Nonabelian Hodge theory in Characteristic $p > 0$

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Hodge theory in characteristic zero

Let $X/C$ be a smooth projective algebraic variety and let $X_{an}$ be its associated analytic space. Then there are canonical isomorphisms:

$$H^n(X_{an}, \mathbb{C}) \cong H^n(X, \Omega_X^{\cdot}/C) \cong \bigoplus_{i+j=n} H^i(X, \Omega^j_X/C).$$

The grading can be thought of as an action of the algebraic group $\mathbb{C}^*$ on $H^n(X_{an}, \mathbb{C})$. Simpson’s nonabelian analog attempts to construct such an action on the fundamental group of $X_{an}$, or at least on the category of its linear representations.
The Riemann-Hilbert correspondence

Equivalences of $\otimes$-categories, compatible with cohomology:

\[ \text{Rep}(\pi_1(X_{\text{an}})) \equiv LC(X_{\text{an}}, \mathbb{C}) \equiv MIC(X/\mathbb{C}) \]

\[ V \mapsto (V \otimes_{\mathbb{C}} \mathcal{O}_{X_{\text{an}}}, \text{id} \otimes d), \quad (E, \nabla) \mapsto E_{an}^\nabla \]
\[ H^n(X_{\text{an}}, V) \cong H^n(X, E \otimes \Omega^j_{X/\mathbb{C}}) \]

**Hodge theory:** If $V$ is **constant**

\[ H^n(X_{\text{an}}, V) \cong \bigoplus_{i+j=n} H^i(X, E \otimes \Omega^j_{X/\mathbb{C}}). \]

- This isomorphism is not algebraic and involves complex conjugation.

- **Fails** if $V$ is not constant.
Simpson’s’ nonabelian Hodge theory

A Higgs bundle (or sheaf) on $X/\mathbb{C}$ is any of the following equivalent sets of data

- $\theta: E' \to E' \otimes \Omega^1_{X/\mathbb{C}}, \mathcal{O}_X$-linear, such that $\theta \wedge \theta = 0$.  

- Action of $S^* T_{X/\mathbb{C}}$ on $E'$

- Sheaf $\tilde{E}'$ on cotangent space $T^*_{X/\mathbb{C}}$ of $X$.

$\mathbb{C}^*$ acts on $T^*_{X/\mathbb{C}}$, hence on $HIG(X/\mathbb{C})$.

**Theorem.** [Simpson et al] There is a natural equivalence of $\otimes$-categories (modulo a stability condition), compatible with cohomology:

$$\text{Rep}(\pi_1(X_{an})) \equiv \text{MIC}(X/\mathbb{C}) \equiv HIG(X/\mathbb{C})$$

$$H^n(X_{an}, V) \cong H^n(X, E \otimes \Omega^*_{X/\mathbb{C}}) \cong H^n(X, E' \otimes \Omega^*_{X/\mathbb{C}})$$

if $(E', \theta')$ corresponds to $(E, \nabla)$.
Proof: Uses PDE, harmonic metrics.

Warnings:

• The correspondence \((E, \nabla) \mapsto (E', \theta)\) is not algebraic or even holomorphic.

• Complex conjugation enters, as in the Hodge decomposition.
**Characteristic $p > 0$: background and introduction**

Fix: $X/S$ smooth morphism of noetherian schemes in characteristic $p > 0$, and its relative Frobenius diagram.

\[
\begin{array}{ccc}
X & \xrightarrow{F} & X' \\
\downarrow & & \downarrow \pi \\
S & \rightarrow & S
\end{array}
\]

$MIC(X/S)$ makes sense and is important. What about Riemann-Hilbert?
Classical Cartier descent

**Theorem. [Cartier]** $E \mapsto E^{\nabla}$ gives an equivalence of $\otimes$-categories:

$$MIC_{\psi=0}(X/S) \equiv MOD(X'/S)$$

$\psi=0$ means, for those objects for which the $p$-curvature vanishes.

$p$-curvature:

$$\psi: T_{X'/S} \to F_\ast \text{End}_{O_X}(E, \nabla) \quad D \mapsto \nabla^p_D - \nabla_{D(p)}$$

$$\psi: E \to E \otimes F^\ast \Omega^1_{X'/S} \quad \text{"F-Higgs field"}$$

**But:** $\psi$ is usually not zero. (Grothendieck, Katz)

**Emerging Theme:**

- Keep $\psi$ in the picture.
- Untwist by $F$ to somehow get Higgs field from $\psi$. 
$p$-curvature and Differential Operators

\[ D_{X/S} := \text{the ring of PD-differential operators on } X/S \]
\[ T_{X/S} \subseteq D_{X/S} \text{ generates } D_{X/S} \text{ over } \mathcal{O}_X \]

Center $\mathcal{Z}_{X/S}$ contains $\mathcal{O}_X$ and in fact all of $S \cdot T_{X'/S}$, via the $p$-curvature mapping:

\[ c: S \cdot T_{X'/S} \xrightarrow{\cong} \mathcal{Z}_{X/S} \quad D \mapsto D^p - D^{(p)}. \]

**Theorem. [BMR]** $D_{X/S}$ is an Azumaya algebra over its center (fppf locally a matrix algebra) of rank $(p^d)^2$.

Consequently, étale locally on $T^*_{X'/S}$, $D_{X/S}$ is $\mathcal{E}nd_{\mathcal{O}_{T^*}}(\mathcal{B})$ for some $\mathcal{B} \in MIC(X/S)$, and $\mathcal{H}om_D(\mathcal{B},)$ gives an equivalence $MIC(X/S) \equiv HIG(X'/S)$, with quasi-inverse $\mathcal{B} \otimes$. 
Explicit étale splitting of $D_{X/S}$

Cartier operator: \( C: F_* Z^1_{X/S} \to \Omega^1_{X'/S} \).

Choose a local section

\[
\zeta: \Omega^1_{X'/S} \to F_* Z^1_{X/S} \subseteq F_* \Omega^1_{X/S} \\
\in T_{X'/S} \otimes F_* Z^1_{X/S} \\
\tilde{\zeta}: F^* \Omega^1_{X'/S} \to \Omega^1_{X/S} \\
\pi^* \tilde{\zeta}: F^* \Omega^1_{X'/S} \to \Omega^1_{X'/S} \\
\phi: T_{X'/S} \to F^* T_{X'/S} \]

\[\alpha := \text{Spec}(\phi) \circ F - \text{id}\]

(Here \( F \) is the relative Frobenius for \( T^*_{X'/S} \) over \( X' \).)

\( \alpha \) is a surjective étale morphism of group schemes over \( X' \).

**Theorem.** The Azumaya algebra \( D_{X/S} \) splits when pulled back by \( \alpha \).
Proof: Suffices to construct a splitting module.

\[(\mathcal{B}, \theta) := F^* S^* T_{X'/S} \in HIG(X'/S)\]

\[\exists! \nabla \text{ on } \mathcal{B} : \nabla(1) = \zeta \in T_{X'/S} \otimes F_* Z_{X/S}^1 \text{ and } \theta \text{ is horizontal.} \quad \]

\[(\mathcal{B}, \theta) \text{ is a } D_{X/S}\text{-module, projective of rank } p^d \text{ over } T^*_{X'/S}.\]

Compute: \[(\mathcal{B}, \psi) = \alpha_*(\mathcal{B}, \theta).\]

So the action of \(D_{X/S}\) on \(\mathcal{B}\) extends to an action of \(\alpha^* D_{X/S} := D_{X/S} \otimes Z_{X/S, \alpha} S^* T_{X'/S}.\) \(\square\)
Local Consequences

\( p \)-curvature functor: \( \text{MIC}(X/S) \xrightarrow{\Psi} F-HIG(X/S) \)

Not fully faithful or essentially surjective.

**Theorem.** 1. If \( \Psi(E_1, \nabla_1) \cong \Psi(E_2, \nabla_2) \), and \( E_i \) are coherent, then \( (E_1, \nabla_1) \cong (E_2, \nabla_2) \) locally on \( X \).

2. Étale locally on \( X \), a coherent object of \( F-HIG(X/S) \) is in the essential image of \( \Psi \) if and only if it descends to an object of \( HIG(X'/S) \).

**Abelian analog:** \( E = \mathcal{O}_X, \omega := \nabla(1) \in Z^1_{X/S} \).

Classical exact sequence: (0th step in \( p \)-adic Hodge theory)

\[
0 \longrightarrow \mathcal{O}_{X'}^* \xrightarrow{F^*} \mathcal{O}_X^* \xrightarrow{d\log} Z^1_{X/S} \xrightarrow{\pi^*-C_{\Psi}} \Omega^1_{X'/S} \xrightarrow{} 0
\]
Idea of the proof:

(1) is elementary, (2) more subtle.

There is a 2-commutative diagram:

\[
\begin{array}{ccc}
HIG(X'/S) & \xrightarrow{\alpha \zeta^*} & HIG(X'/S) \\
\downarrow \phi \otimes & & \downarrow F^\ast_X/S \\
MIC(X/S) & \xrightarrow{\Psi} & F\cdot HIG(X/S)
\end{array}
\]

For modules whose support in \( T_{X'/S}^\ast \) is finite over \( X' \), can split \( \alpha \zeta \) étale locally over \( X' \). (2) follows from this.

Remark: \( \alpha \) has a canonical section in the formal neighborhood of the zero section, and is \(-id\) in its Frobenius neighborhood.
Riemann-Hilbert and the global Cartier transform

$X/S$ smooth, relative dimension $d$.
$\tilde{X}'/\tilde{S}$ a lifting of $X'/S \mod p^2$.
Construct: Analog of Riemann-Hilbert and Simpson’s theory, a la Fontaine-Galois.

$$D^\gamma_{X/S} := D_{X/S} \otimes_{S \cdot T_{X'/S}} \hat{\Gamma} \cdot T_{X'/S}.$$ 

An action of $D^\gamma_{X/S}$ on $E$ amounts to connection $\nabla$ and a
$$\theta: \hat{\Gamma} \cdot T_{X'/S} \to \mathcal{E}nd(E, \nabla),$$
extending the $p$-curvature. $\otimes$-Categories:

$$MIC_\gamma(X/S) \quad (D^\gamma_{X/S} - \text{modules, } \otimes)$$

$$HIG_\gamma(X'/S) \quad (\hat{\Gamma} \cdot T_{X'/S} - \text{modules, } \boxtimes)$$
**Theorem.** Given \( \mathcal{X}/S := (X/S, \tilde{X}/\tilde{S}) \), there exists a canonical

\[
(\mathcal{A}_{\mathcal{X}/S}, \nabla_A, \theta_A) \in MIC_{\gamma}(X/S)
\]

and an equivalence of categories:

\[
MIC_{\gamma}(X/S) \equiv HIG_{\gamma}(X'/S)
\]

\[
C_{\mathcal{X}/S} : (E, \nabla, \theta) \mapsto (E \otimes \mathcal{A}_{\mathcal{X}/S})^{\nabla, \theta}, \text{id} \otimes \theta_A
\]

\[
C_{\mathcal{X}/S}^{-1} : (E', \theta') \mapsto (E' \otimes \mathcal{A}_{\mathcal{X}/S})^\theta, \text{id} \otimes \nabla_A, \text{id} \otimes \theta_A
\]

**Variant:** \( MIC_p(X/S) \equiv HIG_p(X'/S) \): ( \( p \) means where action of \( p \)th power of \( S^+ T_{X'/S} \) is zero.)

The dual of \( \mathcal{A}_{\mathcal{X}/S} \) is a splitting module for \( D^\gamma_{X/S} \).

\( \mathcal{A}_{\mathcal{X}/S} \) has a ring structure, which makes \( C_{\mathcal{X}/S} \) an equivalence of \( \otimes \)-categories.
Construction and geometric interpretation of $\mathcal{A}_{X/S}$

Say $\tilde{T} \in \text{Cris}(X/\tilde{S})$ lifts $T \in \text{Cris}(X/S)$, thickening of $X$ (or an open subset).

\[
\begin{array}{ccc}
T & \xrightarrow{f_{T/S}} & X' \\
\downarrow & & \downarrow \text{inc} \\
T' & & \\
\end{array}
\]

- $\mathcal{L}_{X/S}(\tilde{T}) := \{\tilde{T} \to \tilde{X}' \text{ lifting } f_{T/S}\}$.

- $\mathcal{L}_{X/S}(\tilde{T})$ depends only on $T$, forms a crystal on $X/S$.

- $\text{Spec}(\mathcal{A}_{X/S})$ represents $\mathcal{L}_{X/S}$. 

Explicitly:

\[ \mathcal{E}_{X/S, \tilde{T}} := \text{conormal sheaf of} \]

\[
\begin{array}{ccc}
T & \xrightarrow{\Gamma f_{T/S}} & T \times X' \xrightarrow{\text{inc}} \tilde{T} \times \tilde{X}' \\
\end{array}
\]

• Depends only on \( T \), forms a crystal on \( X/S \), fitting into an exact sequence

\[
0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{E}_{X/S} \rightarrow F^* \Omega^1_{X'/S} \rightarrow 0.
\]

• \( \mathcal{A}_{X/S} = \lim_{\longrightarrow} S^n \mathcal{E}_{X/S} \)

• \( \text{Spec } \mathcal{A}_{X/S} \cong \text{splittings of } \mathcal{E}_{X/S} \cong \text{liftings of } f_{T/S} \)
The derived Cartier transform

**Theorem.** Suppose $X/S$ is smooth of relative dimension $d$, and $(E, \nabla) \in \text{MIC}_{p-d}(X/S)$ corresponds to $(E', \theta') \in \text{HIG}_{p-d}(X'/S)$. Then there is an isomorphism in the derived category:

$$(E \otimes \Omega_{X/S}^\cdot, d) \cong (E' \otimes \Omega_{X'/S}^\cdot, \psi')$$

- This generalize the Hodge decomposition of Deligne-Illusie, and also of Faltings, Illusie, Ogus.

- If $(E, \nabla)$ underlies a Fontaine-module on a lifting of $X/\tilde{S}$, then

$$C_{X/S}(E, \nabla) \cong (\text{Gr}_A E'_{X'}, \text{Gr}_A \nabla')$$

- Uses the action of $G_m$ on $T^*_{X'/S}$ and can be realized by an explicit and canonical double complex.
\( \mathcal{A}^{i,j}_n := A_n \otimes \Omega^i_{X/S} \otimes F^* \Omega^j_{X'/S} \)

\[
\begin{array}{cccccccccccccccccc}
E' \otimes \Omega^2_{X'/S} & \rightarrow & E \otimes \mathcal{A}^{0,2}_{n-2} & \rightarrow & E \otimes \mathcal{A}^{1,2}_{n-2} & \rightarrow & E \otimes \mathcal{A}^{2,2}_{n-2} & \rightarrow & \cdots \\
\uparrow \psi' & & \uparrow \text{id} \otimes \psi & & \uparrow \text{id} \otimes \psi & & \uparrow \text{id} \otimes \psi & & \uparrow \text{id} \otimes \psi \\
E' \otimes \Omega^1_{X'/S} & \rightarrow & E \otimes \mathcal{A}^{0,1}_{n-1} & \rightarrow & E \otimes \mathcal{A}^{1,1}_{n-1} & \rightarrow & E \otimes \mathcal{A}^{2,1}_{n-1} & \rightarrow & \cdots \\
\uparrow \psi' & & \uparrow \text{id} \otimes \psi & & \uparrow \text{id} \otimes \psi & & \uparrow \text{id} \otimes \psi & & \uparrow \text{id} \otimes \psi \\
E' & \rightarrow & E \otimes \mathcal{A}^{0,0}_n & \rightarrow & E \otimes \mathcal{A}^{1,0}_n & \rightarrow & E \otimes \mathcal{A}^{2,0}_n & \rightarrow & \cdots \\
\uparrow \nabla & & \uparrow \nabla & & \uparrow \nabla & & \uparrow \nabla & & \uparrow \nabla \\
E & \rightarrow & E \otimes \Omega^1_{X/S} & \rightarrow & E \otimes \Omega^2_{X/S} & \rightarrow & \cdots
\end{array}
\]