Remarks on F-spans

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K. Kato has pointed out that the hypothesis of "parallelizability" for the lifting F of F_X in [18] Theorem 2.2 is superfluous. This note is my attempt to understand his comment.

Let X/S be a smooth morphism of schemes in characteristic p, and let $F_{X/S}\colon X \to X'$ be the relative Frobenius morphism. Suppose we are given a lifting $F\colon Z \to Z'$ of $F_{X/S}$, where Z/T and Z'/T are formally smooth formal liftings of X/S and X'/S respectively and T is p-torsion free.

Our main goal is the following result.

Theorem 0.1. *Let (E', ∇') be a p-torsion free coherent sheaf with integrable (and quasi-nilpotent?) connection on Z'/T. Suppose that E is a submodule of $\tilde{E} := F^*E'$ which is invariant under the induced connection $\tilde{\nabla}$ on $\tilde{E}$. Let $\eta\colon E' \to F_*F^*E'$ be the adjunction map. Then the natural map $F^*(\eta^{-1}E) \to E$ is an isomorphism.*

Since F is faithfully flat, the most natural way to attack this problem would be to show that E is necessarily invariant under the descent data for F^*E'. In fact, as we explain later, the result can be deduced from Shiho's Theorem 3.1 in [20] which uses related, but different, descent data. Here we follow a different approach, based on Cartier descent, working in the context of F-spans, which in fact is where the above question found its origin.

We recall some notions from [18]. Let $F\colon Z \to Z'$ be a lifting of $F_{X/S}$ as above. Then a *F-span on X/S* is given by a pair of p-torsion free $\mathcal{O}_Z$-modules with (nilpotent) integrable connection (E', ∇') on Z'/W and (E, ∇) on Z/W together with an injective homomorphism

$$\tilde{\Phi}\colon F^*(E', \nabla') \to (E, \nabla).$$

Typically one assumes that E' and E are locally free of finite rank and that the image of $\tilde{\Phi}$ contains p^nE for some $n > 0$. Here we assume only the latter.

We have natural maps

$$\eta_F\colon E' \to F_*F^*E'$$

$$\Phi_F\colon E' \to E := \tilde{\Phi}_F \circ \eta_F.$$

We define:

$$\begin{aligned} M^i_F F^*E' &:= \tilde{\Phi}_F^{-1}(p^iE) \\ A^i_F E' &:= \Phi_F^{-1}(p^iE) = \eta_F^{-1}(M^i_F F^*E') \end{aligned}$$

$$\begin{aligned} M_F^{[i]}F^*E' &:= \sum_j p^{[j]} M_F^{i-j} F^*E' \\ A_F^{[i]}E' &:= \sum_j p^{[j]} A_F^{i-j} E' \end{aligned}$$

We shall show that the filtration $A_F^{[\cdot]}$, unlike the filtration $A_F^{\cdot}$, is independent of the lifting F, allowing a conceptual simplification of some of the constructions of [19].

The map $p^{-i}\tilde{\Phi}_F$ induces a morphism $M_F^i F^*E' \to E$; we denote by $N_F^{-i}E$ its image. Thus we find an isomorphism

$$\tilde{\Phi}_i\colon M_F^i F^*E' \to N_F^{-i}E.$$

If there is no danger of confusion, we may drop the subscript indicating the choice of the lifting F. We write E_X for E/pE, and similarly for E' and $\tilde{E}$.

Proposition 0.2. *The filtrations $M^{\cdot}\tilde{E}$, $M^{[\cdot]}\tilde{E}$ are stable under $\tilde{\nabla}$, annd the filtration $N^{\cdot}E$ is stable under ∇. The filtrations $A^{\cdot}E'$ and $A^{[\cdot]}E'$ satisfy Griffiths transversality. The isomorphisms $\tilde{\Phi}_i\colon M^i\tilde{E} \to N^{-i}E$ are compatible with the connections, and induce isomorphisms:*

$$\begin{aligned} (\mathrm{gr}^i_M\tilde{E}, \tilde{\nabla}) &\to (N^{-i}E_X, \nabla) \\ (\mathrm{gr}^i_M\tilde{E}_X, \tilde{\nabla}) &\to (\mathrm{gr}^{-i}_N E_X, \nabla) \end{aligned}$$

Proof. Omitted for now. See [19]. □

Theorem 0.3. *With the notation above, and for every i, the following statements hold:*

1. *The natural map $F^*(A^iE') \to M^i\tilde{E}$ is an isomorphism.*
2. *The natural maps $F^*(A^iE'_X) \to M^i\tilde{E}_X$ and $A^iE'_X \to (M^iE_X)^{\nabla}$ are isomorphisms.*
3. *The natural maps $F^*\mathrm{gr}^i_A E'_X \to \mathrm{gr}^i_M\tilde{E}_X$ and $\mathrm{gr}^i_A E'_X \to (\mathrm{gr}^i_M\tilde{E}_X)^{\tilde{\nabla}}$ are isomorphisms.*
4. *The natural map $F^*\mathrm{gr}^i_A E' \to \mathrm{gr}^i_M\tilde{E}$ is an isomorphism.*

Proof. We prove these statements together by induction on i. Suppose $i = 0$. Then statement (1) is true by definition, and it follows that $F^*E'_X \cong \tilde{E}_X$. Cartier descent implies that $E'_X = (F^*E'_X)^{\tilde{\nabla}}$. This proves (2).

Since $A^1E' = \eta^{-1}(M^1\tilde{E})$ and contains pE', it follows that

$$A^1E' = E' \times_{\tilde{E}_X} M^1\tilde{E}_X,$$

and hence that

$$A^1E'_X = E'_X \times_{\tilde{E}_X} M^1\tilde{E}_X.$$

Since $E'_X \times_{\tilde{E}_X} M^1\tilde{E}_X = (M^1\tilde{E}_X)^{\nabla}$ and since the p-curvature of $M^1\tilde{E}_X$ vanishes, it follows by Cartier descent that the natural map $F^*(A^1E'_X) \to M^1\tilde{E}_X$ is an isomorphism. Then statement (2) for $i = 0$ implies that the map $F^*\mathrm{gr}^0_A E' \to \mathrm{gr}^0_M\tilde{E}$ is an isomorphism. By

Cartier descent again, it follows that the map $\mathrm{gr}^0_A E' \to (\mathrm{gr}^0_M \tilde{E})^{\tilde{\nabla}}$ is an isomorphism. This proves (3). Since $\mathrm{gr}^0_A E' = \mathrm{gr}^0_A E'_X$ and $\mathrm{gr}^0_M \tilde{E} = \mathrm{gr}^0_M \tilde{E}_X$, (4) follows as well.

For the induction step, assume that the statements hold for all $j < i$. We have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^* A^i E' & \longrightarrow & F^* A^{i-1} E' & \longrightarrow & F^* \mathrm{gr}^{i-1}_A E' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^i \tilde{E} & \longrightarrow & M^{i-1} \tilde{E} & \longrightarrow & \mathrm{gr}^{i-1}_M \tilde{E} & \longrightarrow & 0
\end{array}$$

Statement (1) for $i-1$ implies that the middle vertical arrow is an isomorphism and statement (4) for $i-1$ implies that the right vertical arrow is an isomorphism. Thus the left vertical arrow is also an isomorphism, proving statement (1) for i.

We also have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^* A^i E'_X & \longrightarrow & F^* A^{i-1} E'_X & \longrightarrow & F^* \mathrm{gr}^{i-1}_A E'_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & M^i \tilde{E}_X & \longrightarrow & M^{i-1} \tilde{E}_X & \longrightarrow & \mathrm{gr}^{i-1}_M \tilde{E}_X & \longrightarrow & 0
\end{array}$$

Statements (2) and (3) for $i-1$ imply that the two vertical maps on the right are isomorphisms, and consequently so is the map on the left. Since the p-curvature of $M^i \tilde{E}_X$ vanishes, statement (2) holds for i.

The main difficulty is in the following lemma, which corresponds to [18] 2.2.1 and will allow us to prove statement (3).

Lemma 0.4. *If the statements of Theorem 0.3 hold for all $j < i$, then the map $\mathrm{gr}^i_A E'_X \to F_* \mathrm{gr}^i_M \tilde{E}_X$ is injective.*

Proof. Suppose $a' \in A^i E'$ lifts an element of the kernel of the map in the lemma. Then $\eta(a') = pb + c$, with $c \in M^{i+1} \tilde{E}$ and $b \in \tilde{E}$. Suppose that in fact

$$\eta(a') = p^j b + c$$

with $c \in M^{i+1} \tilde{E}$ and $j > 0$. Since $\eta(a') \in M^i E$, it follows that $b \in M^{i-j} \tilde{E}$. Note that if $j > i$, in fact $a' \in A^{i+1} E'$ and we are done. On the other hand, if $0 < j \leq i$, we calculate:

$$\begin{array}{rcl}
\tilde{\Phi}(\eta(a')) & = & p^j \tilde{\Phi}(b) + \tilde{\Phi}(c) \\
\tilde{\Phi}_i(\eta(a')) & = & \tilde{\Phi}_{i-j}(b) + p\tilde{\Phi}_{i+1}(c) \\
\nabla \tilde{\Phi}_i(\eta(a')) & = & \nabla \tilde{\Phi}_{i-j}(b) + p \nabla \tilde{\Phi}_{i+1}(c)
\end{array}$$

Since $\nabla \tilde{\Phi}_i(\eta(a')) = \tilde{\Phi}_i(\tilde{\nabla}(\eta(a'))$ is divisible by p, the same is true of $\nabla \tilde{\Phi}_{i-j}(b)$. We saw in Proposition 0.2 that the map $\tilde{\Phi}_{i-j}$ induces a horizontal isomorphism $\mathrm{gr}^{i-j}_M \tilde{E}_X \cong \mathrm{gr}^{j-i}_N E_X$, and we conclude that the image of b in $\mathrm{gr}_M^{i-j}\tilde{E}_X$ is horizontal. Statement (3) for $i-j$ says that $\mathrm{gr}_A^{i-j}E'_X \cong (\mathrm{gr}_M^{i-j}\tilde{E}_X)^{\tilde{\nabla}}$, so there exist $b' \in A^{i-j}E', b'' \in \tilde{E}$, and $b''' \in M^{i-j-1}\tilde{E}$ such that

$$b = \eta(b') + b'' + pb'''.$$

Then

$$\eta(a' - p^j b') = p^j b + c - p^j \eta(b') = p^j b'' + p^{j+1} b''' + c.$$

Since $p^j b'' \in M^{i+1}\tilde{E}$, we have can take $a'' := a' - p^j b'$ and $c' = p^j b'' + c$, so now $\eta(a'') = p^{j+1}b''' + c'$. Continuing by induction, we see that eventually a' may be chosen to lie in $A^{i+1}E'_X$. □

Lemma 0.4 implies that $A^{i+1}E'_X = A^i E'_X \times_{M^i \tilde{E}_X} M^{i+1}\tilde{E}_X$. Since $A^i E'_X = (M^i \tilde{E}_X)^{\tilde{\nabla}}$, it follows that $A^{i+1}E'_X = (M^{i+1}\tilde{E}_X)^{\tilde{\nabla}}$, and since the p-curvature of $\tilde{\nabla}$ on $M^{i+1}\tilde{E}_X$ vanishes, the map $F^*(A^{i+1}E'_X) \to M^{i+1}\tilde{E}_X$ is an isomorphism. Thus (2) holds for $i+1$ and (3) holds for i.

Finally, we have a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & F^*\mathrm{gr}_A^{i-1}E' & \xrightarrow{[p]} & F^*\mathrm{gr}_A^i \tilde{E} & \longrightarrow & F^*\mathrm{gr}_A^i E'_X & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \mathrm{gr}_M^{i-1}\tilde{E} & \xrightarrow{[p]} & \mathrm{gr}_M^i \tilde{E} & \longrightarrow & \mathrm{gr}_M^i \tilde{E}_X & \longrightarrow & 0
\end{array}$$

The left vertical arrow is an isomorphism by the induction assumption and the right vertical arrow is an isomorphism by (3). It follows that the middle arrow is an isomorphism, proving (4) and completing the proof of the theorem. □

Proof. [Proofs of Theorem 0.1] First suppose that there is a natural number n such that $p^n\tilde{E} \subseteq E$. Then multiplication by p^n induces a horizontal map $\tilde{\Phi}\colon \tilde{E} \to E$, defining an F-span, and $E = M^n\tilde{E}$. Since $A^n E' = \eta^{-1}(M^n\tilde{E})$, statement (1) of Theorem 0.3 tells us that the natural map $F^*(\eta^{-1}(E)) \to E$ is an isomorphism.

For the general case, let $E_n := E + p^n\tilde{E}$ for each n. Then E_n is also invariant under $\tilde{\nabla}$. If $E'_n := \eta^{-1}(E_n)$ the previous paragraph tells us that the natural map $F^*E'_n \to E_n$ is an isomorphism. Let $E'' := \eta^{-1}(E)$. Then $E'/E'' \subseteq \tilde{E}/E$, and the Artin-Rees lemma implies that for some $r > 0$, $(p^{n+r}\tilde{E}/E) \cap (E'/E'') \subseteq p^n E'/E'''$ for all $n \geq 0$. Then $E_{n+r} \cap E' \subseteq E'' + p^n E'$ for all $n \geq 0$. Taking the intersection over n, we find that

$$E'' \subseteq \cap\{E'_n : n \geq 0\} = \cap\{E' \cap E_n : n \geq 0\} \subseteq \cap\{E'' + p^n E' : n \geq 0\} = E''.$$

Since F is finite and flat, the natural map

$$F^*E'' = F^*(\cap\{E'_n : n \geq 0\}) \to \cap\{F^*E'_n : n \geq 0\} = \cap\{E + p^n\tilde{E} : n \geq 0\} = E$$

is an isomorphism.

Let us now review Shiho's Theorem 3.1 ([20]) and explain how it implies Theorem 0.1. Shiho's theorem shows that Frobenius pullback defines an equivalence C_F from the category of integrable nilpotent p-connections on Z'/T to the category of integrable nilpotent connections on Z/T. Recall that there is a unique map

$$\zeta_F \colon \Omega^1_{Z'/S} \to F_*\Omega^1_{Z/S}$$

such that $p\zeta_F$ is the differential of F. Then if (E', θ') is a module with p-connection on Z', it is easy to verify that there is a unique connection $\tilde{\nabla}$ on $\tilde{E} := F^*E'$ such that $\tilde{\nabla} \circ \eta = (\zeta_F \otimes \eta) \circ \theta'$. Shiho proves that the functor C_F taking (E', θ') to $(\tilde{E}, \tilde{\nabla})$ is an equivalence by studying the descent data for the PD-thickenings which correspond to the crystalline interpretations of these categories.

To apply Shiho's result, suppose that (E', ∇') is a module with integrable nilpotent connection on Z'/T. Then $\theta' := p\nabla'$ is a nilpotent p-connection on E', and

$$(\zeta_F \otimes \eta) \circ \theta' = (p\zeta_F \otimes \eta) \circ \nabla' = (F^* \otimes \eta) \circ \theta',$$

and so the connection $\tilde{\nabla}$ in Shiho's correspondence is just the Frobenius pullback connection on $\tilde{E} := F^*(E')$. Let $E \subseteq \tilde{E}$ be a submodule which is invariant under $\tilde{\nabla}$. We claim that the induced connection on E is also nilpotent. To see this, let $N^iE := p^i\tilde{E} \cap E$, which is also invariant under $\tilde{\nabla}$, and note that the inclusion map induces an injection $\mathrm{gr}^i_N E \to p^i\tilde{E}/p^{i+1}\tilde{E} \cong \tilde{E}/p\tilde{E}$. It follows that the p-curvature of each $\mathrm{gr}^i_N E$ vanishes. On the other hand, we have a surjection $\mathrm{gr}^i_N E \to (N^iE + pE)/(N^{i+1} + pE)$, so the p-curvature of each of the latter also vanishes. By Artin-Rees, there is a natural number r such that $p^r\tilde{E} \cap E \subseteq pE$. Thus the images of N^iE in E/pE define a finite and exhaustive filtration of E/pE, and so the connection on E is indeed nilpotent. Then the full faithfulness of C_F implies that there is a submodule E'' of E', stable under $\theta' := p\nabla$, such that $F^*E'' = E \subseteq \tilde{E}$. Necessarily $E' \subseteq \eta^{-1}(E)$, and since $F^*E' \cong E$, it follows that $E' = \eta^{-1}(E)$, and the proof is complete. □

Corollary 0.5. *With the notations above, the natural maps*

$$\begin{array}{rcl} F^*(A^iE') & \to & M^i\tilde{E} \\ F^*(A^{[i]}E') & \to & M^{[i]}\tilde{E} \\ A^iE' & \to & \eta^{-1}(M^i\tilde{E}) \\ A^{[i]}E' & \to & \eta^{-1}(M^{[i]}\tilde{E}) \end{array}$$

are isomorphisms.

Proof. The following lemma shows that the first pair of equations of the proposition implies the second pair. (Note: The third equation is true by definition, but we shall need the fourth equation, which does not seem to be so obvious.)

Lemma 0.6. *If A is an $\mathcal{O}_{X'}$-submodule of E' and M is the image of F^*A in $\tilde{E}'$, then $A = \eta_F^{-1}(M)$.*

Proof. Since F is flat, the map $F^*A \to F^*E'$ is injective. Let $A' := \eta_F^{-1}(M)$. Then A' is an $\mathcal{O}_{X'}$-submodule of E' and $A \subseteq A'$. We have injections $F^*A \to F^*A' \to M$ whose composition is an isomorphism. Then the map $F^*A \to F^*A'$ is an isomorphism, and since F is faithfully flat, $A = A'$. □

Statement (1) of Theorem 0.3 proves the first equation of Proposition 0.5. Since F is flat, the natural maps $F^*A^i_F E' \to F^*E'$ and $F^*A^{[i]}_F E' \to F^*E'$ are injective, and moreover

$$F^*A^{[i]}_F E' = \sum_j p^{[i-j]} F^*A^{i-j}_F E' = M^{[i]}_F F^*E'$$

This is the second equation of the proposition, which implies the fourth, by Lemma 0.6. □

If G is another lifting of F_X, the connection ∇' furnishes a horizontal isomorphism

$$\varepsilon_{G,F} \colon G^*(E', \nabla') \to F^*(E', \nabla').$$

and hence a map

$$\tilde{\Phi}_G \colon G^*(E', \nabla') \to (E, \nabla),$$

making the diagram

$$\begin{array}{ccc} G^*E' & \xrightarrow{\tilde{\Phi}_G} & E \\ {\scriptstyle\varepsilon}\downarrow & \nearrow_{\tilde{\Phi}_F} & \\ F^*E' & & \end{array}$$

commutative. It follows that the isomorphism ε takes $M^i_G G^*E'$ isomorphically to $M^i_F F^*E'$, and that $N^{-i}_F E = N^{-i}_G E$.

The diagram:

$$\begin{array}{ccccc} E' & \xrightarrow{\eta_G} & G_*G^*E' & \longrightarrow & G_*E \\ & & {\scriptstyle\varepsilon}\downarrow & \nearrow_{G_*\tilde{\Phi}_F} & \\ & & G_*F^*E' & & \end{array}$$

shows that

$$\Phi_G = G_*(\tilde{\Phi}_F) \circ \varepsilon_{G,F}. \qquad (1)$$

It need not be the case that $A^i_F E' = A^i_G E'$; we give an example below. The following proposition remedies this situation.

Proposition 0.7. *Let F and G be liftings of $F_{X/S}$ to maps $Z \to Z'$. Then for all i, $A^{[i]}_F E' = A^{[i]}_G E'$.*

Proof. For simplicity we write the rest of the proof assuming that X is a curve and that it admits a local coordinate t. Let $\partial := \nabla'(d/dt)$ and suppose that $G^*(t) = F^*(t) + pg$. Then if $e' \in E'$,

$$\varepsilon(\eta_G(e')) = \sum_j p^{[j]} g^j \eta_F(\partial^j(e')) \tag{2}$$

Note that, since the connection ∇' is nilpotent, the sequence $\partial^j(e')$ converges to zero.

To prove that each $A^{[i]}_G E' \subseteq A^{[i]}_F E'$, it will suffice to prove that each $A^i_G E' \subseteq A^{[i]}_F E'$, We work by induction on i. Assume that $e' \in A^i_G E'$. Then $\eta_G(e') \in M^i_G G^* E'$ and hence $\varepsilon(\eta_G(e')) \in M^i_F F^* E'$. Then

$$\eta_F(e') = \varepsilon(\eta_G(e')) - \sum_{j>0} p^{[j]} g^j \eta_F(\partial^j(e')).$$

By Griffiths transversality, $\partial^j(e') \in A^{i-j}_G E'$, hence the induction hypothesis implies that $\partial^j(e') \in A^{[i-j]}_F E'$ when $j > 0$. Then $p^{[j]}\partial^j(e') \in A^{[i]}_F E'$, and we conclude that $\eta_F(e') \in F^*(A^{[i]}_F E') + M^i_F F^* E' = M^{[i]}_F F^* E'$ by the second equation of Proposition 0.5. Its last equation then implies that $e' \in A^{[i]}_F E'$. $\square$

Example: Let $X := \operatorname{Spec} k[t, t^{-1}]$, let Z be the formal completion of $\operatorname{Spec} W[t, t^{-1}]$ and let F send t to t^p. Let E be the free $\mathcal{O}_Z$-module with basis $(e_0, \ldots, e_{p-1})$, let η be the endomorphsm of E sending e_i to e_{i-1} and e_0 to zero, and let

$$\nabla e_i = \eta(e_i) dt/t.$$

Recall that

$$\log(1 + x) = x - x^2/2 + x^3/3 + \cdots.$$

For each i, we have a formal horizontal section

$$\begin{aligned} \tilde{e}_i &:= e^{(-\log t)\eta}(e_i) \\ &= e_i - (\log t)e_{i-1} + (1/2)(\log t)^2 e_{i-2} + \cdots + ((-1)^i/i!)(\log t)^i e_0. \end{aligned}$$

Then

$$\begin{aligned} e_i &= e^{(\log t)\eta}(\tilde{e}_i) \\ &= \tilde{e}_i - (\log t)\tilde{e}_{i-1} + (1/2)(\log t)^2 \tilde{e}_{i-2} + \cdots + ((-1)^i/i!)(\log t)^i \tilde{e}_0. \end{aligned}$$

Let $(E', \nabla') = (E, \nabla)$ and define

$$\tilde{\Phi}_F \colon F^*(E', \nabla') \to (E, \nabla) : e'_i \mapsto p^i e_i.$$

It is immediate to check that this map is horizontal

Now suppose that G is another lift of F_X, sending t to $t^p + pg$. Let $u := (1 + pt^{-p}g)$, so that $t^p + pg = t^p u$. Note that $\log u \in pW\{t, t^{-1}\}$, say $\log u = p\delta$. That is,

$$\delta = t^{-p}g - (p/2)t^{-2p}g^2 + (p^2/3)t^{-3p}g^3 + \cdots .$$

To calculate $\varepsilon := \varepsilon_{G,F}$, we use the fact that it acts as the identity on horizontal sections. Thus

$$\begin{aligned}
\varepsilon(G^*(e_i)) &= \varepsilon(G^*(e^{(\log t)\eta}\tilde{e}_i) \\
&= e^{\log(t^p u)\eta}\varepsilon(G^*(\tilde{e}_i)) \\
&= e^{\log(t^p u)\eta}F^*(\tilde{e}_i) \\
&= e^{\log(t^p u)\eta}F^*(e^{-(\log t)\eta})(e_i) \\
&= e^{\log(t^p u)\eta}(e^{-(\log t^p)\eta})(e_i) \\
&= e^{(\log u)\eta}e_i \\
&= e_i + (\log u)e_{i-1} + (\log u)^2/(2!)e_{i-2} + \cdots + (\log u)^i/i!e_0 \\
&= e_i + p\delta e_{i-1} + p^{[2]}\delta^2 e_{i-2} + \cdots + p^{[i]}\delta^i e_0
\end{aligned}$$

Then

$$\begin{aligned}
\Phi_G(e_i) &= \tilde{\Phi}_F(\varepsilon(G^*(e_i))) \\
&= = p^i(e_i + \delta e_{i-1} + \delta^{[2]}e_{i-2} + \cdots + \delta^{[i]}e_0)
\end{aligned}$$

In particular, $e_i \in A^i_G E'$ if $i < p$, but $\Phi_G(e_p) = p^p(\cdots) + (p^{p-1}/(p-1)!)\delta^p e_0$.

Take for example $p = 2$. Then

$$\begin{aligned}
\Phi_G(e_0) &= e_0 \\
\Phi_G(e_1) &= 2e_1 + 2\delta e_0 \\
\Phi_G(e_2) &= 4e_2 + 4\delta e_1 + 2\delta^2 e_0
\end{aligned}$$

Assume δ is not zero modulo 2 (e.g. if $g = 4t^2, u = 5$, and $\delta = 1/2 \log 5$. Then e_2' does not belong to $A^2_G E'$, although it does belong to $A^2_F E'$. On the other hand, since G lifts F_X and k is perfect, there exists an element δ' of $W\{t, t^{-1}\}$ such that $G^*(\delta') \equiv \delta^2 \pmod 2$. Then $e_2' - 2\delta' e_0' \in A^2_G E'$.

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