Sh_{\Lambda}^{\diamond}(U)$, change as we vary the open subsets $U \subset Z$.

2.1. Singular Support for Categories.

Definition 2.1 (Singular Supports). *Let Λ be a conical Lagrangian in T^*M . We define the **singular support** $SS(Sh_{\Lambda}^{\diamond}) \subset T^*M$ of the (co)sheaf of categories Sh_{Λ}^{\diamond} as follows. A point $(x, \xi) \in T^*M$ is not in $SS(Sh_{\Lambda}^{\diamond})$, if for any open neighborhood U of x , there exists a smaller neighborhood B of x and smooth function $f : B \rightarrow \mathbb{R}$ such that $f(x) = 0$, $df(x) = \xi$ and the natural restriction*

$$\rho_f : Sh_{\Lambda}^{\diamond}(\{f(x) < \epsilon\}) \rightarrow Sh_{\Lambda}^{\diamond}(\{f(x) < -\epsilon\})$$

is an equivalence. We also abbreviate $SS(Sh_{\Lambda}^{\diamond})$ to $SS(\Lambda)$.

Recall that a functor $F : \mathcal{C} \rightarrow \mathcal{D}$ is an equivalence if and only if F is fully-faithful and essentially surjective. And if F admits a left adjoint, we have the following sufficient condition for essential surjectivity.

Lemma 2.2. *Let $F^L : \mathcal{D} \rightarrow \mathcal{C}$ denote the left adjoint of F . If F^L is fully-faithful, then F is essentially surjective.*

Proof. Let X be an object in \mathcal{D} , by adjunction we have morphism $X \rightarrow FF^L X$ in \mathcal{D} . Since for any object Y in \mathcal{D} , we have

$$\text{Hom}(Y, X) \simeq \text{Hom}(F^L Y, F^L X) \simeq \text{Hom}(Y, FF^L X)$$

induced by the morphism $X \rightarrow FF^L X$, we see X is quasi-isomorphic to $FF^L X$, hence F is essentially surjective. \square

Definition 2.3. *As in the setup of Definition 2.1, we define $SS_{Hom}(\Lambda)$ (resp. SS_{Hom}^L) in the same way as $SS(\Lambda)$, except we replace the condition that " ρ_f is an equivalence" by " ρ_f is fully-faithful" (resp. " ρ_f^L is fully-faithful").*

Proposition 2.4.

$$SS(\Lambda) = SS_{Hom}(\Lambda) \cup SS_{Hom}^L(\Lambda)$$

Proof. Since if a functor is an equivalence, then its left-adjoint is also an equivalence. Thus if $(x, \xi) \notin SS(\Lambda)$, then $(x, \xi) \notin SS_{Hom}(\Lambda)$ and $(x, \xi) \notin SS_{Hom}^L(\Lambda)$. Hence the inclusion \supset is clear.

On the other hand, if $(x, \xi) \notin SS_{Hom}(\Lambda) \cup SS_{Hom}^L(\Lambda)$, then by definition 2.3, ρ_f is a fully-faithful and ρ_f^L is fully-faithful, hence ρ_f is an equivalence of category by Lemma 2.2, thus $(x, \xi) \notin SS(\Lambda)$. Hence the inclusion \subset is clear. \square

There is another equivalent definition of SS_{Hom} .

Definition 2.5. Let Λ_1, Λ_2 be two conical Lagrangian skeleton in T^*M , then

$$\mathcal{H}om(\Lambda_1, \Lambda_2) = \bigcup \{SS(\mathcal{H}om(F, G)) \mid F \in Sh_{\Lambda_1}^\diamond(U), G \in Sh_{\Lambda_2}^\diamond(U), U \text{ open in } M\}.$$

Lemma 2.6.

$$SS_{Hom}(\Lambda) = \mathcal{H}om(\Lambda, \Lambda)$$

Proof. Recall $SS_{Hom}(\Lambda)$ measures the failure of fully-faithfulness of the restriction functor, and $\mathcal{H}om(\Lambda, \Lambda)$ measures the failure of the invariance of local section of hom sheaf between any two local objects, hence the two Lagrangian skeleton are the same. \square

Let M be a compact manifold, B any manifold, and $\pi : M \times B \rightarrow B$ be the projection, and $\Lambda \subset T^*(M \times B)$ be a conical Lagrangian. Then we may define a sheaf of categories on B by pushing forward Sh_Λ^\diamond on $M \times B$,

$$\pi_* Sh_\Lambda^\diamond : U \mapsto Sh_{\Lambda_B}^\diamond(\pi^{-1}(U)).$$

Let $SS(\pi_* Sh_\Lambda^\diamond)$ be defined in the same as in Definition 2.1. We have the following standard estimate

$$SS(\pi_* Sh_\Lambda^\diamond) \subset \pi_* SS(Sh_\Lambda^\diamond) := \{(b, \beta) \in T^*B \mid \exists(x, \xi; b, \beta) \in SS(Sh_\Lambda^\diamond), \xi = 0\}$$

2.2. Singular Support for Hom Sheaf. First, we recall two results from [KS13] about upper bound of singular support of hom sheaf.

Lemma 2.7 ([KS13], Proposition 5.4.14). Let F, G are constructible sheaves on M . If $SS^\infty(F) \cap SS^\infty(G) = \emptyset$, then we have

$$SS(\mathcal{H}om(F, G)) \subset SS(G) + SS(F)^a,$$

where $(-)^a : (x, \xi) \mapsto (x, -\xi)$ is the anti-podal map on T^*M and $+$ is the fiberwise sum in T^*M .

Lemma 2.8 ([KS13], Corollary 6.4.5, Remark 6.2.8). Let F, G are constructible sheaves on M . Then we have

$$SS(\mathcal{H}om(F, G)) \subset SS(G) \hat{+} SS(F)^a$$

where $(x_o, \xi_o) \in SS(G) \hat{+} SS(F)^a$ if and only if there exists sequences $\{(x_n, \xi_n)\}$ in $SS(G)$ and $\{(y_n, \eta_n)\}$ in $SS(F)$, such that as $n \rightarrow \infty$, we have (i) $x_n \rightarrow x_o$, $y_n \rightarrow x_o$, (ii) $\xi_n - \eta_n \rightarrow \xi_o$, and (iii) $|x_n - y_n| |\xi_n| \rightarrow 0$

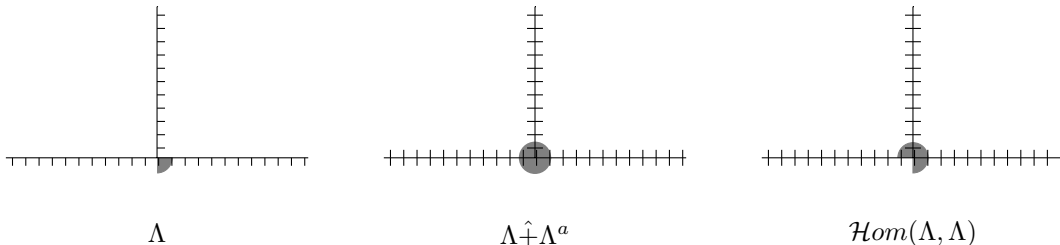
Immediately, we get

Corollary 2.9.

$$\mathcal{H}om(\Lambda, \Lambda) \subset \Lambda \hat{+} \Lambda^a$$

However, this bounded is not sharp.

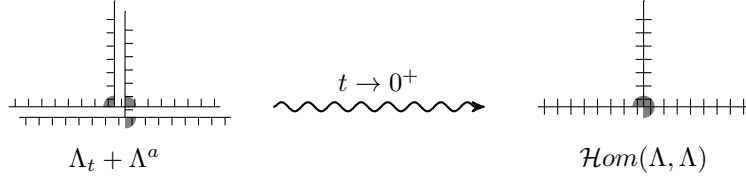
Example 2.10. Consider the following Lagrangian skeleton. We have $\mathcal{H}om(\Lambda, \Lambda) \subsetneq \Lambda \hat{+} \Lambda^a$.



where the shaded sector means the corresponding direction in the cotangent fiber $T_0^*\mathbb{R}^2$. \triangle

The idea to improve is to perturb G by a positive flow (e.g. a geodesic flow), so that $SS(G)^\infty \cap SS(F) = \emptyset$, then apply Lemma 2.7.

Example 2.11. Consider again the setup in Example 2.10. The translation of Λ^∞ in the direction $(1, -1)$ is a positive isotopy of Legendrian.



\triangle

Let $I = (0, \epsilon)$ for some $\epsilon > 0$. Consider the open immersion and closed embedding

$$j : M \times (0, \epsilon) \hookrightarrow M \times [0, \epsilon] \hookrightarrow M \times \{0\} : i.$$

We define the **boundary value functor** to be

$$\lim_{t \rightarrow 0^+} = i^{-1}j_* : Sh(M \times I) \rightarrow Sh(M), \quad F_I \mapsto F_0 = i^{-1}j_*(F_I).$$

Consider space $M_1 \times M_2 \times I$, where $M_i = M$ and the subscript is only for bookkeeping sake, and π_{M_i} or $\pi_{M_1 \times I}$ the corresponding projection.

Definition 2.12. Let $A_I \subset M_1 \times M_2 \times I$ be a sub-analytic closed subset, $A_t := A_I \cap \pi_I^{-1}(t)$. Let $K = \mathbb{C}_{A_I}$ and $K_t = K|_{\pi_I^{-1}(t)}$. We say K is a **(upper-shriek) kernel for positive isotopy** if the following is true

- (1) For any $x \in M, t \in (0, \epsilon), i = 1, 2, \pi_{M_i \times I}^{-1}(x, t) \cap A_I$ is a compact contractible set containing x .
- (2) For $0 < t < s < \epsilon$, we have $A_t \subset A_s$.
- (3) The intersection $A_0 := \bigcap_{0 < t < \epsilon} A_t$ is the diagonal subset $\Delta_M \subset M \times M$.
- (4) If $(x, \xi; y, \eta; t, \tau) \in SS(K)$ and is not contained in the zero section, then

$$\tau \geq 0, \quad \xi \neq 0, \quad \eta \neq 0.$$

The kernel of positive isotopy acts as follows (see [STW16] Appendix for a summary of the functorial operation of sheaves)

$$K^! : Sh(M_1) \rightarrow Sh(M_2 \times I), \quad K^!G := (\pi_{M_2 \times I})_* \mathcal{H}om(K, \pi_{M_1}^!G)$$

and

$$K_t^! : Sh(M_1) \rightarrow Sh(M_2), \quad K_t^!G := (\pi_{M_2})_* \mathcal{H}om(K_t, \pi_{M_1}^!G)$$

We have

$$G_t := K_t^!G = (K^!G)|_{M_2 \times \{t\}}.$$

We call G_t a **positive isotopy of G** . And G is the boundary value of G_I by construction

$$G = \lim_{t \rightarrow 0^+} G_t.$$

For $0 < t < s < \epsilon$, we have canonical morphisms

$$\rho_{t,s} : G_t \rightarrow G_s$$

called **continuation morphisms** and is compatible with composition. Hence we have a projective system of sheaves.

Proposition 2.13. If $\{G_t\}_{0 < t < \epsilon}$ is a positive isotopy of sheaves for $G =: G_0$, then the followings are true

(1) The projective limit of $\{G_t\}_{0 < t < \epsilon}$ in $Sh^\diamond(M)$ exists and is G ,

$$G = \varprojlim G_t,$$

(2) For any constructible sheaf $F \in Sh^\diamond(M)$, we have projective limit of the projective system $\varprojlim \mathcal{H}om(F, G_t)$,

$$\mathcal{H}om(F, G) = \varprojlim \mathcal{H}om(F, G_t)$$

Proof. For (1), we observe that $K_0 = \mathbb{C}_{\Delta_M}$ is the inductive limit of K_t , $\mathcal{H}om(K_t, \pi_{M_1}^! G)$ is the projective limit of G , and finally π_{M_2*} preserves projective limit. (2) follows from (1), since hom and internal hom preserves projective limit. \square

We can use the projective limit to get an upper bound of the singular support.

Proposition 2.14. *If $G = \varprojlim G_t$ is the projective limit of a family of sheaves G_t , then we have*

$$SS(G) \subset \limsup_{t \rightarrow 0^+} SS(G_t) := \bigcap_{t_0 > 0} \overline{\bigcup_{0 < t < t_0} SS(G_t)}$$

Proof. Suppose (x, ξ) is not in $\limsup_{t \rightarrow 0^+} SS(G_t)$, then there exists a conic open set U , and a $t_0 > 0$, such that for all $t < t_0$ U is disjoint from $SS(G_t)$. Hence we may choose an open ball B around x , such that $B \subset \pi(U)$, and $f : B \rightarrow \mathbb{R}$, with $f(x) = 0$, $df(x) = \xi$ and $\Gamma_{df} \subset U$. For small enough $\delta > 0$, we consider the cone $P := \mathbb{C}_{\{f(x) < -\delta\}} \rightarrow \mathbb{C}_{\{f(x) < \delta\}}$, then we have $\mathcal{H}om(P, G_t) = 0$ for $t < t_0$. Since $\mathcal{H}om(P, -)$ commutes with projective limit, we have $\mathcal{H}om(P, G) = 0$. This shows (x, ξ) is not in $SS(G)$. \square

Theorem 2.15. *Let F, G be constructible sheaves over M and $\{G_t\}_{t \in (0, \epsilon)}$ a positive isotopy of G . Then*

$$SS(\mathcal{H}om(F, G)) \subset \limsup_{t \rightarrow 0^+} SS(\mathcal{H}om(F, G_t)).$$

In particular, if $SS^\infty(F) \cap SS^\infty(G_t) = \emptyset$ for $t \in (0, \epsilon)$ then

$$SS(\mathcal{H}om(F, G)) \subset \limsup_{t \rightarrow 0^+} SS(F)^a + SS(G_t).$$

Proof. From Proposition 2.13, we see $\mathcal{H}om(F, G)$ is the projective limit of $\mathcal{H}om(F, G_t)$. Then the conclusion follows from Proposition 2.14. The special case follows from Lemma 2.7. \square

2.3. Left non-characteristic condition. Here we introduce a technical condition that ensures $\mathcal{H}om(F_t, G)$ to be invariant for an isotopy of sheaves $\{F_t\}$, even if $SS^\infty(F_t) \cap SS^\infty(G) \neq \emptyset$ for some t .

First, we recall what is an isotopy of sheaves. Let $I \subset \mathbb{R}$ be an open interval, M be a compact manifold, F_I a constructible sheaf on $M \times I$. We say F_I is an **isotopy of sheaves** if $SS^\infty(F_I) \cap SS^\infty(\mathbb{C}_{M \times \{t\}}) = \emptyset$ for all $t \in I$.

Definition 2.16. *Let F_I be an isotopy of sheaves, G a constructible sheaf on M . We say F_I is left non-characteristic (LNC) with respect to G , if there exists a positive isotopy of G , $\{G_s\}$, $s \in (0, \epsilon)$, such that*

$$SS^\infty(F_t) \cap SS^\infty(G_s) = \emptyset, \forall t \in I, s \in (0, \epsilon).$$

Proposition 2.17. *As in the above setup, if F_I is LNC with respect to G , then $\mathcal{H}om(F_t, G)$ is invariant for $t \in I$.*

Proof. By non-characteristic deformation lemma (see e.g. [1]), we have $\mathcal{H}om(F_t, G_s)$ to be invariant (up to quasi-isomorphism) for all $t \in I$ and $s \in (0, \epsilon)$.

Then, we have $\mathcal{H}om(F_t, G) = \varprojlim_s \mathcal{H}om(F_t, G_s) = \mathcal{H}om(F_t, G_{s_0})$ for some fixed s_0 . Hence $\mathcal{H}om(F_t, G)$ are invariant. \square

3. THE WINDOW SKELETONS

In this section, we will apply CCC and describe skeletons mirror to $\text{Coh}(\mathcal{X})$, $\text{Coh}(\mathcal{X}_\pm)$ and \mathcal{B}_W . They will be called **full** skeleton, positive (or negative) **cylindrical** skeleton and **window** skeleton, respectively.

3.1. The full skeleton and the GIT quotient skeleton. First, we recall the fan Σ_N for \mathbb{C}^N . Σ_N consist of all the faces of the closed positive quadrant $\mathbb{R}_{\geq 0}^N$

$$\Sigma_N = \{\sigma_I \mid I \subset [N]\}, \quad \sigma_I = \text{cone}(e_i^\vee, i \in I).$$

Consider the 1-PS $\mathbb{C}^* \hookrightarrow (\mathbb{C}^*)^N$ given by $a \in (\mathbb{Z}^N)^\vee$. It induces $\mathbb{Z}^\vee \hookrightarrow (\mathbb{Z}^n)^\vee$ and $\mu_{\mathbb{Z}} : \mathbb{Z}^N \rightarrow \mathbb{Z}$, and we define M and N by

$$\begin{aligned} 0 \rightarrow \mathbb{Z} \xrightarrow{a} (\mathbb{Z}^N)^\vee \xrightarrow{q_{\mathbb{Z}}} \mathbb{N} \rightarrow 0, \\ 0 \rightarrow M \rightarrow \mathbb{Z}^N \xrightarrow{\mu_{\mathbb{Z}}} \mathbb{Z} \rightarrow 0. \end{aligned}$$

Since a is a primitive vector, N and M are free abelian group. We denote $N_A = N \otimes_{\mathbb{Z}} A$ and similarly for M_A , for abelian group $A = \mathbb{R}, \mathbb{C}^*, \mathbb{T} = \mathbb{R}/\mathbb{Z}$.

We define sub-fans $\Sigma_N^\pm \subset \Sigma_N$:

$$\Sigma_N^\pm := \{\sigma_I \mid I \subset [N], \text{ s. t. } [N]_\pm \not\subset I\}.$$

Consider the quotient map

$$q_{\mathbb{R}} : (\mathbb{R}^N)^\vee \rightarrow N_{\mathbb{R}}$$

Define the image of $(\Sigma_N^\pm, \{e_i^\vee : \sigma_{\{i\}} \in \Sigma_N^\pm\})$ under the quotient by $(\Sigma_\pm, \{v_i\})$. This is a smooth stacky fan.

Let \mathcal{X}_\pm denote the $(N-1)$ -dimensional smooth toric DM stack for Σ_\pm . And let $\tilde{\mathcal{X}}_\pm = (\mathbb{C}^N)_{\pm}^{ss} \subset \mathbb{C}^N$ denote the smooth open subvariety for Σ_N^\pm .

Consider the torus fibration

$$M_{\mathbb{T}} \rightarrow \mathbb{R}^N/M \xrightarrow{\mu} \mathbb{R}.$$

Denote

$$\mathcal{Y} = \mathbb{R}^N/M, \quad \mathcal{Y}_t = \mu^{-1}(t).$$

For $t \in \mathbb{Z}$, $\mathcal{Y}_t \simeq M_{\mathbb{T}}$ canonically.

The $(\mathbb{C}^*)^N$ -equivariant skeleton for \mathbb{C}^N is $\Lambda_N \subset T^*\mathbb{R}^N$

$$\Lambda_N = \bigcup_{\sigma_I \in \Sigma_N} (\mathbb{Z}^N + \sigma_I^\perp) \times (-\sigma_I) \subset T^*\mathbb{R}^N.$$

The \mathbb{C}^* -equivariant skeleton for \mathbb{C}^N is the skeleton for $\mathcal{X} = [\mathbb{C}^N/\mathbb{C}^*]$,

$$\Lambda_N/M = \bigcup_{\sigma_I \in \Sigma_N} [(\mathbb{Z}^N + \sigma_I^\perp)/M] \times (-\sigma_I) \subset T^*Y.$$

The \mathbb{C}^* -equivariant skeleton for $\tilde{\mathcal{X}}_\pm$ is the skeleton for $\mathcal{X}_\pm = [\tilde{\mathcal{X}}_\pm/\mathbb{C}^*]$,

$$\Lambda_N^\pm/M = \bigcup_{\sigma_I \in \Sigma_N^\pm} [(\mathbb{Z}^N + \sigma_I^\perp)/M] \times (-\sigma_I) \subset T^*Y.$$

Equivalently, we can consider the non-equivariant skeleton for the stacky fan Σ_\pm

$$\Lambda_\pm := \Lambda_{\Sigma_\pm} = \bigcup_{\sigma \in \Sigma_\pm} M_{\mathbb{T}}^\sigma \times (-\sigma) \subset T^*M_{\mathbb{T}}.$$

Then Λ_\pm is the reduction of Λ_N^\pm/M on \mathcal{Y}_t for $t \in \mathbb{Z}$.

We call Λ_N/M the **full skeleton**, Λ_N^\pm/M the **cylindrical GIT quotient skeleton** and Λ_\pm simply the **GIT quotient skeleton**.

3.2. The Window Skeleton: Periodic region. Let W be a window, i.e., a set of consecutive integers. We define the window skeleton as a union of singular supports of mirror to the objects in the B-model mirror.

Under equivariant CCC, a $(\mathbb{C}^*)^N$ -equivariant line bundle \mathcal{L}_k on \mathbb{C}^N , i.e. $(\mathbb{C}^*)^N$ acts on the fiber over 0 by weight k , goes to a constructible sheaf Q_{-k} , where

$$Q_v = \mathbb{C}_{v+(\mathbb{R}_{>0})^N}$$

the constant sheaf supported on the shifted open quadrant.

Hence, we define a subset of \mathbb{Z}^N by

$$\mathbb{Z}^N(W) := \mu_{\mathbb{Z}}^{-1}(-W).$$

Then we define the window skeleton upstairs

$$\tilde{\Lambda}_W = \bigcup_{v \in \mathbb{Z}^N(W)} SS(Q_v), \quad \Lambda_W = \tilde{\Lambda}_W / \mathbf{M}.$$

Theorem 3.1. *Let W be a window and $-W = \{a, a+1, \dots, b\}$.*

(1) *If $|W| \geq \eta_+$, then*

$$\tilde{\Lambda}_W|_{(b, \infty)} = \Lambda_N^+|_{(b, \infty)}$$

(2) *If $|W| \geq \eta_-$, then*

$$\tilde{\Lambda}_W|_{(-\infty, a)} = \Lambda_N^+|_{(-\infty, a)}$$

Proof. We only prove (1) since the proof for (2) is similar. To show $\tilde{\Lambda}_W|_{(b, \infty)} \subset \Lambda_N^+|_{(b, \infty)}$, we only need to show that $SS(Q_v)|_{(b, \infty)} \subset \Lambda_N^+|_{(b, \infty)}$ for all $v \in \mathbb{Z}^N(W)$. We have decomposition

$$SS(Q_v) = \bigcup_{I \subset [N]} (v + \kappa_{I^c}) \times (-\sigma_I)$$

If $[N]_+ \subset I$, then $I^c \subset [N]_-$, hence $\mu(v + \kappa_{I^c}) \leq \mu v \leq b$, thus

$$SS(Q_v)|_{(b, \infty)}|_{(b, \infty)} = \bigcup_{I \subset [N], [N]_+ \not\subset I} (v + \kappa_{I^c}) \times (-\sigma_I).$$

Hence $SS(Q_v)|_{(b, \infty)} \subset \Lambda_N^+|_{(b, \infty)}$.

Next, we show $\tilde{\Lambda}_W|_{(b, \infty)} \supset \Lambda_N^+|_{(b, \infty)}$. We only need to show that, for any $u \in \mathbb{Z}^N$, for any $I \subset [N]$ with $[N]_+ \not\subset I$,

$$\tilde{\Lambda}_W|_{(b, \infty)} \supset (u + \sigma_I^\perp)|_{(b, \infty)} \times (-\sigma_I).$$

Again, we only need to show that for any $x \in (u + \sigma_I^\perp)|_{(b, \infty)}$, there exists a $v \in \mathbb{Z}^N(W)$, such that

$$SS(Q_v) \supset x \times (-\sigma_I). \quad (3.1)$$

Since $x - u \in \sigma_I^\perp$, we have

$$x_i \in \mathbb{Z} \quad \text{if } i \in I.$$

We may write $x_i = [x_i] + \{x_i\}$, where $[x_i] \in \mathbb{Z}$ and $\{x_i\} \in [0, 1)$. Then

$$\mu([x]) = \mu(x) - \mu(\{x\}) > b - \sum_j a_j \{x_j\} = b - \sum_{j \in I^c} a_j \{x_j\} > b - \sum_{j \in I^c \cap [N]_+} a_j \{x_j\} > b - \sum_{j \in [N]_+} a_j = b - \eta_+$$

Since $\mu([x])$ is an integer, we have $\mu([x]) \geq b - \eta_+ + 1$. If $\mu([x]) \leq b$, then we may choose $v = [x]$; otherwise, we may choose $v = [x] - k(\sum_{j \in I^c \cap [N]_+} e_j)$ for certain positive integer k such that $\mu(v) \in [a, b]$. Indeed this is possible, since $I^c \cap [N]_+ \neq \emptyset$, and $\eta_+ \geq \mu(\sum_{j \in I^c \cap [N]_+} e_j) > 0$, hence $\mu([x] - k(\sum_{j \in I^c \cap [N]_+} e_j))$ decrease with step size at most η_+ , hence will eventually fall into the interval $[a, b]$ which contains at least η_+ integers.

Now that we have a $v \in x + (\sigma_I)^\perp$ and $v_i \leq x_i$ for all i , we have Eq 3.1 as desired.

□

3.3. Window Skeleton near a Lattice Point . Let W be a window with $-W = \{a, \dots, b\}$. We now study the window skeleton $\tilde{\Lambda}_W|_{(a-\epsilon, b+\epsilon)}$ for some small ϵ . This is determined by the behavior near a lattice point $v \in \mathbb{Z}^N(W)$.

In general, for any $v \in \mathbb{Z}^N$, we may consider the specialization of $\tilde{\Lambda}_W$ at v

$$\tilde{\Lambda}_{W,v} \subset T^*(T_v \mathbb{R}^N) = \mathbb{R}^N \times (\mathbb{R}^N)^\vee.$$

Since the strata of $\tilde{\Lambda}_W$ is already conical near v , $\tilde{\Lambda}_{W,v}$ can be obtained by first restricting to a small ball centered at v then extend back to \mathbb{R}^N .

We need some notation. For $I \subset [N]$, we define

$$P_I := \{x \in \mathbb{R}^N : x_i > 0, \text{ if } i \notin I\}, \quad \Lambda_I = SS(\mathbb{C}_{P_I}).$$

We call I positive-type (resp. negative-type) if $I \subset I_+$ (resp. $I \subset I_-$), and we call I mixed-type, if I intersects with both $[N]_\pm$ nontrivially.

If $\Lambda_I \subset \tilde{\Lambda}_{W,v}$, we say Λ_I **appears in** $\tilde{\Lambda}_{W,v}$. We denote the set of I where Λ_I appears in $\tilde{\Lambda}_{W,v}$ as $\mathcal{I}_{W,v}$.

Proposition 3.2. Λ_I appears in $\tilde{\Lambda}_{W,v}$ if and only if there exists a vertex $w \in \mathbb{Z}^N(W)$, such that

$$\begin{cases} w_i < v_i & i \in I \\ w_i = v_i & i \notin I \end{cases}$$

Proof. Since $\tilde{\Lambda}_W$ is a union of Λ_w for each $w \in \mathbb{Z}^N(W)$. And only if $w \leq v$ can Λ_w contribute to $\Lambda_w|_v$. For Λ_w contribute as Λ_I , we need the quadrant Q_w to equal to Q_I near v , hence the conditions on w_i . \square

Lemma 3.3. Let $v \in \mathbb{Z}^N(W)$ and $I \subset [N]$. If I is of mixed type, or if $I = \emptyset$, then Λ_I appears in $\tilde{\Lambda}_{W,v}$.

Proof. By previous proposition, it suffices to choose w with stated property. If $I = \emptyset$, then we may choose $w = v$. If I is of mixed type, then it suffices to find positive integers c_i for all $i \in I$, such that $\sum_{i \in I} a_i c_i = 0$. It is easy to see such solution exists. Then, we may choose $w = v - \sum_{i \in I} c_i e_i$. \square

Lemma 3.4. Let $v \in \mathbb{Z}^N(W)$ and $I \subset [N]$. If I is of positive-type or negative-type, then Λ_I appears in $\tilde{\Lambda}_{W,v}$ if and only if the vertex $w = v - \sum_{i \in I} a_i$ is in $\mathbb{Z}^N(W)$.

Proof. From Proposition 3.2, we may verify the 'if' part of the statement using the w given. For the 'only if' part, suppose Λ_I appears in $\tilde{\Lambda}_{W,v}$, then there is $w' = v - \sum_{i \in I} a_i c_i$ for positive integers c_i , and $\mu(w') \in -W$. If I is of positive-type, we have $\mu(v) > \mu(w) \geq \mu(w')$; and if I is of negative-type, we have $\mu(v) < \mu(w) \leq \mu(w')$, in either case we have $\mu(w)$ sandwiched between two integers in a set of consecutive integers $-W$, hence $\mu(w) \in -W$. \square

Lemma 3.5. Let $v \in \mathbb{Z}^N$ and $I \subset [N]$.

- (1) If $|W| \geq \eta_+$ and $\mu(v) > \max(-W)$, then Λ_I appears in $\tilde{\Lambda}_{W,v}$ if and only if I is of mixed-type or I is of positive-type.
- (2) If $|W| \geq \eta_-$ and $\mu(v) < \min(-W)$, then Λ_I appears in $\tilde{\Lambda}_{W,v}$ if and only if I is of mixed-type or I is of negative-type.

Proof. We only prove the statement (1), the other one is similar.

If I is of mixed-type or positive-type, then we just need to find a $w \in \mathbb{Z}^N(W)$ satisfies the conditions in Proposition 3.2. Suppose $w = v - \sum_{i \in I} c_i e_i$, for positive integers c_i , then $\mu(w) = \mu(v) - \sum_{i \in I} c_i a_i$. We need to show that there exists c_i s such that $\mu(w) \in -W$. Let

$$\alpha_I = \sum_{i \in I} a_i, \quad \beta_I = \min(\mathbb{Z}_{>0} \cap \{\sum_{i \in I} a_i c_i \mid c_i \geq 0\}).$$

Then $\beta_I \leq \eta_+$ and $\alpha_I \leq \eta_+$, and

$$\{\mu(v) - \sum_{i \in I} c_i a_i \mid c_i \in \mathbb{Z}_{>0}\} \cap -W = \{\mu(v) - \alpha_I - n\beta_I \mid n \geq 0\} \cap -W \neq \emptyset.$$

□

4. PROOF OF THEOREM 0.3

So far we have defined the window skeleton Λ_W for a given window W , and considered the category $Sh^w(\mathcal{Y}, \Lambda_W)$. There is another potentially different definition of window category, which is the CCC image of \mathcal{B}_W

$$\mathcal{A}_W := \langle \{\tau(\mathcal{L}_k), k \in W\} \rangle \subset Sh^w(\mathcal{Y}, \Lambda_{\mathcal{Y}}).$$

There is a fully-faithful embedding

$$\iota_W : \mathcal{A}_W \hookrightarrow Sh^w(\mathcal{Y}, \Lambda_W).$$

Our goal in this section is to prove the Theorem 0.3, which says, ι_W is an equivalence if $|W| \geq \eta_-$.

Proof. Let W be a window, $-W = \{a, a+1, \dots, b\}$. Assume $|W| \geq \eta_-$.

For $v \in \mathbb{Z}^N$, let $\mathcal{F}(v)$ denote the probe sheaf for point $v + \epsilon \mathbf{1}$ in $Sh^w(\mathbb{R}^N, \tilde{\Lambda}_W)$. The collection of probe sheaves $C = \{\mathcal{F}(v) \mid v \in \mathbb{Z}^N\}$ classically generates⁵ $Sh^w(\mathbb{R}^N, \tilde{\Lambda}_W)$. Our goal is to show that a sub-collection $C(-W)$ also generates, where we use notation

$$C(J) = \{\mathcal{F}(v) \mid v \in \mathbb{Z}^N, \mu(v) \in J\}, \quad J \subset \mathbb{Z}.$$

We will prove in two steps

- (1) First, we show that $C((-\infty, b] \cap \mathbb{Z})$ is generated by $C(-W)$.
- (2) Next, we show that C is generated by $C((-\infty, b] \cap \mathbb{Z})$.

In both steps, we induct on $\mu(v)$. Our induction hypothesis is that, for $\mu(v) \in [k, b]$ where $k \leq a$, $\mathcal{F}(v)$ is generated by $C(-W)$. For the base case, the hypothesis holds for $k = a$. We show if it holds for k , then it holds for $k-1$. Let $v \in \mathbb{Z}^N$ with $\mu(v) = k-1$. Consider the microlocal stalk functor for point $(x, \xi) = (v + \epsilon e_{[N]_+}, -e_{[N]_-}^\vee)$, it is co-represented by the following chain of probe sheaves

$$\mathcal{F}(v) \rightarrow \bigoplus_{J \subset [N]_-, |J|=1} \mathcal{F}(v - e_J) \rightarrow \cdots \rightarrow \bigoplus_{J = [N]_-} \mathcal{F}(v - e_J).$$

Since the point $(x, \xi) \notin \tilde{\Lambda}_W$, the above chain complex is acyclic. Hence $\mathcal{F}(v)$ is generated by $\mathcal{F}(v - e_J)$ for $\emptyset \neq J \subset [N]_-$. The μ -weight of the generators are

$$\mu(v - e_J) = \mu(v) - \sum_{j \in J} a_j e_j = \mu(v) + \sum_{j \in J} |a_j| e_j$$

Thus $\mu(v - e_J) \in [k, k-1 + \eta_-] \subset [k, b]$, hence $\mathcal{F}(v)$ is generated by $C([k, b] \cap \mathbb{Z})$ and in turn is generated by $C(-W)$.

The second step is similar. Our induction hypothesis is that, for $k \geq b$, $C((-\infty, k] \cap \mathbb{Z})$ is generated by $C((-\infty, b] \cap \mathbb{Z})$. It holds for the base case $k = b$. Suppose it holds for k , we now consider $\mu(v) = k+1$. By similar argument, we have acyclic chain complex

$$\mathcal{F}(v) \rightarrow \bigoplus_{J \subset [N]_+, |J|=1} \mathcal{F}(v - e_J) \rightarrow \cdots \rightarrow \bigoplus_{J = [N]_+} \mathcal{F}(v - e_J).$$

Hence $\mathcal{F}(v)$ is generated by $\mathcal{F}(v - e_J)$ with $\emptyset \neq J \subset [N]_+$. Since for such J , we have

$$\mu(v - e_J) \in \mu(v) + [-\eta_+, -1],$$

⁵In what follows, we will simply say ‘generate’ instead of ‘classically generate’, not to be confused with ‘compactly generate’.

$\mathcal{F}(v)$ is generated by $C((-\infty, k])$, hence by $C((-\infty, b] \cap \mathbb{Z})$. \square

5. FULLY-FAITHFULNESS OF RESTRICTION FUNCTOR

Let W be a window, $\Lambda_W \subset T^*Y$ be the window skeleton, and let $\pi : Y \rightarrow \mathbb{R}$ be the torus fibration with fiber $M_{\mathbb{T}}$. If I is an open interval, we define

$$\mathcal{C}(I) := Sh_{\Lambda_W}^{\diamond}(\pi^{-1}(I)), \mathcal{C}^w(I) := Sh_{\Lambda_W}^w(\pi^{-1}(I)).$$

If $I \subset J$ are two intervals, we have restriction functor $\rho_{I,J}$, its left adjoint $\rho_{I,J}^L$ and the restriction $\rho_{I,J}^w$ to compact objects:

$$\rho_{I,J} : \mathcal{C}(J) \rightarrow \mathcal{C}(I), \quad \rho_{I,J}^L : \mathcal{C}(I) \rightarrow \mathcal{C}(J), \quad \rho_{I,J}^w : \mathcal{C}^w(I) \rightarrow \mathcal{C}^w(J).$$

In this section, we shall study when the restriction functor $\rho_{I,J}$ is fully-faithful. In the next section, we shall study when the co-restriction functor $\rho_{I,J}^L$ (or equivalently $\rho_{I,J}^w$) is fully-faithful.

5.1. Hom-Sheaf's Singular Support. We shall use Theorem 2.15 to obtain an upper bound.

First, we define a kernel to create positive isotopy. Since the space is \mathbb{R}^N and the skeleton Λ_N is rectilinear, we can use a convolution kernel

$$K = \mathbb{C}_{\{(x,y,t) | t \in (0,1), x,y \in \mathbb{R}^n, |x_i - y_i| \leq t\}}.$$

If $F \in Sh^{\diamond}(\mathbb{R}^N, \Lambda_N)$, then $K_t^! F = (T_t)_* F$ (the dimension 1 case is clear), where $T_t : \mathbb{R}^N \rightarrow \mathbb{R}^N$ is translation by $-t(1, \dots, 1)$. If $\Lambda \subset \Lambda_N$, then let $T_t \Lambda$ denote the positively translated skeleton.

Since the conormal to the boundary of the open set $\pi^{-1}((a, b))$ are in the direction $\pm a$, hence we are only interested in whether $(x, \pm a) \in \mathcal{H}om(\tilde{\Lambda}_W, \tilde{\Lambda}_W)$. Since a is not conormal to any coordinate planes of any positive dimension, hence we only need to look at the lattice points.

Recall the notation of Λ_I from section 3.3.

Lemma 5.1. *Let $v \in \mathbb{Z}^N$, and $\tilde{\Lambda}_{W,v}$ be the specialization of $\tilde{\Lambda}_W$ at v . Then*

- (1) $a \in \mathcal{H}om(\tilde{\Lambda}_W, \tilde{\Lambda}_W)_v$ only if there exists $I, J \subset [N]$, such that Λ_I, Λ_J appears in $\tilde{\Lambda}_{W,v}$ and $I = [N]_-, J \subset [N]_+$.
- (2) Dually, $-a \in \mathcal{H}om(\tilde{\Lambda}_W, \tilde{\Lambda}_W)_v$ only if there exists $I, J \subset [N]$, such that Λ_I, Λ_J appears in $\tilde{\Lambda}_{W,v}$ and $I = [N]_+, J \subset [N]_-$.

Proof. We only prove the statement about $+a$. Let $\mathcal{I}_{W,v}$ denote the collection of subset $I \subset [N]$, such that $\Lambda_I \subset \tilde{\Lambda}_{W,v}$. Then we have

$$\tilde{\Lambda}_{W,v} = \bigcup_{I \in \mathcal{I}_{W,v}} \Lambda_I.$$

Hence

$$\mathcal{H}om(\tilde{\Lambda}_{W,v}, \tilde{\Lambda}_{W,v}) \subset \bigcup_{I, J \in \mathcal{I}_{W,v}} \lim_{t \rightarrow 0^+} [(\Lambda_I)^a + T_t \Lambda_J]. \quad (5.1)$$

We are interested in which pair I, J can produce a co-vector a . Recall that $T_t \Lambda_J$ is the outward conormal to the open set

$$T_t P_J = \{x \in \mathbb{R}^N : x_i > -t \forall i \in J^c\},$$

and Λ_I^a is the inward conormal to

$$P_I = \{x \in \mathbb{R}^N : x_i > 0 \forall i \in I^c\}.$$

Hence for $(\Lambda_I)^a + T_t \Lambda_J$ to contain a covector that has each component nonzero, we need to have $T_t P_J \cap P_I$ to be a vertex corner, i.e. $I^c \cup J^c = [N]$. The corner \mathbf{x} has coordinate $x_i = 0$ if $i \in I^c$ and $x_i = -t$ if $x \in I \cap J^c$. The fiber of $(\Lambda_I)^a + T_t \Lambda_J$ over \mathbf{x} is then $\sigma_{I^c} - \sigma_I$, hence it can contain a if and only if $I^c = [N]_+$ and $I = [N]_-$.

Hence, if $a \in \mathcal{H}om(\tilde{\Lambda}_W, \tilde{\Lambda}_W)_v$, we have $I, J \subset \mathcal{I}_{W,v}$, with $I = [N]_-$ and $J \cap I = \emptyset$, thus $J \subset [N]_+$. \square

We conclude this section with the following theorem.

Theorem 5.2. *Let W be a window.*

- (1) $(v, +a) \in \mathcal{H}om(\tilde{\Lambda}_W, \tilde{\Lambda}_W)$ if and only if $v \in \mathbb{Z}^N(W)$ and $\mu(v) + \eta_- \in -W$.
- (2) If $(v, -a) \in \mathcal{H}om(\tilde{\Lambda}_W, \tilde{\Lambda}_W)$, then $v \in \mathbb{Z}^N(W)$ and $\mu(v) - \eta_+ \in -W$.

Proof. We prove only (1).

The ‘only if’ part. By assumption there exists $I, J \in \mathcal{I}_{W,v}$ and $I = [N]_-$ $J \subset [N]_+$. If $v \notin \mathbb{Z}^N(W)$, then by Lemma 3.5, even if we enlarge W so that $|W|$ is large enough, we cannot have both negative-type and positive-type (or emptyset) appears in $\tilde{\Lambda}_{W,v}$, hence $v \in \mathbb{Z}^N(W)$. Then, by Lemma 3.3, we may take $J = \emptyset$, and by Lemma 3.4 $I = [N]_-$ requires $\mu(v) + \eta_- \in -W$.

The ‘if’ part. Consider $\mathcal{F} = Q_v$, $\mathcal{G} = Q_{v-e_{[N]_-}}$, by assumption $SS(\mathcal{F}), SS(\mathcal{G}) \subset \tilde{\Lambda}_W$. Then a direct calculation verifies that

$$\mathcal{H}om(\mathcal{G}, \mathcal{F}) = (\boxtimes_{i \in [N]_+} \mathbb{C}_{\{x \geq v_i\}}) \boxtimes (\boxtimes_{i \in [N]_-} \mathbb{C}_{\{x > v_i\}}).$$

Hence the singular support is a product form, and its fiber over v is

$$SS(\mathcal{H}om(\mathcal{G}, \mathcal{F}))_v = \sigma_{[N]_+} \times (-\sigma_{[N]_-})$$

Hence $a \in SS(\mathcal{H}om(\mathcal{G}, \mathcal{F}))_v \subset \mathcal{H}om(\tilde{\Lambda}_W, \tilde{\Lambda}_W)_v$. □

6. FULLY-FAITHFULNESS FOR THE CO-RESTRICTION FUNCTOR

We continue with the setup of the last section. Our main result is that,

Theorem 6.1. *If the $|W| \geq \eta_+$, then the co-restriction functor*

$$\rho_{I,J}^L : \mathcal{C}(I) \rightarrow \mathcal{C}(J)$$

is fully-faithful, for any $I \subset J \subset \mathbb{R}$.

More precisely, we have the following proposition.

Proposition 6.2. *Let W be a window, and $v \in \mathbb{Z}^N$. Then we have*

- (1) *If $\mu(v) \in -W$, then the co-vectors $(v, \pm a)$ are not in $SS_{\mathcal{H}om}^L(\tilde{\Lambda}_W)$.*
- (2) *If $\mu(v) > \max(-W)$, and $|W| \geq \eta_+$, then the co-vectors $(v, \pm a)$ are not in $SS_{\mathcal{H}om}^L(\tilde{\Lambda}_W)$.*
- (3) *If $\mu(v) < \min(-W)$, and $|W| \geq \eta_-$, then the co-vectors $(v, \pm a)$ are not in $SS_{\mathcal{H}om}^L(\tilde{\Lambda}_W)$.*

If any of the above three conditions are satisfied, we say that **the window W is thick enough for v** .

We are going to study the probe sheaves locally near each lattice point. To verify a cotangent vector $(v, a) \in T^*\mathbb{R}^N$ is not in $SS_{\mathcal{H}om}^L(\tilde{\Lambda}_W)$, it suffices to check that $a \notin SS_{\mathcal{H}om}^L(\tilde{\Lambda}_{W,v})_0$, where $\tilde{\Lambda}_{W,v}$ is the specialization of the Lagrangian skeleton $\tilde{\Lambda}_W$ at v and lives in $T^*(T_v\mathbb{R}^N)$.

6.1. Stalk-Probe Sheaves Near a Lattice point. The skeleton $\tilde{\Lambda}_W$ is contained in Λ_N . We shall first study the stalk probe sheaf for $\Lambda_{N,v}$. The projection image of $\Lambda_{N,v}$ cut \mathbb{R}^N into 2^N different quadrants, which we label as

$$Q_I = \{x \in \mathbb{R}^N \mid x_i > 0 \text{ if } i \notin I; x_i < 0 \text{ if } i \in I\}, \quad \forall I \subset [N].$$

For example

$$Q_\emptyset = (\mathbb{R}_{>0})^N, \quad Q_{[N]} = (\mathbb{R}_{<0})^N.$$

We also define some ‘wedge’ spaces

$$P_I = \{x \in \mathbb{R}^N \mid x_i > 0 \text{ if } i \notin I\}, \quad \forall I \subset [N].$$

Since if $(x, \xi) \in \Lambda_{N,v}$, then $\xi_i \leq 0$, the stalk at x is quasi-isomorphic to the stalk at $x + \epsilon(1, \dots, 1)$ for any small enough positive ϵ . Hence, for any point $x \in \mathbb{R}^N$, we say $x + 0 \in Q_I$ if $x + \epsilon(1, \dots, 1) \in Q_I$ for any small enough positive ϵ . For example if $x = 0 \in \mathbb{R}^N$, then $x + 0 \in Q_\emptyset$.

For any fixed Q_I , all the point p with $p + 0 \in Q_I$ has the same stalk probe sheaves, hence we will talk about stalk probe sheaves for region Q_I instead for individual points.

Proposition 6.3. *Let $I \subset [N]$. The stalk probe sheaf for region Q_I in category of $Sh^\diamond(\mathbb{R}^N, \Lambda_{N,v})$ is \mathbb{C}_{P_I} .*

Proof. The category $Sh^\diamond(\mathbb{R}^N, \Lambda_{N,v})$ is equivalent to the module category over the poset of $\{J : J \subset [N]\}$, hence the collection of objects $S := \{\mathbb{C}_{P_I} : I \subset [N]\}$ forms a compact generator of $Sh^\diamond(\mathbb{R}^N, \Lambda_{N,v})$. Let p be an interior point of Q_I . Since for any $\mathbb{C}_{P_J} \in S$, we have restriction map that is an isomorphism

$$\text{Hom}(\mathbb{C}_{P_I}, \mathbb{C}_{P_J}) \simeq \mathbb{C}_{P_J}|_p \simeq \begin{cases} \mathbb{C} & I \subset J \\ 0 & \text{else} \end{cases}.$$

Hence $\text{Hom}(\mathbb{C}_{P_I}, -)$ is the stalk functor at p . □

Recall that $\Lambda_I = SS(\mathbb{C}_{P_I})$.

Lemma 6.4. *Let W be a window, $v \in \mathbb{Z}^N$ and $I \subset [N]$. If $\Lambda_I \subset \tilde{\Lambda}_{W,v}$, then the stalk probe sheaf for Q_I is \mathbb{C}_{P_I} .*

Proof. Since $\tilde{\Lambda}_{W,v} \subset \Lambda_{N,v}$, hence $Sh^\diamond(\mathbb{R}^N, \tilde{\Lambda}_{W,v})$ embeds fully-faithfully into $Sh^\diamond(\mathbb{R}^N, \Lambda_{N,v})$. Hence a stalk probe sheaf \mathcal{P} in $Sh^\diamond(\mathbb{R}^N, \Lambda_{N,v})$ with $SS(\mathcal{P}) \subset \tilde{\Lambda}_{W,v}$ is automatically a stalk probe sheaf for $Sh^\diamond(\mathbb{R}^N, \tilde{\Lambda}_{W,v})$ for the same region. □

To see what happens if Λ_I does not appear in $\tilde{\Lambda}_W$, we first consider a simpler situation.

Lemma 6.5. *Consider the following skeleton over \mathbb{R}^m*

$$\Lambda'_{m,0} = \bigcup_{\emptyset \neq I \subset [m]} SS(\mathbb{C}_{P_I}).$$

Then the stalk probe sheaf for Q_\emptyset in $Sh^\diamond(\mathbb{R}^m, \Lambda'_{m,0})$ is

$$\mathcal{F}_\emptyset := \underbrace{\bigoplus_{|I|=1} \mathbb{C}_{P_I}}_{\text{underlined}} \rightarrow \bigoplus_{|I|=2} \mathbb{C}_{P_I} \rightarrow \cdots \rightarrow \bigoplus_{|I|=m} \mathbb{C}_{P_I}$$

where the underlined entry is at cohomological degree 0, and the differential is given by the natural inclusion with alternating signs.

Furthermore, the stalks of \mathcal{F}_\emptyset are

$$(\mathcal{F}_\emptyset)_p = \begin{cases} \mathbb{C} & p \in (\mathbb{R}_{>0})^m \\ \mathbb{C}[-m+1] & p \in (\mathbb{R}_{\leq 0})^m \\ 0 & \text{elsewhere} \end{cases}$$

Proof. Consider the skeleton

$$\Lambda_{m,0} = \bigcup_{I \subset [m]} SS(\mathbb{C}_{P_I}).$$

Then $\Lambda_{m,0} \setminus \Lambda'_{m,0}$ is a smooth Lagrangian strata L in the cotangent fiber $T_0^*(\mathbb{R}^m)$. The microstalk functor for L (unique up to a degree shift) in $Sh^\diamond(\mathbb{R}^m, \Lambda_{m,0})$ can be represented as a chain complex of stalk functors Φ_I for region Q_I :

$$\Phi_L := \Phi_\emptyset \rightarrow \bigoplus_{|I|=1} \Phi_I \rightarrow \cdots \rightarrow \bigoplus_{|I|=m} \Phi_I.$$

After restricting to the subcategory $Sh^\diamond(\mathbb{R}^m, \Lambda'_{m,0})$, the microstalk functor Φ_L is zero, hence the chain complex of stalk functors become acyclic. Thus the co-representing objects \mathcal{F}_I for Φ_I in $Sh^\diamond(\mathbb{R}^m, \Lambda'_{m,0})$ forms an acyclic chain complex:

$$\mathcal{F}_\emptyset \rightarrow \bigoplus_{|I|=1} \mathcal{F}_I \rightarrow \cdots \rightarrow \bigoplus_{|I|=m} \mathcal{F}_I$$

After recognizing $\mathcal{F}_I = \mathbb{C}_{P_I}$ for $|I| > 0$ (c.f. Lemma 6.4), we have the claimed resolution of \mathcal{F}_\emptyset .

Next, we verify the claim for the stalks of \mathcal{F}_\emptyset . If $p \in (\mathbb{R}_{>0})^m$, then the augmented complex

$$\mathbb{C} \rightarrow \left[\bigoplus_{|I|=1} \mathbb{C}_{P_I}|_p \rightarrow \bigoplus_{|I|=2} \mathbb{C}_{P_I}|_p \rightarrow \cdots \rightarrow \bigoplus_{|I|=m} \mathbb{C}_{P_I}|_p \right]$$

become the Koszul complex $\boxtimes_{i=1}^m (\mathbb{C} \rightarrow \mathbb{C})$, hence is acyclic. Thus, the stalk of the complex is \mathbb{C} . If $p \in (\mathbb{R}_{\leq 0})^m$, then the only non-zero entry is the last one $\mathbb{C}_{\mathbb{R}^m}|_p$, sitting at degree $m - 1$. If p is elsewhere, then the stalk of the complex reduces to a Koszul complex, hence is acyclic. \square

Definition 6.6. We call the above stalk probe sheaf \mathcal{F}_\emptyset on \mathbb{R}^m as the **hourglass sheaf of dimension m** , and denoted as Δ_m .

Now, we are ready to continue to describe the stalk probe sheaf for $\tilde{\Lambda}_W$ near v .

Let $[N] = \{1, \dots, N\}$, and the subsets $[N]_+ = \{i : a_i > 0\}$ and $[N]_- = \{i : a_i < 0\}$. For any subset $I \subset [N]$, we denote $I_\pm = I \cap [N]_\pm$. We denote $I_\pm^c = [N]_\pm \setminus I_\pm$. We decompose \mathbb{R}^N as

$$\mathbb{R}^N = \mathbb{R}^{N_+} \times \mathbb{R}^{N_-}.$$

If $\mathcal{F}_+ \in Sh(\mathbb{R}^{N_+})$ and $\mathcal{F}_- \in Sh(\mathbb{R}^{N_-})$, we will write $\mathcal{F}_+ \boxtimes \mathcal{F}_-$ for the product sheaf in \mathbb{R}^N . In particular, if $J_+ \subset [N]_+$, we will write $P_{J_+} \in Sh(\mathbb{R}^{N_+})$.

Proposition 6.7. Let $v \in \mathbb{Z}^N$, $I \subset [N]$. Assume the window W is thick enough for v , and Λ_I does not appear in $\tilde{\Lambda}_W$. Then exactly one of following cases happen 3.5)

- (1) $I \subset [N]_+$ and for any $\emptyset \neq K \subset [N]_-$, $\Lambda_{I \sqcup K} \subset \tilde{\Lambda}_{W,v}$.
- (2) $I \subset [N]_-$ and for any $\emptyset \neq K \subset [N]_+$, $\Lambda_{I \sqcup K} \subset \tilde{\Lambda}_{W,v}$.

In case (1) (resp. (2)), we say (v, I) is **of positive (resp. negative) type**.

Proof. We enumerate all possible cases of (v, I) . If $v \in \mathbb{Z}^N(W)$, and Λ_I does not appear, then $I \neq \emptyset$ and $I \subset [N]_+$ or $I \subset [N]_-$. Suppose $I \subset [N]_+$, then for any $\emptyset \neq K \subset [N]_-$, $I \sqcup K$ is of mixed type, hence $\Lambda_{I \sqcup K}$ appears in $\tilde{\Lambda}_{W,v}$ by Lemma 3.3.

If $v \notin \mathbb{Z}^N(W)$, and if $\mu(v) > \max(-W)$, then $I = \emptyset$ or I is of negative type, by Lemma 3.5. We claim that we are in case (2), since for any $\emptyset \neq K \subset [N]_+$, $I \sqcup K$ is of mixed type, hence $\Lambda_{I \sqcup K}$ appears in $\tilde{\Lambda}_{W,v}$. We cannot be in case (1), since even if $I = \emptyset$ then adding a non-empty negative type subset K still makes $I \sqcup K$ a negative type, hence $\Lambda_{I \sqcup K} \not\subset \tilde{\Lambda}_{W,v}$.

The case for $v \notin \mathbb{Z}^N(W)$ and $\mu(v) < \min(-W)$ is similar. We claim that we are in case (1) and omit the proof. \square

Proposition 6.8. Let W be a window, $v \in \mathbb{Z}^N$ and $I \subset [N]$. Assume the window W is thick enough for v , and Λ_I does not appear in $\tilde{\Lambda}_{W,v}$. Then

- (1) If (v, I) is of positive type, then the stalk probe sheaf for region Q_I is

$$\mathcal{F}_I = \bigoplus_{|J_+|=1} \mathcal{F}_{J_+} \rightarrow \bigoplus_{|J_+|=2} \mathcal{F}_{J_+} \rightarrow \cdots \rightarrow \bigoplus_{|J_+|=N_+} \mathcal{F}_{J_+} \simeq \mathbb{C}_{P_{J_+}} \boxtimes \Delta_{N_-},$$

where $J = J_+ \sqcup J_-$, and $J_+ = I_+ = I$.

(2) If (v, I) is of negative type, then the stalk probe sheaf for region Q_I is

$$\mathcal{F}_I = \bigoplus_{|J_+|=1} \mathcal{F}_J \rightarrow \bigoplus_{|J_+|=2} \mathcal{F}_J \rightarrow \cdots \rightarrow \bigoplus_{|J_+|=N_+} \mathcal{F}_J \simeq \Delta_{N_+} \boxtimes \mathbb{C}_{P_{J_-}},$$

where $J = J_+ \sqcup J_-$, and $J_- = I_- = I$.

Proof. We only prove (1) since the proof of (2) is exactly the same. Let $p \in Q_I$ with $p_i = +1$ if $i \notin I$ and $p_i = -1$ if $i \in I$. Consider the affine subspace $M = p + \mathbb{R}^{N_-}$. The restriction of Λ_W on M is the same as in Lemma 6.5, with $m = N_-$. Hence, we have an acyclic chain complex of stalk functors

$$\Phi_I \rightarrow \bigoplus_{|J_-|=1} \Phi_J \rightarrow \bigoplus_{|J_-|=2} \Phi_J \rightarrow \cdots \rightarrow \bigoplus_{|J_-|=N_-} \Phi_J$$

where J has the same positive part J_+ as I , and J_- runs through all subsets of $[N]_-$. Hence the stalk probe sheaves forms the similar chain complex, replacing Φ_J by \mathcal{F}_J . Thanks for the condition that (v, I) is of positive type, we see for $|J_-| \geq 1$, $\Lambda_J = \Lambda_{I \sqcup J_-}$ appears in $\tilde{\Lambda}_{W,v}$, and

$$\mathcal{F}_J = \mathbb{C}_{P_J} = \mathbb{C}_{P_{J_+}} \boxtimes \mathbb{C}_{P_{J_-}}, \quad \forall |J_-| \geq 1.$$

Thus, we may pull out the common factor $\mathbb{C}_{P_{J_+}}$ and get $\mathcal{F}_I \simeq \mathbb{C}_{P_{J_+}} \boxtimes \Delta_{N_-}$. \square

Corollary 6.9. *Let $v \in \mathbb{Z}^N$ and W is a window thick enough for v . For any $I \subset [N]$, let \mathcal{F}_I be the probe for region Q_I . The category $Sh^w(\mathbb{R}^N, \tilde{\Lambda}_{W,v})$ is split-generated by \mathcal{F}_I for those I such that Λ_I appears in $\tilde{\Lambda}_{W,v}$.*

Proof. Since $Sh^w(\mathbb{R}^N, \tilde{\Lambda}_{W,v})$ is split-generated by all $\{\mathcal{F}_I : I \subset [N]\}$, and any \mathcal{F}_I can be written as a chain complex using only $\{\mathcal{F}_I : \Lambda_I \subset \tilde{\Lambda}_{W,v}\}$, hence the smaller set of \mathcal{F}_I split-generates. \square

6.2. Co-restriction of stalk-probe sheaves near a lattice point. Let W be a window, $v \in \mathbb{Z}^N$. We define half-spaces in $T_v \mathbb{R}^N \simeq \mathbb{R}^N$

$$B_{\pm} = \{x \in \mathbb{R}^N : \pm x \cdot a > 0\}.$$

Let \mathcal{F}_I denote the stalk probe sheaf for region Q_I .

As a warm-up, we consider the full skeleton's specialization $\Lambda_{N,v}$.

Lemma 6.10. *The stalk probe sheaf for $Q_I \cap B_{\pm}$ in $Sh^{\diamond}(B_{\pm}, \Lambda_{N,v})$ is $\mathbb{C}_{P_I|_{B_{\pm}}}$, if $Q_I \cap B_{\pm} \neq \emptyset$.*

Proof. Consider the restriction to B_- only, since the other case is similar. $Q_I \cap B_- = \emptyset$ if and only if $I = [N]_-$. The category $Sh^{\diamond}(B_{\pm}, \Lambda_{N,v})$ is equivalent to the module category over the poset $\{I \subset [N] : I \neq [N]_-\}$, hence $\{\mathbb{C}_{P_I|_{B_-}} \mid I \subset [N], I \neq [N]_-\}$ compactly generate the category. The rest of the proof is the same as Proposition 6.3. \square

Proposition 6.11. *Assume W is thick enough for v . Then the stalk probe sheaf for $Q_I \cap B_{\pm}$ is $\mathcal{F}_I|_{B_{\pm}}$, if $Q_I \cap B_{\pm} \neq \emptyset$.*

Proof. Consider the restriction to B_- only. If Λ_I appears in $\tilde{\Lambda}_{W,v}$ and $I \neq [N]_-$, then $\mathcal{F}_I = \mathbb{C}_{P_I}$, $\mathcal{F}_I|_{B_-}$ is the probe for region $Q_I \cap B_-$ in $Sh^{\diamond}(B_{\pm}, \Lambda_{N,v})$, hence is also the probe for the same region in $Sh^{\diamond}(B_{\pm}, \tilde{\Lambda}_{W,v})$.

If $\Lambda_I \notin \tilde{\Lambda}_{W,v}$, then we consider the following two cases.

(1) (v, I) is of positive type, i.e $I \subset [N]_+$ and $I \sqcup K \in \mathcal{I}_{W,v}$ for any non-empty $K \subset [N]_-$.

(a) Assume $I \neq \emptyset$. Consider $p \in Q_I$ of the following form, $p = -e_I + \epsilon e_{I^c}$. Then $\mu(p) = -\sum_{i \in I} a_i + O(\epsilon)$, hence for small enough ϵ , $\mu(p) < 0$ and $p \in Q_I \cap B_-$. And the slice $S = p + \mathbb{R}^{N_-}$ contains the point p' where $p'_i = p_i$ for $i \in [N]_+$ and $p'_i = 0$ for $i \in [N]_-$. There are 2^{N_-} strata on the slice S near p' , they belong to Q_J with $J_+ = I$ and $J_- \subset [N]_-$ respectively. Hence, we have an acyclic chain complex of stalk functors, resulting in an acyclic chain complex of probe sheaves

$$\mathcal{F}_{Q_I \cap B_-} \rightarrow \bigoplus_{|J_-|=1} \mathbb{C}_{P_J \cap B_-} \rightarrow \bigoplus_{|J_-|=2} \mathbb{C}_{P_J \cap B_-} \rightarrow \cdots \rightarrow \bigoplus_{|J_-|=N_-} \mathbb{C}_{P_J \cap B_-}$$

where we plugged in the probe sheaf for $Q_J \cap B_-$ by $\mathbb{C}_{P_J \cap B_-}$ since $J \in \mathcal{I}_{W,v}$. This acyclic complex can be viewed as a resolution of the first term $\mathcal{F}_{Q_I \cap B_-}$, and the rest of the chain complex is $\mathcal{F}_I|_{B_-}$, hence we get

$$\mathcal{F}_{Q_I \cap B_-} \simeq \mathcal{F}_I|_{B_-}.$$

- (b) Assume $I = \emptyset$, this is only possible if $v \notin \mathbb{Z}^N(W)$, and more precisely we have $\mu(v) < \min(-W)$. We claim

$$SS(\mathbb{C}_{P_\emptyset}|_{B_-}) \subset \bigcup_{i \in [N]_-} \Lambda_{\{i\}}|_{B_-} \subset \tilde{\Lambda}_{W,v},$$

Indeed, if $x \in \partial(P_\emptyset) \cap B_-$, then if we define $K = \{i : x_i \neq 0\}$, then $K \not\subset [N]_+$, and $SS(\mathbb{C}_{P_\emptyset})|_x \subset \Lambda_J|_x$ if and only if $K^c \subset J^c$, i.e. $J \subset K$. Hence we may choose $J = \{j\}$ for some $j \in K \cap [N]_-$. This verifies the claim. Finally, we see

$$\mathbb{C}_{P_\emptyset}|_{B_-} = [\mathbb{C}_{\mathbb{R}_+^{N_+}} \boxtimes \mathbb{C}_{\mathbb{R}_+^{N_-}}]|_{B_-} = [\mathbb{C}_{\mathbb{R}_+^{N_+}} \boxtimes \Delta_{N_-}]|_{B_-} = \mathcal{F}_\emptyset|_{B_-}$$

Since B_- does not intersects with the product $\mathbb{R}_+^{N_+} \times \mathbb{R}_{\leq 0}^{N_-}$, hence Δ_{N_-} acts as $\mathbb{C}_{\mathbb{R}_+^{N_-}}$.

- (2) (v, I) is of negative type, i.e. $I \subset [N]_-$ and $I \sqcup K \in \mathcal{I}_{W,v}$ for any non-empty $K \subset [N]_+$. Since $Q_I \cap B_- \neq \emptyset$, $I \neq [N]_-$.

- (a) Assume $I \neq \emptyset$. Consider $p \in Q_I$ of the following form, $p = -\epsilon e_I + e_{[N]_- \setminus I} + \epsilon e_{[N]_+}$. Then $\mu(p) = \sum_{i \in [N]_- \setminus I} a_i + O(\epsilon)$, hence for small enough ϵ , $\mu(p) < 0$ and $p \in Q_I \cap B_-$. And the slice $S = p + \mathbb{R}^{N_+}$ contains the point p' where $p'_i = p_i$ for $i \in [N]_-$ and $p'_i = 0$ for $i \in [N]_+$. There are 2^{N_+} strata on the slice S near p' , they belong to Q_J with $J_- = I$ and $J_+ \subset [N]_+$ respectively. The rest of the argument is similar to case (v, I) of positive type, $I \neq \emptyset$, and we omit the details.
- (b) The case $I = \emptyset$ is impossible by the proof in Proposition 6.7.

□

6.3. Proof of Proposition 6.2. Let W be a window, $v \in \mathbb{Z}^N$, assume W is thick enough for v . Consider the restrictions

$$\rho_{v,\pm a} : Sh^\diamond(T_v \mathbb{R}^N, \tilde{\Lambda}_{W,v}) \rightarrow Sh^\diamond(B_{\mp}, \tilde{\Lambda}_{W,v}), \quad B_{\pm} = \{x \in T_v \mathbb{R}^N : \pm x \cdot a > 0\}$$

and the left-adjoint $\rho_{v,\pm a}^L$. We are going to show that $id \rightarrow \rho_{v,\pm a} \circ \rho_{v,\pm a}^L$ are isomorphisms.

This is precisely what Proposition 6.11 is about. Consider the co-restriction from B_- to \mathbb{R}^N . Suppose $Q_I \cap B_- \neq \emptyset$, then by Proposition 6.11, $\mathcal{F}_I|_{B_-}$ is the probe sheaf for $Sh^\diamond(B_-, \tilde{\Lambda}_{W,v})$ for region $Q_I \cap B_-$. Tautologically, we have

$$\rho_{v,a}^L(\mathcal{F}_I|_{B_-}) = \mathcal{F}_I,$$

hence

$$\rho_{v,a} \circ \rho_{v,a}^L(\mathcal{F}_I|_{B_-}) = \mathcal{F}_I|_{B_-}.$$

This holds for all the compact generators of $Sh^\diamond(B_-, \tilde{\Lambda}_{W,v})$, hence $\rho_{v,a}^L$ is an isomorphism. Similarly for $\rho_{v,a}^L$.

7. PROOF OF THEOREM 0.6

In previous two sections, we studied the under what conditions the restriction and the co-restriction functors are fully-faithful: roughly speaking, if $|W|$ is too large, then the restriction functor will not be fully-faithful; if $|W|$ is too small, then the co-restriction functor will not be fully-faithful.

In this section, we will let $|W| = \eta_+$, the minimum width of the window to ensure the co-restriction functor to be fully-faithful. To be concrete, let

$$-W = \{a, a+1, \dots, b\}, \quad b = a + \eta_+ - 1.$$

Since $SS_{Hom}^L = \emptyset$, from Proposition 2.4 and Lemma 2.6, we have

$$SS(\tilde{\Lambda}_W) = \mathcal{H}om(\tilde{\Lambda}_W, \tilde{\Lambda}_W).$$

From Theorem 5.2, we see $(v, a) \in SS(\tilde{\Lambda}_W)$ if $v \in \mathbb{Z}^N$ and $\mu(v) \in [a, a + \eta)$, and $(v, -a)$ is not in $SS(\tilde{\Lambda}_W)$ for any v .

Let $[v] = p(v) \in \mathbb{R}^N / M$. The above statement also holds if we replace v by $[v]$ and $\tilde{\Lambda}_W$ by Λ_W . Recall from section 5 that $\mathcal{C}(I) = Sh^\diamond(\pi^{-1}(I), \Lambda_W)$.

Proposition 7.1. (1) As t increases from $-\infty$ to $+\infty$, $\mathcal{C}((-\infty, t))$ is locally constant except when $t = a, a + 1, \dots, a + \eta_1$.

$$\mathcal{C}((-\infty, t)) \simeq \begin{cases} Sh^\diamond(M_{\mathbb{T}}, \Lambda_-) & \text{if } t \ll 0 \\ Sh^\diamond(M_{\mathbb{T}}, \Lambda_+) & \text{if } t \gg 0 \end{cases}$$

(2) As t decreases from $+\infty$ to $-\infty$, $\mathcal{C}((t, +\infty))$ remains invariant, and equals $Sh^\diamond(M_{\mathbb{T}}, \Lambda_+)$.

Proof. This are consequences of the description of the singular support $SS(\Lambda_W)$. □

Let $\mathcal{C}(< t)$ denote $\mathcal{C}((-\infty, t))$.

Proposition 7.2. For any $t \in \mathbb{R}$, we have

$$\mathcal{C}(< t) \simeq \mathcal{C}(t) := Sh^\diamond(M_{\mathbb{T}}, \Lambda_W|_t)$$

Proof. (1) If $t \notin \{a, a + 1, \dots, a + \eta - 1\}$, then for any ϵ small enough, we have

$$\mathcal{C}(< t) \simeq \mathcal{C}((t - \epsilon, t + \epsilon)).$$

Since $\mathcal{C}((t - \epsilon, t + \epsilon)) \simeq \mathcal{C}(I)$ for any open $I \subset (t - \epsilon, t + \epsilon)$, hence the restriction $\mathcal{C}((t - \epsilon, t + \epsilon))$ to $\mathcal{C}(t)$ is an equivalence.

(2) If $t \in \{a, a + 1, \dots, a + \eta - 1\}$, we claim that $\mathcal{C}(t) \simeq \mathcal{C}(t - \epsilon)$ for some small ϵ . Given the claim, we then have $\mathcal{C}(t) \simeq \mathcal{C}(t - \epsilon) \simeq \mathcal{C}(< t - \epsilon) \simeq \mathcal{C}(< t)$. Now we prove the claim, let $[v]$ denote the vertex in slice $\pi^{-1}(t)$, then the stratification of \mathcal{Y}_t compared with $Y_{t-\epsilon}$ has one region less. If we lift $[v]$ to v and work in the tangent space of v , then the region $Q_{[N]_+}$ does not intersect with the slice B_0 where $B_\delta = \{x \cdot a = \delta\}$. However, $\Lambda_{[N]_+}$ does not appear in $\tilde{\Lambda}_{W,v}$, hence the stalk in region $Q_{[N]_+}$ can be deduced from the stalks in other regions, Q_I where I is of mixed type. Thus $\mathcal{C}(t) \simeq \mathcal{C}(t - \epsilon)$ and the claim holds. □

Now we consider how $\mathcal{C}((-\infty, t))$ jump as t cross one of the critical moments.

Proposition 7.3. Let $t \in \{a, a + 1, \dots, a + \eta - 1\}$, then we have semi-orthogonal decomposition

$$\mathcal{C}((-\infty, t + \epsilon)) = \langle \text{Vect}, \mathcal{C}((-\infty, t - \epsilon)) \rangle$$

for small enough ϵ .

Proof. The co-restriction functor $\rho^L : \mathcal{C}(< t - \epsilon) \rightarrow \mathcal{C}(< t + \epsilon)$ is fully-faithful, and let S denote the image of ρ^L . WE claim that the right-orthogonal S^\perp is generated by one simple object, thus $S^\perp \simeq \text{Vect}$ and we get the desired semi-orthogonal decomposition. Consider the Lagrangian $\tilde{\Lambda}_{W,v}$, for $v \in \mathbb{Z}^N(W)$. By assumption, $\mu(v) + \eta_- \in -W$, hence for any $I \subset [N]_-$, $\Lambda_I \text{ in } \tilde{\Lambda}_{W,v}$. Thus, consider the object in $Sh^\diamond(\mathbb{R}^N, \tilde{\Lambda}_{W,v})$

$$\mathbb{C}_{P_0} \rightarrow \bigoplus_{|I|=1} \mathbb{C}_{P_I} \rightarrow \bigoplus_{|I|=2} \mathbb{C}_{P_I} \rightarrow \cdots \rightarrow \underbrace{\bigoplus_{|I|=N_-} \mathbb{C}_{P_I}}_{\text{underlined}}$$

where $I \subset [N]_-$ and the underlined cell is in degree 0. We see it is actually $\mathbb{C}_{\mathbb{R}_{>0}^{N_+}} \boxtimes \mathbb{C}_{\mathbb{R}_{\leq 0}^{N_-}}$, hence is not supported on B_- and is a simple object. □

By repeat this decomposition η times, and using Proposition 7.1, we get

Corollary 7.4.

$$Sh^\diamond(M_T, \Lambda_+) \simeq \mathcal{C}(a + \eta) \simeq \langle \text{Vect}, \mathcal{C}(a + \eta - 1) \rangle \simeq \cdots \simeq \underbrace{\langle \text{Vect}, \cdots, \text{Vect} \rangle}_{\eta \text{ times}}, Sh^\diamond(M_T, \Lambda_-).$$

We can be more precise about the relations of these Vects. For $t \in \{a, a+1, \dots, a+\eta-1\}$, and $v \in \mu_{\mathbb{Z}}^{-1}(t)$, we may consider the chain complex

$$\mathcal{P}_I(v) := Q_v \rightarrow \bigoplus_{J \subset I, |J|=1} Q_{v-e_J} \rightarrow \bigoplus_{J \subset I, |J|=2} Q_{v-e_J} \rightarrow \cdots \rightarrow Q_{v-e_I},$$

where $I = [N]_-$. Then $SS(\mathcal{P}_I(v)) \in \tilde{\Lambda}_W$. And $\mathcal{P}_I(v)$ is the microstalk skyscraper sheaf for the Lagrangian smooth strata $(v + (0, 1)^{I^c}) \times (-\sigma_I)$ over the corresponding face of the cube $v + [0, 1]^N$. These collections of $p_i \mathcal{P}_I(v)$ corresponds to the Vect factors, and under CCC, they go to structure sheaf for the unstable loci of the $(-)$ GIT quotient, with appropriate \mathbb{C}^* -weights. We omit the detail of the computation here, since it is of the same structure as the following B-model computation.

7.1. Window subcategory in B-model . In this section, we describe semi-orthogonal decomposition in B-model. We follow closely [Seg11] and [Ker17, Theorem 2].

Let V_+ be the unstable locus for the GIT quotient corresponding to \mathcal{X}_- , thus

$$V_+ = \{z \in \mathbb{C}^N \mid z_i = 0, i \in [N]_-\},$$

and V_+ is parametrized by $z_i, i \in [N]_+$.

We have Koszul resolution of \mathcal{O}_{V_+} :

$$\mathcal{O}_{V_+} = (\boxtimes_{j \in [N]_+} \mathcal{O}_{\mathbb{C}}) \boxtimes \boxtimes_{j \in [N]_-} [\mathcal{O}_{\mathbb{C}} \xrightarrow{z_j} \mathcal{O}_{\mathbb{C}}]$$

Then the equivariant version is

$$\mathcal{O}_{V_+}([0, \eta_-]) = (\boxtimes_{j \in [N]_+} \mathcal{O}_{\mathbb{C}}) \boxtimes (\boxtimes_{j \in [N]_-} [\mathcal{O}_{\mathbb{C}}(-a_j) \xrightarrow{z_j} \mathcal{O}_{\mathbb{C}}(0)])$$

where $\mathcal{O}_{\mathbb{C}}(k)$ is the \mathbb{C}^* -equivariant line bundle on \mathbb{C} with weight k , and the interval $[0, \eta_-]$ means \mathcal{O}_{V_+} has a resolution with weights lying in this interval.

Lemma 7.5. For any $k \in \mathbb{Z}$, we have

- (1) $\text{Hom}(\mathcal{O}_{V_+}([0, \eta_-]), \mathcal{O}_{\mathbb{C}^N}(0)) \simeq \text{Sym}^*(\oplus_{i \in [N]_+} \mathbb{C}(-a_i))[-N_-](-\eta_-)$
- (2) $\text{Hom}(\mathcal{O}_{\mathbb{C}^N}(0), \mathcal{O}_{V_+}([0, \eta_-])) = \text{Sym}^*(\oplus_{i \in [N]_+} \mathbb{C}(-a_i))$
- (3) $\text{Hom}(\mathcal{O}_{V_+}([0, \eta_-]), \mathcal{O}_{V_+}([0, \eta_-])) = \text{Sym}^*(\oplus_{i \in [N]_+} \mathbb{C}(-a_i)) \otimes \text{Sym}^*(\oplus_{i \in [N]_-} \mathbb{C}[-1](a_i))$

Proof. This boils down to one-dimensional case. If \mathbb{C}^* acts on \mathbb{C} with weight a , then we have

$$\text{Hom}(\mathcal{O}_{\mathbb{C}}(0), \mathcal{O}_{\mathbb{C}}(0)) = \text{Sym}^*(\mathbb{C}(-a))$$

Consider the Koszul resolution of the skyscraper sheaf on 0:

$$\mathcal{O}_0 := \mathcal{O}_{\mathbb{C}}(-a) \xrightarrow{z} \mathcal{O}_{\mathbb{C}}(0),$$

Then we may check that

$$\begin{aligned} \text{Hom}(\mathcal{O}_0, \mathcal{O}_{\mathbb{C}}(0)) &= \mathbb{C}[-1](a) \\ \text{Hom}(\mathcal{O}_{\mathbb{C}}(0), \mathcal{O}_0(0)) &= \mathbb{C}(0) \\ \text{Hom}(\mathcal{O}_0(0), \mathcal{O}_0(0)) &= \mathbb{C}(0) \oplus \mathbb{C}[-1](a) \end{aligned}$$

□

Hence, for \mathbb{C}^* -equivariant hom, we have

$$\mathrm{Hom}_{\mathbb{C}^*}(\mathcal{O}_{\mathbb{C}^N}(i), \mathcal{O}_{V_+}(j)) = \mathrm{Sym}^*(\oplus_{i \in [N]_+} \mathbb{C}(-a_i))_{i-j}$$

In particular, if $i > j$ the hom space vanishes.

Suppose $W = \{0, \dots, \eta_+ - 1\}$. Then, we have semi-orthogonal decomposition of

$$\mathcal{B}_W = \mathrm{Coh}(\mathcal{X}_+) \simeq \underbrace{\langle \mathcal{O}_{V_+}(0), \mathcal{O}_{V_+}(1), \dots, \mathcal{O}_{V_+}(\eta - 1) \rangle}_{\eta \text{ many}} \underbrace{\langle \{\mathcal{O}_{\mathbb{C}^N}(\eta), \mathcal{O}_{\mathbb{C}^N}(\eta + 1), \dots, \mathcal{O}_{\mathbb{C}^N}(\eta_+ - 1)\} \rangle}_{\simeq \mathrm{Coh}(\mathcal{X}_-)}$$

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