Physical Brownian Motion in Magnetic Field as Rough Path

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September 20, 2016
Brownian Motion

Brownian motion—The movements that a massless particle takes under the impact of random forces (think speck of dust floating in water).

Source: https://quanttutorials.files.wordpress.com/2013/05/bm-sample-path.png
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(iii) for every $t, h \geq 0$, $W_{t+h} - W_t$ is independent of $(W_u : 0 \leq u \leq t)$ and has normal distribution with mean 0 and variance $h$. 
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Q: What happens to physical Brownian motion as \( m \to 0 \)?
Question: Why care about (non-physical) Brownian motion when actual particles have mass?
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Answer: Because it’s still a good approximation for particles with low mass. It is the limit of physical Brownian motion as $m \to 0$. 
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Introduce Magnetic Field

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A particle with charge $q$ moving in a constant magnetic field $B$ experiences a force

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for some anti-symmetric matrix $B$. Now,

$$m\ddot{x} = -A\dot{x} + qB\dot{x} + \dot{W}$$
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where all eigenvalues of $M$ have strictly positive real part.
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as a system of first-order DEs

\[ dX = \frac{1}{m} Pdt \]

\[ dP = -\frac{1}{m} MPdt + dW \]

where we think of \( P \) as momentum.
Q: Now that we’ve added a magnetic field, what happens as $m \to 0$?
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As $m \to 0$, physical Brownian motion under a magnetic field converges to Brownian motion under a magnetic field...

...plus an extra area term (in the rough path sense).
Motivation

In general, we have controlled ODEs of the form

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$$Y_{i+1} = Y_i + hf_0(Y_i) + hf(Y_i)(X_{t+1} - X_t).$$

Furthermore, when $X$ is smooth, $Y$ is continuous as a function of $X$ (a stability result).
If $X$ is Brownian motion, we may take the limit as $h \to 0$ of

$$Y_{i+1} = Y_i + hf_0(Y_i) + \sqrt{hf}(Y_i)\xi_{i+1},$$

where $(\xi_i)$ are i.i.d. standard Gaussian random variables.
Motivation II

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$$Y_{i+1} = Y_i + hf_0(Y_i) + \sqrt{hf(Y_i)} \xi_{i+1},$$

where $(\xi_i)$ are i.i.d. standard Gaussian random variables. $Y$ is not continuous as a function of $W$, i.e., stability is lost. The problem is that $W$ is a.s. not $\alpha$-Hölder continuous for $\alpha = 1/2$. 
### Definition

A rough path is a continuous map $X : [0, T] \to V$ along with a continuous "second-order process" $\mathbf{X} : [0, T]^2 \to V \otimes V$.
## Rough Paths

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$$\mathbb{X}_{s,t} - \mathbb{X}_{u,t} - \mathbb{X}_{s,u} = X_{s,u} \otimes X_{u,t}.$$

The terms $\mathbb{X}_{s,t}$ are "inspired by" the terms $\int_s^t X_r \otimes dX_r$ which appear in second-order Euler processes. Generally, one may not deduce $\mathbb{X}_{s,t}$ from $X_{s,t}$. Chen's relation still holds when $X_{s,t}$ is replaced by $X_{s,t} + F_t - F_s$. When $X_t$ is $\alpha$-Hölder continuous for $\alpha > 1/2$, we may take Young's integrals for $X_{s,t}$, but otherwise, no canonical choice exists.
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A rough path is a continuous map \( X : [0, T] \rightarrow V \) along with a continuous "second-order process" \( \bar{X} : [0, T]^2 \rightarrow V \otimes V \) such that

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\bar{X}_{s,t} - \bar{X}_{u,t} - \bar{X}_{s,u} = X_{s,u} \otimes X_{u,t}.
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- The terms \( \bar{X}_{s,t} \) are "inspired by" the terms \( \int_s^t X_{s,r} \otimes dX_r \) which appear in second-order Euler processes.
- Generally, one may not deduce \( \bar{X} \) from \( X \).
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- Generally, one may not deduce $\mathbb{X}$ from $X$.
- Chen's relation still holds when $\mathbb{X}_{s, t}$ is replaced by $\mathbb{X}_{s, t} + F_t - F_s$.
- When $X_t$ is $\alpha$-Hölder continuous for $\alpha > 1/2$, we may take Young's integrals for $\mathbb{X}_{s, t}$, but otherwise, no canonical choice exists.
Rough Paths II

Definition

For $\alpha \in (1/3, 1/2]$ and rough paths $X, Y$, define the $\alpha$-Hölder rough path metric

$$\rho_{\alpha}(X, Y) := \sup_{s \neq t \in [0, T]} \frac{|X_{s,t} - Y_{s,t}|}{|t - s|^{\alpha}} + \sup_{s \neq t \in [0, T]} \frac{|X_{s,t} - Y_{s,t}|}{|t - s|^{2\alpha}}.$$ 

Denote the set of rough paths $X$ with $\rho_{\alpha}(X, 0) < \infty$ as $C^\alpha([0, T])$. 

Brownian motion is $\alpha$-Hölder continuous for $\alpha < 1/2$ but not for $\alpha = 1/2$ a.s.
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Itô-Lyons Map

While the Itô solution map $S : \mathcal{W} \to Y$ is \textit{not} continuous, the Itô-Lyons map $\hat{S} : (\mathcal{W}, \mathcal{W}) \to Y$ is.
While the Itô solution map $S : W \to Y$ is *not* continuous, the Itô-Lyons map $\hat{S} : (W, \mathbb{W}) \to Y$ is.

$$W(\omega) \xrightarrow{\Psi} (W, \mathbb{W})(\omega) \xrightarrow{\hat{S}} Y(\omega),$$

where $\Psi$ is measurable.
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**Theorem (Kolmogorov criterion)**

Let $q \geq 1$, $\beta > 1/q$ and assume for all $s, t \in [0, 1]$ that

$$|X_{s,t}|_{L^q(\omega)} \leq C|t - s|^\beta. \quad (3)$$

Then, for all $\alpha \in [0, \beta - 1/q)$, there exists a random variable $K_\alpha(\omega) \in L^q(\omega)$ such that

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For Brownian motion, equation (3) follows from the normal distribution of $W_{s,t}$. 
Let $D_n = \{k2^{-n} : 0 \leq k \leq 2^n\} \subset (0, 1)$. Set

$$K_n = \sup_{t \in D_n} |X_{t, t+2^{-n}}|.$$
Proof of Kolmogorov Criterion

Let $D_n = \{k2^{-n} : 0 \leq k \leq 2^n\} \subset (0, 1)$. Set

$$K_n = \sup_{t \in D_n} |X_{t, t+2^{-n}}|.$$

By Equation (3), we have

$$E(K_n^q) \leq E \sum_{t \in D_n} |X_{t, t+2^{-n}}|^q \leq \frac{1}{|D_n|} C^q |D_n|^\beta q = C^q |D_n|^\beta q - 1.$$
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(Remember: $\beta > 1/q$.)
Proof of Kolmogorov Criterion II

Fix $s < t$ in $\bigcup_n D_n$ and choose $m$ so $|D_{m+1}| < t - s \leq |D_m|$. (Other points will follow by continuity.)
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(Other points will follow by continuity.) Then,

\[
|X_{s,t}| \leq \sum_{i=0}^{n-1} |X_{\tau_i, \tau_{i+1}}| \leq 2 \sum_{n \geq m+1} K_n,
\]

where \( \tau_i \in \bigcup_n D_n \) and no three \( |\tau_{i+1} - \tau_i| \) are equal.
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\[
\frac{|X_{s,t}|}{|t - s|^{\alpha}} \leq \sum_{n \geq m+1} \frac{2K_n}{|D_n|^{\alpha}} \leq K_{\alpha} := \sum_{n \geq 0} \frac{2K_n}{|D_n|^{\alpha}}.
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$$\frac{|X_{s,t}|}{|t - s|^{\alpha}} \leq \sum_{n \geq m+1} \frac{2K_n}{|D_n|^{\alpha}} \leq K_{\alpha} := \sum_{n \geq 0} \frac{2K_n}{|D_n|^{\alpha}}.$$

This means $|X_{s,t}| \leq K_{\alpha}|t - s|^{\alpha}$, so we need only show $K_{\alpha} \in L^q$. 
(Reminder: $K_\alpha := \sum_{n\geq 0} \frac{2K_n}{|D_n|^\alpha}$.)

Since $\alpha < \beta - \frac{1}{q}$ and $E(K_q^n) \leq C q |D_n|^{\beta q - 1}$, we have

$$||K_\alpha||_{L^q} \leq \sum_{n\geq 0} 2^{K_n} |D_n|^\alpha \cdot E(K_q^n)^{1/q} \leq \sum_{n\geq 0} 2^{K_n} |D_n|^\alpha |D_n|^{\beta - 1/q} < \infty.$$
Proof of Kolmogorov Criterion III

(Reminder: \( K_\alpha := \sum_{n \geq 0} \frac{2K_n}{|D_n|^\alpha} \).)

Since \( \alpha < \beta - 1/q \) and

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E(K_n^q) \leq C^q |D_n|^\beta q^{-1},
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Proof of Kolmogorov Criterion III

(Reminder: \( K_\alpha := \sum_{n \geq 0} \frac{2K_n}{|D_n|^{\alpha}} \).)

Since \( \alpha < \beta - 1/q \) and

\[ \mathbb{E}(K_n^q) \leq C^q |D_n|^{\beta q - 1}, \]

we have

\[ \|K_\alpha\|_{L^q} \leq \sum_{n \geq 0} \frac{2}{|D_n|^{\alpha}} \mathbb{E}(K_n^q)^{1/q} \leq \sum_{n \geq 0} \frac{2}{|D_n|^{\alpha}} |D_n|^{\beta - 1/q} < \infty. \]

Thus, \( K_\alpha \in L^q \), completing the proof.
Theorem (Kolmogorov criterion for rough paths)

Let \( q \geq 2, \beta > 1/q \) and assume for all \( s, t \in [0, 1] \) that

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|X_{s,t}|_{L^q(\omega)} \leq C|t - s|^\beta \quad \text{and} \quad |\bar{X}_{s,t}|_{L^{q/2}} \leq C|t - s|^{2\beta}.
\]

Then, for all \( \alpha \in [0, \beta - 1/q) \), there exist random variables \( K_\alpha \in L^q \) and \( \bar{K}_\alpha \in L^{q/2} \) such that

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|X_{s,t}| \leq K_\alpha(\omega)|t - s|^\alpha \quad \text{and} \quad |\bar{X}_{s,t}| \leq \bar{K}_\alpha(\omega)|t - s|^{2\alpha}.
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Kolmogorov Criterion for Rough Paths

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In particular, \( (X, \underline{X}) \in C^\alpha \) a.s. whenever \( \beta - 1/q > 1/3 \) and \( \alpha \in (1/3, \beta - 1/q) \).
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Then, for all \( \alpha \in [0, \beta - 1/q) \), there exist random variables \( K_\alpha \in L^q \) and \( \nabla K_\alpha \in L^{q/2} \) such that

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In particular, \((X, \nabla X) \in C^\alpha \) a.s. whenever \( \beta - 1/q > 1/3 \) and \( \alpha \in (1/3, \beta - 1/q) \).

For Brownian motion, we may take \( q \to \infty \), so \((W, \nabla W) \in C^\alpha \) for all \( \alpha < 1/2 \).
Problem: How to choose $W$ for Brownian rough path $W$?
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Solution 1: Itô integration

\[ W_{s,t}^{\text{Itô}} = \int_s^t W_{s,r} \otimes dW_r = \lim_{n \to \infty} \sum_{[t_{i-1}, t_i] \in D_n} W_{t_{i-1}} \otimes (W_{t_i} - W_{t_{i-1}}), \]

where $D_n$ is a sequence of partitions with mesh going to 0.
Problem: How to choose $\mathbb{W}$ for Brownian rough path $W$? 

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where $D_n$ is a sequence of partitions with mesh going to 0.

Lemma

For any $\alpha \in (1/3, 1/2)$, $T > 0$, $W = (W, \mathbb{W}^{\text{Itô}}) \in C^\alpha([0, T])$ a.s.
Solution 2: Stratonovich integration
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We define

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W_{s,t}^{\text{Strat}} = \int_s^t W_{s,r} \otimes \circ dW_r
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where \( |D_{n}| \to 0 \). One may show

\[ W_{s,t}^{\text{Strat}} = \int_{s}^{t} W_{s,r} \otimes \circ dW_{r} = W_{s,t}^{\text{Itô}} + \frac{1}{2} I(t - s) \]
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Note: Only Stratonovich integration gives geometric rough paths.
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- We adjoin an "area" term to our paths, which is computed with either Itô or Stratonovich stochastic integration.
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- We adjoin an "area" term to our paths, which is computed with either Itô or Stratonovich stochastic integration.

- The resulting *enhanced Brownian motion* does lead to stability in the solution (Itô-Lyons) map.
### Main Result

**Theorem**

*Let $M, m, X, P, W$ be as before.*
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Let $M, m, X, P, W$ be as before. Define $\hat{W} = (W, \hat{W})$, where

$$\hat{W}_{s,t} = W_{s,t}^{\text{Strat}} + (t - s)\frac{1}{2}(MC - CM^*),$$

and $C = \int_0^\infty e^{-Ms}e^{-M^*s}ds$. 
Main Result

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Let $M, m, X, P, W$ be as before. Define $\hat{W} = (W, \hat{W})$, where

$$\hat{W}_{s,t} = W_{s,t}^{\text{Strat}} + (t - s)\frac{1}{2}(MC - CM^*),$$

and $C = \int_{0}^{\infty} e^{-Ms} e^{-M^*s} ds$. Then, as $m \to 0$, $MX$ converges to $\hat{W}$ in $L^q$ and $\rho_\alpha$ for $q \geq 1$ and $\alpha \in (1/3, 1/2]$. 

Main Result

Theorem

Let $M, m, X, P, W$ be as before. Define $\hat{W} = (W, \hat{\hat{W}})$, where

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$$\|\rho_\alpha(MX(\omega), \hat{W}(\omega))\|_{L^q(\omega)} \to 0.$$
Comments

- If the magnetic field is zero, then $B = 0$, so $M$ is a diagonal (symmetric) matrix. Thus, $M = M^*$, $MC = CM^*$, and we have no extra area term in the limit.
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- A similar result holds for Itô integration due to the relation $W_{s,t}^{\text{Strat}} = W_{s,t}^{\text{Itô}} + \frac{1}{2} I(t - s)$. 
Proof Idea: Rescaling

Inspired by "Brownian scaling"

$$(W_{\lambda^2 t} : t \geq 0) \overset{D}{=} (\lambda W_t : t \geq 0)$$

let $m = \epsilon^2$ and $Y^\epsilon = P/\epsilon$ and solve

$$dY^\epsilon = -\epsilon^2 MY^\epsilon dt + \epsilon^{-1} dW$$

$$dX^\epsilon = \epsilon^{-1} Y^\epsilon dt.$$
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Also, setting \(\tilde{B}_t = \epsilon B_{\epsilon^{-2} t}\),

\[d\tilde{Y} = -MY\tilde{Y} dt + d\tilde{B}, \quad d\tilde{X} = \tilde{Y} dt,\]

so \((Y^\epsilon_t, \epsilon^{-1} X^\epsilon_t) = (\tilde{Y}_{\epsilon^{-2} t}, \tilde{X}_{\epsilon^{-2} t})\).
Proof Idea: Convergence

To prove $\rho_\alpha(X^\epsilon, X) \to 0$, it suffices to show that $X^\epsilon \to X$ pointwise and

$$\sup_\epsilon \rho_\beta(X^\epsilon, 0) < \infty,$$

where $\beta > \alpha$. This idea is called interpolation.
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where $\beta > \alpha$. This idea is called interpolation. In our case, we want to show pointwise convergence and

$$\sup_{0 < \epsilon \leq 1} E[||MX^\epsilon||^q_\alpha] < \infty, \quad \sup_{0 < \epsilon \leq 1} E \left[\left\| \int MX^\epsilon \otimes d(MX^\epsilon) \right\|^{q}_{2\alpha} \right] < \infty.$$
Proof Idea: Convergence II

By the Kolmogorov criterion for rough paths, to show

\[
\sup_{0 < \epsilon \leq 1} E[\| MX^\epsilon \|_q] < \infty, \quad \sup_{0 < \epsilon \leq 1} E \left[ \left\| \int MX^\epsilon \otimes d(MX^\epsilon) \right\|_2 \right] < \infty.
\]
Proof Idea: Convergence II

By the Kolmogorov criterion for rough paths, to show

$$\sup_{0<\epsilon \leq 1} \mathbb{E}[\| MX^\epsilon \|_\alpha^q] < \infty, \quad \sup_{0<\epsilon \leq 1} \mathbb{E} \left[ \left\| \int MX^\epsilon \otimes d(MX^\epsilon) \right\|_{2\alpha}^q \right] < \infty.$$ 

it suffices to show

$$\sup_{0<\epsilon \leq 1} \mathbb{E}[|X^\epsilon_{s,t}|^q] \lesssim |t-s|^{q/2} \quad \sup_{0<\epsilon \leq 1} \mathbb{E} \left[ \left\| \int_s^t X^\epsilon_s \otimes dX^\epsilon \right\|_q \right] \lesssim |t-s|^q.$$
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Since $X$ is a Gaussian process, we may take $q = 2$. 
Theorem

Let $M, m, X, P, W$ be as before. Define $\hat{W} = (W, \hat{W})$, where

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$$\|\rho_\alpha (MX(\omega), \hat{W}(\omega))\|_{L^q(\omega)} \to 0.$$
Proof of Pointwise Convergence I

Making a substitution in our system,

\[ dM^\epsilon = \epsilon^{-1} MY^\epsilon \, dt = dW - \epsilon dY^\epsilon, \]
Proof of Pointwise Convergence I

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By assumptions on \( M \), \( \sup_{0 \leq t < \infty} \mathbb{E}|\tilde{Y}_t^2| < \infty \), so \( \epsilon \tilde{Y}_{-2t}^\epsilon = \epsilon Y_t^\epsilon \to 0 \) in \( L^2 \) and all \( L^q \) \((q < \infty)\) uniformly.
Proof of Pointwise Convergence I

Making a substitution in our system,

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By assumptions on \( M \), \( \sup_{0 \leq t < \infty} \mathbb{E} |\tilde{Y}_t^2| < \infty \), so
\[ \epsilon \tilde{Y}_{\epsilon^{-2} t} = \epsilon Y_t^\epsilon \rightarrow 0 \text{ in } L^2 \text{ and all } L^q \ (q < \infty) \text{ uniformly}. \]

Letting \( \epsilon \rightarrow 0 \) gives \( M^\epsilon X \rightarrow W \) pointwise.
A stationary solution is one for which the distribution of the process $(X, dW)$ is invariant under time shifts.
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\tilde{Y}_{t}^{\text{stat}} = \int_{-\infty}^{t} e^{-M(t-s)} dW_s.
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\(\tilde{Y}_{t}^{\text{stat}}\) has law \(\nu \sim \mathcal{N}(0, C)\), for

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\]

By the ergodic theorem

\[
\frac{1}{t} \int_{0}^{t} f(Y^\epsilon_t) dt \rightarrow \int f(y) \nu(dy)
\]

in \(L^q\) for test functions \(f\).
Proof of Pointwise Convergence III

\[
\int_0^t MX_s^\epsilon \otimes d(MX^\epsilon)_s = \int_0^t MX_s^\epsilon \otimes dW_s - \epsilon \int_0^t MX_s^\epsilon \otimes dY_s^\epsilon
\]
Proof of Pointwise Convergence III

\[ \int_0^t MX_s^\varepsilon \otimes d(MX^\varepsilon)_s = \int_0^t MX_s^\varepsilon \otimes dW_s - \varepsilon \int_0^t MX_s^\varepsilon \otimes dY_s^\varepsilon \]

\[ = \int_0^t MX_s^\varepsilon \otimes dW_s - MX_t^\varepsilon \otimes (\varepsilon Y_t^\varepsilon) + \varepsilon \int_0^t d(MX^\varepsilon)_s \otimes Y_s^\varepsilon \]
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\[
\rightarrow \int_0^t W_s \otimes W_s - 0 + t \int M y \otimes y \nu(dy)
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Proof of Pointwise Convergence III

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\[ \rightarrow \int_0^t W_s \otimes W_s - 0 + t \int My \otimes y \nu(dy) \]

\[ = \int_0^t W_s \otimes dW_s + tMC \]
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\quad \quad \rightarrow \int_0^t W_s \otimes W_s - 0 + t \int My \otimes y\nu(dy)
\]

\[
= \int_0^t W_s \otimes dW_s + tMC
\]

\[
= W_{0,t} + t(MC - \frac{1}{2}I) = W_{0,t} + \frac{1}{2}(MC - CM^*)
\]
Last Remarks

- Details of uniform bounds follow from properties of $M$ and probability.
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There is no additional area term when driving noise is $\alpha$-Hölder continuous for $\alpha > 1/2$. 
Thank you
