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Math 110

Review Solutions

Week 1.

1. Show that the set of differentiable real-valued functions f on the interval $(-4,4)$ such that $f'(-1) = 3f(2)$ is a subspace of $\mathbb{R}^{(-4,4)}$.

   1. Show it contains 0. Let f_0 denote the zero function, where $f_0(x) = 0 \quad \forall x \in \mathbb{R}$. $0 = f_0'(-1) = 3f(2) = 0$, so f_0 is in this set.

   2. Show it's closed under addition and scalar multiplication. Let $c \in \mathbb{R}$ be a scalar, and let f, g be differentiable real-valued functions on $(-4,4)$ such that $f'(-1) = 3f(2)$ and $g'(-1) = 3g(2)$. Then $(f+cg)'(-1) = f'(-1) + cg'(-1) = 3f(2) + 3cg(2) = 3(f+cg)(2)$ by linearity of differentiation. Thus $f + cg$ is also in the set, so the set is indeed closed under addition, scalar multiplication, and contains zero, and is thus a subspace.

   (If we check this property first, we don't need to check (1.) separately, since that is included if $f = g = f_0$ and $c = 0$.)

2. Let $b \in \mathbb{R}$. Show that the set of continuous real-valued functions f on the interval $[0,1]$ such that $\int_0^1 f = b$ is a subspace of $\mathbb{R}^{[0,1]}$ if and only if $b = 0$.

   Check that this set contains f_0 (the zero function). $\int_0^1 f_0 = 0$, so if the set is a subspace, then necessarily $b = 0$.

   Now we show that if $b = 0$, the set is a subspace. Let $c \in \mathbb{R}$ be a scalar, and let f, g be continuous real-valued functions on $[0,1]$ such that $\int_0^1 f = 0$ and $\int_0^1 g = 0$. Then $\int_0^1 (f+cg) = \int_0^1 f + c\int_0^1 g = 0 + c0 = 0$ by linearity of integration. Thus $f + cg$ is also in the set, so the set is indeed closed under addition, scalar multiplication, and contains zero, and is thus a subspace when $b = 0$.

3. Is $\mathbb{R}^2$ a subspace of complex vector space $\mathbb{C}^2$ over $F = \mathbb{R}$? Over $F = \mathbb{C}$?

   $\mathbb{R}^2$ is a subspace of $\mathbb{C}^2$ over $\mathbb{R}$, since for $(x,y), (z,w) \in \mathbb{R}^2$ and scalar $c \in \mathbb{R}$, $(x,y) + c(z,w) = (x+cz, y+cw) \in \mathbb{R}^2$. $\mathbb{R}^2$ is not a subspace of $\mathbb{C}^2$ over $\mathbb{C}$, since it is not closed under scalar multiplication: for example, $(1,1) \in \mathbb{R}^2$, but $i(1,1) = (i,i) \notin \mathbb{R}^2$.

4. Give an example of a nonempty subset U of $\mathbb{R}^2$ such that U is closed under scalar multiplication, but U is *not* a subspace of $\mathbb{R}^2$.

   U is the union of the 1st and 3rd quadrant of $\mathbb{R}^2$, i.e. the set $\{(x,y) : xy \geq 0\}$. (In general, any *cone* will have the property of being closed under scalar multiplication, where a cone is a set bounded by subspaces of $\mathbb{R}^n$.)Even simpler, the *union of a set of subspaces* is closed under scalar multiplication, and not necessarily a subspace. For instance, U can be the union of the line $y = 2x$ and $y = x$.

5. Give an example of a nonempty subset U of $\mathbb{R}$ such that U is closed under addition, but U is *not* a subspace of $\mathbb{R}$.

   $U = \{1, 2, 3, \ldots\}$, the set of positive integers.

6. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

   $$U_1 + W = U_2 + W,$$

   then $U_1 = U_2$.

   Counterexample: $U_1 = \{0\}$ and $U_2 = W$.

Week 2.

1. Every subspace $U \subseteq V$ has a "complement" $W \subseteq V$ such that $U \oplus W = V$.

   It is important to say V is finite dimensional here, otherwise the problem becomes much harder.

   Let $u_1, \dots, u_k$ be a basis for U. We can complete that to a basis for V by the Replacement Theorem, so that now $u_1, \dots, u_k, v_1, \dots, v_m$ is a basis for V. I claim that $W = \operatorname{Span}(v_1, \dots, v_m)$ is the complement to U such that $U \oplus W = V$. This is easy to check: certainly $U + W = V$ as any $v \in V$ is in the span of the union of the basis vectors of U and W. Moreover, $U \cap W = \{0\}$ by linear independence of the vectors $\{u_1, \dots, u_k, v_1, \dots, v_m\}$.

2. Suppose $v_1, \dots, v_m$ are linearly independent in V and $w \in V$. Prove that $\dim \operatorname{Span}(v_1 + w, \dots, v_m + w) \leq m - 1$.

   This problem contains a typo!! It should be $\dim \operatorname{Span}(v_1 + w, \dots, v_m + w) \geq m - 1$. By plugging in $w = 0$, you could have seen that the statement of the problem is impossible.

   Suppose $\dim \operatorname{Span}(v_1 + w, \dots, v_m + w) \leq m - 2$. Then there are at least two vectors that can be written as a linear combination of the others. Without loss of generality, let them be $v_1 + w$ and $v_2 + w$. Thus we can write:

$$\begin{aligned} v_1 + w &= a_3(v_3 + w) + \cdots + a_m(v_m + w) \\ v_2 + w &= b_3(v_3 + w) + \cdots + b_m(v_m + w) \end{aligned}$$

   Isolate w to get:

$$\begin{aligned} w(1 - a_3 - \ldots - a_m) &= -v_1 + a_3 v_3 + \cdots + a_m v_m \\ w(1 - b_3 - \ldots - b_m) &= -v_2 + b_3 v_3 + \cdots + b_m v_m \end{aligned}$$

   Note that $1 - a_3 - \ldots - a_m \neq 0$ and $1 - b_3 - \ldots - b_m \neq 0$ since otherwise v_1 and v_3 would both be written as linear combinations of $v_3, \dots, v_m$.

   Thus:

$$\begin{aligned} w &= \frac{1}{1 - a_3 - \ldots - a_m}(-v_1 + a_3 v_3 + \cdots + a_m v_m) \\ w &= \frac{1}{1 - b_3 - \ldots - b_m}(-v_2 + b_3 v_3 + \cdots + b_m v_m) \end{aligned}$$

   Subtracting one from the other:

$$\frac{1}{1 - a_3 - \ldots - a_m}(-v_1 + a_3 v_3 + \cdots + a_m v_m) - \frac{1}{1 - b_3 - \ldots - b_m}(-v_2 + b_3 v_3 + \cdots + b_m v_m) = 0$$

   with not all coefficients zero (in particular, the coefficients of v_1 and v_2 are nonzero. This contradicts the linear independence of $v_1, \dots, v_m$. Thus $\dim \operatorname{Span}(v_1 + w, \dots, v_m + w) \geq m - 1$.

3. Find a number t such that

$$(3, 1, 4), \quad (2, -3, 5), \quad (5, 9, t)$$

   is not linearly independent in $\mathbb{R}^3$.

   We set up a matrix with the three vectors as columns and do row reduction to obtain zeros in the bottom row.

Note: We did not have to complete the matrix row reduction to get to this point! Eliminating the last row was enough to show that its rank is less than 3.

$$\begin{pmatrix} 3 & 2 & 5 \\ 1 & -3 & 9 \\ 4 & 5 & t \end{pmatrix} \to \begin{pmatrix} 3 & 2 & 5 \\ 1 & -3 & 9 \\ 0 & -6 & t-14 \end{pmatrix} \to \begin{pmatrix} 0 & 11 & -22 \\ 1 & -3 & 9 \\ 0 & -6 & t-14 \end{pmatrix} \to \begin{pmatrix} 0 & 11 & -22 \\ 1 & -3 & 9 \\ 0 & 0 & (11/6)(t-14)-22 \end{pmatrix}$$

We want $(11/6)(t-14)-22$ to equal 0 in order to eliminate the last row. Solving for t, we get $t = 26$.

4. Suppose $v_1, \dots, v_m$ is linearly independent in V and $w \in V$. Prove that if $v_1 + w, \dots, v_m + w$ is linearly dependent, then $w \in \text{Span}(v_1, \dots, v_m)$.

   Suppose $v_1 + w, \dots, v_m + w$ is linearly dependent. Then there are some $a_1, \dots, a_m$ that are not all 0 such that

   $$a_1(v_1 + w) + \cdots + a_m(v_m + w) = 0 \quad \to \quad a_1v_1 + \cdots + a_mv_m + w(a_1 + \cdots + a_m) = 0.$$

   Note that $a_1 + \cdots + a_m \neq 0$ since otherwise we would have $a_1v_1 + \cdots + a_mv_m = 0$, which contradicts the linear independence of $v_1, \dots, v_m$. Thus

   $$w = -\frac{1}{a_1 + \cdots + a_m}(a_1v_1 + \cdots + a_mv_m),$$

   so $w \in \text{Span}(v_1, \dots, v_m)$.

5. If $\{v_1, \dots, v_n\}$ *generate* or *span* vector space V, then is $v_1, \dots, v_n$ necessarily linearly independent? What can you say about the dimension of V?

   No, they are linearly independent if and only if they are a basis of V. For example, $\{1, 2\}$ span $\mathbb{R}$ but are not linearly independent. $\dim V \leq n$.

6. Explain why there does not exist a list of six polynomials that is linearly independent in $\mathbb{P}_4(\mathbb{R})$.

   $\dim \mathbb{P}_4(\mathbb{R}) = 5$, and by the Dimension Theorem, there cannot be 6 linearly independent vectors in a vector space of dimension 5.

7. Prove that $\mathbb{P}(\mathbb{R})$ is infinite-dimensional.

   Suppose it were finite dimensional with basis $p_1, \dots, p_n$. Let M be the maximal degree of $p_1, \dots, p_m$. Then $x^M \notin \text{Span}(p_1, \dots, p_n)$, so that contradicts that they are a basis for $\mathbb{P}(\mathbb{R})$. Thus it must be infinite dimensional.

8. Suppose $p_0, \dots, p_m$ are polynomials in $\mathbb{P}_m(\mathbb{R})$ such that $p_j(2) = 0$ for each j. Prove that $p_0, \dots, p_m$ is not linearly independent in $\mathbb{P}_m(\mathbb{R})$.

Week 3.

1. Let

   $$S = \text{Span}\left((4, 2, -1, 1), (2, 6, -7, 1), (2, -4, 6, 0), (-1, 0, -2, 3), (1, 2, -3, 2)\right)$$

   be a subspace of $\mathbb{R}^4$. Find a basis for S.

   This is a problem to check that you can easily do row reduction on matrices and get all the information you need from that result.

We set up a matrix using the vectors $\{v_1, \dots, v_5\}$ in S as columns so that the rank of the matrix can tell us how many of these columns are linearly independent, and so that the columns that contain the pivots can tell us which of the vectors we can choose as a basis.

$$\begin{pmatrix} \overset{v_1}{4} & \overset{v_2}{2} & \overset{v_3}{2} & \overset{v_4}{-1} & \overset{v_5}{1} \\ 2 & 6 & -4 & 0 & 2 \\ -1 & -7 & 6 & -2 & -3 \\ 1 & 1 & 0 & 3 & 2 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 3 & 2 \\ 0 & -2 & 2 & -13 & -7 \\ 0 & 4 & -4 & -6 & -2 \\ 0 & -6 & 6 & 1 & -1 \end{pmatrix} \to \begin{pmatrix} 1 & 1 & 0 & 3 & 2 \\ 0 & 2 & -2 & 13 & 7 \\ 0 & 0 & 0 & -32 & -16 \\ 0 & 0 & 0 & 40 & 20 \end{pmatrix} \to$$

$$\begin{pmatrix} \overset{v_1}{1} & \overset{v_2}{1} & \overset{v_3}{0} & \overset{v_4}{3} & \overset{v_5}{2} \\ 0 & 2 & -2 & 13 & 7 \\ 0 & 0 & 0 & 2 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}$$

We take the vectors v_1, v_2, v_4 corresponding to the columns containing the pivots as the basis for S.

2. Let V be an infinite dimensional vector space. Show that V contains an infinite set of linearly independent vectors.

   We prove this by contradiction. (In general, to show something is infinite, the easiest way to prove it is to suppose it is finite of maximal size n, and then show that we can add another element to it, which contradicts the maximality of n.)

   Suppose the largest set of linearly independent vectors in V is $v_1, \dots, v_n$ of size n. Since V is infinitely dimensional, $\{v_1, \dots, v_n\}$ cannot span V. Thus there is some vector $w \notin \text{Span}(v_1, \dots, v_n)$. Thus $\{v_1, \dots, v_n, w\}$ is also linearly independent, which contradicts the maximality assumption. Thus V must contain an infinite set of linearly independent vectors.

3. Find an example of a linear transformation $T : \mathbb{R}^3 \to \mathbb{R}^2$ such that $(1, 1, 1) \in N(T)$ and T is onto.

   Let us choose a basis b_1, b_2, b_3 for $\mathbb{R}^3$ and T such that $T(b_1) = 0, T(b_2) = (1, 0)$, and $T(b_3) = (0, 1)$. The easiest is to let $b_1 = (1, 1, 1)$, and so we can choose $b_2 = (1, 0, 0)$ and $b_3 = (0, 1, 0)$ for a valid basis. Then for $v = a_1b_1 + a_2b_2 + a_3b_3$, $T(v) = (a_2, a_3, 0)$. This is sufficient to define T. (Of course, you can also write T in terms of the standard basis for $\mathbb{R}^3$, but that is not necessary and more time consuming.)

   We show these choices satisfy the conditions. The set $\{T(b_1), T(b_2)\} = \{(1, 0), (0, 1)\}$ is linearly independent since these vectors are the standard basis for $\mathbb{R}^2$. Thus $\dim R(T) = 2$ so T is onto, and $(1, 1, 1) \in N(T)$, as desired.

4. Find an example of a linear transformation $T : \mathbb{R}^2 \to \mathbb{R}^3$ such that $(1, 1, 1), (-1, 2, 0) \in R(T)$. What is $N(T)$?

   Let T be a linear transformation such that $T(1, 0) = (1, 1, 1)$ and $T(0, 1) = (-1, 2, 0)$. Then $T(x, y) = (x - y, x + 2y, x)$. By the dimension theorem, $\dim N(T) = 2 - \dim R(T) = 0$, so $N(T) = \{0\}$.

5. Suppose $T \in \mathcal{L}(V, W)$ and $v_1, \dots, v_m$ is a list of vectors in V such that $\{Tv_1, \dots, Tv_m\}$ is a linearly independent list in W. Prove that $v_1, \dots, v_m$ is linearly independent.

   Suppose they are linearly dependent, so there are some $a_1, \dots, a_m$ that are not all 0 such that $a_1v_1 + \dots + a_mv_m = 0$. Then $T(a_1v_1 + \dots + a_mv_m) = 0$, but by linearity this means that $a_1T(v_1) + \dots + a_mT(v_m) = 0$, which contradicts that $T(v_1), \dots, T(v_m)$ is linearly independent. Thus $v_1, \dots, v_m$ must be linearly independent.

6. Give an example of a transformation that satisfies homogeneity (i.e. $cf(v) = f(cv)$ for a scalar c and vector v) but is not linear. And additivity (i.e. $f(v+w) = f(v) + f(w)$ for vectors v, w) but is not linear.

   $V = \mathbb{R}$, let $T(x) = 1/x$ for $x \neq 0$ and $T(0) = 0$: this is homogeneous but not linear.

   $V = \mathbb{N}$, the set of natural numbers, and T is the identity map. This satisfies additivity, but is not linear. Also V is not a vector space.

7. Find a linear transformation $T : \mathbb{R}^4 \to \mathbb{R}^4$ such that $R(T) = N(T)$.

   Let $(1,0,0,0), (0,1,0,0) \in N(T)$. Let $T(0,0,1,0) = (1,0,0,0)$ and $T(0,0,0,1) = (0,1,0,0)$. That is, $T(x,y,z,w) = (z,w,0,0)$. Since $\{(1,0,0,0), (0,1,0,0), (0,0,1,0), (0,0,0,1)\}$ is a basis for $\mathbb{R}^4$, $R(T)$ is the span of T applied to these 4 vectors, so $R(T) = \text{Span}((1,0,0,0), (0,1,0,0))$, and $N(T) = \text{Span}((1,0,0,0), (0,1,0,0))$, so $R(T) = N(T)$.

8. Compute the null space and range of linear transformation $T : \mathbb{P}_4(\mathbb{R}) \to \mathbb{P}_4(\mathbb{R})$, defined by:

$$\begin{aligned} T(1) &= 4 + 2x + 2x^2 - x^3 + x^4 \\ T(x) &= 2 + 6x - 4x^2 + 2x^4 \\ T(x^2) &= -1 - 7x + 6x^2 - 2x^3 - 3x^4 \\ T(x^3) &= 1 + x + 3x^3 + 2x^4 \end{aligned}$$

   (Hint: use a previous exercise to avoid computations)

   With the standard basis β for $\mathbb{P}_4$, the matrix representation for T has columns given by $[T(\beta_i)]_\beta$, which is the matrix

$$[T]_\beta^\beta = \begin{pmatrix} 4 & 2 & -1 & 1 \\ 2 & 6 & -7 & 1 \\ 2 & -4 & 6 & 0 \\ -1 & 0 & -2 & 3 \\ 1 & 2 & -3 & 2 \end{pmatrix}$$

   ..to be continued!

9. (a) Let $U = \{p \in \mathbb{P}_4(\mathbb{R}) : p(2) = p(5)\}$. Find a basis of U.

   $\mathbb{P}_4(\mathbb{R})$ has basis $\{1, x, x^2, x^3, x^4\}$. For $p(x) = a_0 + a_1x + a_2x^2 + a_3x^3 + a_4x^4$, $p(2) = a_0 + 2a_1 + 4a_2 + 8a_3 + 16a_4$ and $p(5) = a_0 + 5a_1 + 25a_2 + 125a_3 + 625a_4$. Setting these equal, we obtain $3a_1 + 21a_2 + 117a_3 + 609a_4 = 0$. This equation has 4 free variables a_0, a_2, a_3, a_4, and $a_1 = -\frac{1}{3}(21a_2 + 117a_3 + 609a_4)$.

   ...to be continued!

   (b) Extend the basis to a basis of $\mathbb{P}_4(\mathbb{R})$.

   (c) Find a subspace W of $\mathbb{P}_4(\mathbb{R})$ such that $\mathbb{P}_4(\mathbb{R}) = U \oplus W$.

Week 5.

1. Suppose $D \in \mathcal{L}(\mathbb{P}(\mathbb{R}))$ is the differentiation map and $T \in \mathcal{L}(\mathbb{P}(\mathbb{R}))$ is the multiplication by x^2 map. Show that $TD \neq DT$. Let $Q \in \mathcal{L}(\mathbb{P}(\mathbb{R}))$ be the integration map. Is $DQ = QD$?

   We show this holds for $\mathbb{P}(\mathbb{R})$ for any integer n. Let $\beta = \{1, x, \ldots\}$ be the standard basis for $\mathbb{P}(\mathbb{R})$. After looking at some examples, it's easy to see that $D(\beta_i) = i\beta_{i-1}$ for $i \geq 1$ and $D(\beta_0) = D(1) = 0$. Also, $T(\beta_i) = \beta_{i+2}$ for $i \geq 0$.

Thus we can check that (when we restrict to dimension n)

$$[DT]_\beta^\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 2 & 0 & \cdots & 0 \\ 0 & 3 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad [TD]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 2 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}$$

Thus TD and DT are not equal. We could also show they are not equal by showing their action on the $\{\beta_i\}$ elements is not equal.

Also, $Q(\beta_i) = \beta_{i+1}/i$ for $i \geq 1$ and $Q(\beta_0) = Q(1) = x = \beta_1$. Thus

$$[Q]_\beta^\beta = \begin{pmatrix} 0 & 0 & \cdots & 0 \\ 1 & 0 & \cdots & 0 \\ 0 & 1/2 & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix}, \qquad [QD]_\beta^\beta = \begin{pmatrix} 1 & 0 & \cdots & 0 & \\ 0 & 1 & \cdots & 0 & \\ 0 & 0 & \cdots & 0 & \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{pmatrix}, \qquad [DQ]_\beta^\beta = \begin{pmatrix} 0 & 0 & 0 & \cdots & 0 \\ 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 0 \end{pmatrix}.$$

So in fact $QD = I$, but $DQ \neq I$! This is an interesting example of two non-invertible linear transformations where a *left* inverse exists, but not a *right* inverse. Make note of this example!

2. Suppose transformation $T : V \to W$ has an empty null space. Can T be a linear transformation?

   No, because the nullspace must always contain 0, and can never be *empty*. Empty means $\emptyset$, which is not the same as $\{0\}$.

3. Let $T, S \in \mathcal{L}(V, W)$. Suppose $T + S$ is one to one. Prove or show a counterexample that T and S must also be one to one.

   Counterexample. Let $V = \mathbb{R}^2$ and $W = \mathbb{R}^2$. Let $T(x, y) = (x, 0)$, $S(x, y) = (0, y)$.

4. Let $T, S \in \mathcal{L}(V, W)$. Suppose $T + S$ is onto. Prove or show a counterexample that T and S must also be onto.

   Counterexample. Let $V, W = \mathbb{R}$, and let T be the 0 map and S the identity map.

5. Suppose V is finite-dimensional. Prove that every linear map on a subspace of V can be extended to a linear map on V. I.e. show that if U is a subspace of V and $S \in \mathcal{L}(V, W)$, then there exists $T \in \mathcal{L}(V, W)$ such that $Tu = Su$ for all $u \in U$.

   Let $u_1, \ldots, u_k$ be a basis for U, so we can extend that to a basis for V by taking $u_1, \ldots, u_k, v_1, \ldots, v_m$. Define $T \in \mathcal{L}(V, W)$ by $T(u_i) = S(u_i)$ and $T(v_i) = 0$. (This is sufficient for the definition of T since its action is completely determined by its action on the basis elements. However, you do need to say a couple sentences to prove T satisfies the conditions.)

6. Suppose V is finite-dimensional with $\dim V > 0$ and suppose W is infinite-dimensional. Prove that $\mathcal{L}(V, W)$ is infinite-dimensional.

   Suppose $\mathcal{L}(V, W)$ is finite-dimensional with basis $\{T_1, \ldots, T_n\}$. $R(T_i) \leq \dim V$. Since W is infinite dimensional, we can find some $w \in W$ such that $w \notin (R(T_1) \cup \cdots \cup R(T_n))$. Let $\{v_1, \ldots, v_k\}$ be a basis for V, and define $T \in \mathcal{L}(V, W)$ by $T(v_i) = w$. Then T is a new linear transformation that is not in the span of $T_1, \ldots, T_n$, which contradicts that they are a basis for $\mathcal{L}(V, W)$. Thus this vector space must be infinite-dimensional.

   (There are several ways to solve this problem, this is one example.)

Week 4.

1. Show that for each $q \in \mathbb{P}(\mathbb{R})$ there exists $p \in \mathbb{P}(\mathbb{R})$ such that $((x^2+5x+7)p)'' = q$.

   We show that for any integer n, the map $T \in \mathcal{L}(\mathbb{P}_n(\mathbb{R}))$ defined by $T(p(x)) = ((x^2+5x+7)p)''$ is onto. This is the composition of two maps: $U(p(x)) = (x^2+5x+7)p(x)$ composed with $S(r(x)) = r''(x)$. It is easy to show $S : \mathbb{P}_{n+2}(\mathbb{R}) \to \mathbb{P}_n(\mathbb{R})$ is onto and $U : \mathbb{P}_n(\mathbb{R}) \to \mathbb{P}_{n+2}(\mathbb{R})$ is one-to-one (I won't show this here, but you should know how to do this). Thus $T = SU$ is onto (you did that exercise in the homework). Since this is true for all n, the map on $\mathbb{P}(\mathbb{R})$ has the same property.

2. Suppose $D \in \mathcal{L}(\mathbb{P}_3(\mathbb{R}), \mathbb{P}_2(\mathbb{R}))$ is the differentiation map. Find a basis of $\mathbb{P}_3(\mathbb{R})$ such that the matrix of D with respect to these bases is
$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}.$$

   For this problem it is important to specify the basis of $\mathbb{P}_2(\mathbb{R})$! Since the problem didn't say, we can choose that to be the standard basis $\beta = \{1, x, x^2\}$. Thus the above matrix is $[D]_\gamma^\beta$ for some γ which we compute below.

   Let $\gamma_1, \ldots, \gamma_4$ be a basis of $\mathbb{P}_3(\mathbb{R})$. Let $\gamma_1 = a_0 + a_1x + a_2x^2 + a_3x^3$, $\{b_i\}$ for γ_2, $\{c_i\}$ for γ_3, and $\{d_i\}$ for γ_4. Then
$$\begin{aligned} D(\gamma_1) &= a_1 + 2a_2x + 3a_3x^2 = 1 \\ D(\gamma_2) &= b_1 + 2b_2x + 3b_3x^2 = x \\ D(\gamma_3) &= c_1 + 2c_2x + 3c_3x^2 = x^2 \\ D(\gamma_4) &= d_1 + 2d_2x + 3d_3x^2 = 0 \end{aligned}$$
   according to the basis β.

   Thus $a_1 = 1$, $b_2 = 1/2$, $c_3 = 1/3$, and all other terms for $1 \le i \le 3$ are 0. Since γ_4 cannot equal 0 as it is one of the basis vectors, we must let $d_0 \neq 0$. Thus we can set $\gamma_1 = x$, $\gamma_2 = x^2/2$, $\gamma_3 = x^3/3$, and $\gamma_4 = 1$.

3. Let $T : \mathbb{R}^2 \to \mathbb{R}^3$ be defined by $T(x,y) = (x+y, x-y, 2x+y)$, and let β be the basis $\{(-1,2),(3,4)\}$ on $\mathbb{R}^2$ and let $\mathbb{R}^3$ have the standard basis γ. Denote the standard basis on $\mathbb{R}^2$ by σ. Find the change of basis matrix $[\psi]_\sigma^\gamma$. Find $[T]_\beta^\gamma$.

   This problem has a typo, it should be: $[\psi]_\sigma^\beta$.

   It is easiest to find:
$$[\psi^{-1}]_\beta^\sigma = \begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix}.$$
   Thus
$$[\psi]_\sigma^\beta = ([\psi^{-1}]_\beta^\sigma)^{-1} = -\frac{1}{10}\begin{pmatrix} 4 & -3 \\ -2 & -1 \end{pmatrix}.$$
   Also,
$$[T]_\sigma^\gamma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}.$$
   Then
$$[T]_\beta^\gamma = [T]_\sigma^\gamma[\psi^{-1}]_\beta^\sigma = \begin{pmatrix} 1 & 1 \\ 1 & -1 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} -1 & 3 \\ 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 7 \\ -1 & -1 \\ 0 & 10 \end{pmatrix}.$$

4. Suppose V is finite dimensional and $\dim V > 1$. Prove that the set of noninvertible operators on V is not a subspace of $L(V)$. Prove that the set of noninvertible operators on V is a subspace of $\mathcal{L}(V)$ when $\dim V = 1$.

   Discussed in section. The set of noninvertible operators is not closed under addition. If $\dim V > 1$, choose a nonzero noninvertible operator T, say projection onto the first basis vector v_1, represented by the matrix with a 1 in the upper left corner and zeros everywhere else. Now let $S = I - T$, represented by the identity matrix minus $[T]$. S is noninvertible as well (the first basis vector v_1 is in $N(S)$), but $T + S = I$, which is invertible.

   When $\dim V = 1$, the only vector contained in the set of noninvertible operators is the 0 transformation. The set $\{0\}$ is a subspace. (See proof of Quiz problem 3)

5. Prove that if $T \in \mathcal{L}(V, W)$ is injective and surjective, then T has an inverse in $\mathcal{L}(V, W)$.

   Let $S : W \to V$ be such that if $T(v) = w$, then $S(w) = v$. Since T is surjective (onto), every $w \in W$ has some v such that $T(v) = w$, and since T is injective (one-to-one), every $w \in W$ has a *unique* $v \in V$ such that $T(v) = w$. Thus $S(w)$ is well-defined. We show that S is a linear transformation.

   1. $T(0) = 0$, so $S(0) = 0$.

   2. Let $w, u \in W$ and let c be a scalar. Let $v, t \in V$ be such that $T(v) = w$ and $T(t) = u$. Then $T(v + ct) = T(v) + cT(t) = w + cu$, so $S(w + cu) = v + ct = S(w) + cS(t)$. Thus S is indeed a linear transformation.

6. Let $T \in \mathcal{L}(\mathbb{P}_n(\mathbb{R}), \mathbb{R})$ be defined by $T(f(x)) = f(1)$. Find the matrix representation of T with respect to the standard basis β of $\mathbb{P}_n(\mathbb{R})$. Let $S \in \mathcal{L}(\mathbb{R}^m, \mathbb{R})$ be such that $[S]_\gamma^\sigma = [T]_\beta^\sigma$ where γ is the standard basis of $\mathbb{R}^m$ and σ is the standard basis of $\mathbb{R}$. What is m and what is S?

   Let $R \in \mathcal{L}(D_{m\times m}(\mathbb{R}), \mathbb{R})$ where $D_{m\times m}(\mathbb{R})$ is the vector space of diagonal $m \times m$ matrices. Suppose $[R]_\alpha^\sigma = [T]_\beta^\sigma$ where α is the standard basis of $D_{m\times m}(\mathbb{R})$. What is R?

   First part: $\beta = \{1, x, x^2, \ldots, x^n\}$. $T(1) = 1$, $T(x) = 1$, and we can see that $T(x^i) = 1$ for all i. Thus $[T]_\beta^\beta = [1, \ldots, 1]$. Naturally, m must equal $n + 1$ since γ must have the same dimension as β (this was supposed to be an "obvious" question). Since $[S]_\gamma^\sigma = [1, \ldots, 1]$, we have that $S(e_i) = 1$ where $\gamma = \{e_1, \ldots, e_{n+1}\}$ is the standard basis of $\mathbb{R}^{n+1}$. Thus $S(x_1, \ldots, x_{n+1}) = x_1 + \cdots + x_{n+1}$.

   Second part. Just like the above, if $x_1, \ldots, x_{n+1}$ are the entries on the diagonal of a matrix $A \in D_{m\times m}(\mathbb{R})$, then $R(A) = x_1 + \ldots + x_{n+1}$.

Week 6.

1. What is the dual basis of the standard basis $e_1, \ldots, e_n$ of F^n?

   We find $\{f_i\} \in \mathcal{L}(F^n, F)$ such that $f_i(e_j) = \begin{cases} 1 & i = j \\ 0 & i \neq j \end{cases}$. To solve for f_i, we have the system of n equations $\{f_i(e_0) = 0, \ldots, f_i(e_{i-1}) = 0, f_i(e_i) = 1, f_i(e_{i+1} = 0, \ldots\}$, which has the solution $f_i(x_1, \ldots, x_n) = x_i$, the projection onto the i'th coordinate. (You should probably write this out in more detail, but I won't here since computations are simple.)

2. Let $V = \mathbb{P}_3(\mathbb{R})$ have the standard ordered basis $\beta = \{1, x, \ldots, x^3\}$. What is the dual basis of β in V^*? Write the dual basis as an *operator* on $f(x) \in \mathbb{P}_3(\mathbb{R})$.

   Just like in the previous problem, when an element $p(x) = a_0 + \cdots + a_3x^3 \in \mathbb{P}_3$ is written as a vector $(a_0, \ldots, a_3)$, we get that f_i is the projection map onto the i'th coordinate a_i. Thus each f_i picks out the i'th coefficient of $p(x)$. As an operator, this can be written as taking the i'th derivative and scaling by $\frac{1}{i!}$, that is, $f_i(p(x)) = \frac{1}{i!}p^{(i)}(0)$.

Note: we need $p^{(i)}(0)$ so that we can isolate the coefficient that corresponds to x^i.

3. Let $V = \mathbb{P}_2(\mathbb{R})$ with some ordered basis γ. Suppose the basis of V^* is

$$\begin{aligned}\phi_1(f(t)) &= \int_0^1 f(t)dt \\ \phi_2(f(t)) &= f'(1) \\ \phi_3(f(t)) &= f(0)\end{aligned}$$

Find the ordered basis that corresponds to the ordered basis ϕ_1, ϕ_2, ϕ_3.

Let $\gamma_1 = a_0 + a_1x + a_2x^2, \gamma_2 = b_0 + b_1x + b_2x^2, \gamma_3 = c_0 + c_1x + c_2x^2$ be the ordered basis γ. Since we know $\phi_i(\gamma_j) = \delta_{ij}$, we get the following 9 equations:

$$\begin{aligned}\phi_1(\gamma_1) &= a_0 + a_1/2 + a_2/3 = 1 & \phi_2(\gamma_1) &= a_1 + 2a_2 = 0 & \phi_3(\gamma_1) &= a_0 = 0 \\ \phi_1(\gamma_2) &= b_0 + b_1/2 + b_2/3 = 0 & \phi_2(\gamma_2) &= b_1 + 2b_2 = 1 & \phi_3(\gamma_2) &= b_0 = 0 \\ \phi_1(\gamma_3) &= c_0 + c_1/2 + c_2/3 = 0 & \phi_2(\gamma_3) &= c_1 + 2c_2 = 0 & \phi_3(\gamma_3) &= c_0 = 1\end{aligned}$$

Solving for the a_i's, b_i's, and c_i's, we get:

$$\gamma_1 = 3x - \frac{3}{2}x^2 \qquad \gamma_2 = -\frac{1}{2}x + \frac{3}{4}x^2 \qquad \gamma_3 = 1 - 3x + \frac{3}{2}x^2$$

4. Define $T : \mathbb{P}_1(\mathbb{R}) \to \mathbb{R}^3$ by $T(p(x)) = (p(0), p(1), p(2))$. Let β, γ be the standard ordered bases for $\mathbb{P}_1(\mathbb{R})$ and $\mathbb{R}^2$, respectively. Compute $[T^t]_{\gamma^*}^{\beta^*}$ directly. Confirm that it equals $([T]_\beta^\gamma)^t$.

Let $f_1, f_2 \in \mathcal{L}(\mathbb{P}_1, \mathbb{R})$ be the basis β^*, and let $g_1, g_2, g_3 \in \mathcal{L}(\mathbb{R}^3, \mathbb{R})$ be the basis γ^*. As before, f_i and g_i are the projection maps onto the i'th coordinate since β and γ are both standard bases.

Column i of $[T^t]_{\gamma^*}^{\beta^*}$ is $[T^t(g_i)]_{\beta^*}$. We compute:

$$\begin{aligned}T^t(g_1)(p(x)) &= (g_1 \circ T)(p(x)) = g_1(p(0), p(1), p(2)) = p(0) \\ T^t(g_2)(p(x)) &= (g_2 \circ T)(p(x)) = g_2(p(0), p(1), p(2)) = p(1) \\ T^t(g_3)(p(x)) &= (g_3 \circ T)(p(x)) = g_3(p(0), p(1), p(2)) = p(2)\end{aligned}$$

If we write $p(x) = a_0 + a_1x$, then $p(0) = a_0$, $p(1) = a_0 + a_1$, and $p(2) = a_0 + 2a_1$. Recall that $f_i(p(x)) = a_i$. Thus

$$\begin{aligned}T^t(g_1)(p(x)) &= f_1(p(x)) \\ T^t(g_2)(p(x)) &= f_1(p(x) + f_2(p(x)) \\ T^t(g_3)(p(x)) &= f_1(p(x)) + 2f_2(p(x)).\end{aligned}$$

Thus

$$[T^t(g_1)]_{\beta^*} = \begin{pmatrix}1\\0\end{pmatrix}, \qquad [T^t(g_2)]_{\beta^*} = \begin{pmatrix}1\\1\end{pmatrix}, \qquad [T^t(g_3)]_{\beta^*} = \begin{pmatrix}1\\2\end{pmatrix}$$

and so

$$[T^t]_{\gamma^*}^{\beta^*} = \begin{pmatrix}1 & 1 & 1\\0 & 1 & 2\end{pmatrix}.$$

Now,

$$[T]_\beta^\gamma = \begin{pmatrix}1 & 0\\1 & 1\\1 & 2\end{pmatrix},$$

which is indeed the transpose of $[T^t]_{\gamma^*}^{\beta^*}$.

5. Suppose W is finite-dimensional and $T \in \mathcal{L}(V, W)$. Prove that $T^t = 0$ if and only if $T = 0$.

   Suppose $T = 0$, i.e. $T(v) = 0 \quad \forall v \in V$. By definition, $T^t(\phi) = \phi \circ T = \phi \circ 0 = 0$, so $T^t = 0$.

   Suppose $T^t = 0$. Then $T^t(\phi) = 0 = \phi \circ T \quad \forall \phi \in W^*$. Suppose T is not 0, so there is some $v_1 \in V$ such that $T(v_1) = w_1 \neq 0$ for $w_1 \in W$. Let $w_1, \ldots, w_n$ be a basis for W and let $f_1, \ldots, f_n$ be the corresponding basis for W^*. Then $f_1(w_1) = 1$. We plug this into T^t to get $T^t(f_1) = f_1 \circ T$. Recall that $T^t(f_1) \in V^*$, so this linear functional equals 0 if and only if $T^t(f_1)(v) = 0 \quad \forall v \in V$. But $T^t(f_1)(v_1) = f_1(T(v_1)) = f_1(w_1) = 1$, a contradiction. Thus $T = 0$ if $T^t = 0$.

   Thus $T^t = 0$ if and only if $T = 0$.

Extra (I'm not writing solutions for these.)

1. Prove or give a counterexample: if U_1, U_2, W are subspaces of V such that

$$V = U_1 \oplus W \qquad \text{and} \qquad V = U_2 \oplus W,$$

then $U_1 = U_2$.

2. Suppose the list v_1, v_2, v_3, v_4 is linearly independent in V. Prove that the list

$$v_1 - v_2, v_2 - v_3, v_3 - v_4, v_4$$

is also linearly independent.

3. Prove or give a counterexample. If $v_1, \ldots, v_m$ and $w_1, \ldots, w_m$ are linearly independent lists of vectors in V, then $v_1 + w_1, \ldots, v_m + w_m$ is linearly independent.

4. Suppose $v_1, \ldots, v_m$ is linearly independent in V and $w \in V$. Prove that $v_1, \ldots, v_m, w$ is linearly independent if and only if $w \notin \text{Span}(v_1, \ldots, v_m)$.

5. Let $T_1, \ldots, T_n \in \mathcal{L}(\mathbb{P}(\mathbb{R}), \mathbb{P}(\mathbb{R}))$, where $T_i(f(x))$ is the i'th derivative of f. Show that $T_1, \ldots, T_n$ is linearly independent.

6. Let $T_1, \ldots, T_n \in \mathcal{L}(\mathbb{P}(\mathbb{R}), \mathbb{P}(\mathbb{R}))$, where $T_i(f(x)) = x^i f(x)$. Show that $T_1, \ldots, T_n$ is linearly independent.

7. Suppose $S, T \in L(V)$. Prove ST is invertible if and only if both S and T are invertible.

8. Suppose W is finite dimensional and $T_1, T_2 \in L(V, W)$. Prove that $N(T_1) = N(T_2)$ if and only if there exists an invertible operator $S \in L(W)$ such that $T_1 = S(T_2)$.

9. Suppose V finite dimensional and $S, T \in \mathcal{L}(V)$. Prove that $ST = I$ if and only if $TS = I$.

10. Show that V and $\mathcal{L}(F, V)$ are isomorphic vector spaces.

11. Let c, d be scalars, and let V be a vector space of dimension $n > 0$ with ordered basis $\beta_1, \ldots, \beta_n$. Let $f_1, \ldots, f_n$ be the corresponding ordered basis of V^*. Let $u = c\beta_1 + \beta_2 \in V$, $v = \beta_1 + d\beta_2 \in V$, and $g = cf_1 + f_2 \in V^*$, $h = f_1 + df_2 \in V^*$. Show g and h must be linearly independent if u and v are.