Higgs cohomology, p-curvature, and the Cartier isomorphism

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Abstract

Let X/S be a smooth morphism of schemes in characteristic p and let (E,∇) be a sheaf of $\mathcal{O}_X$-modules with integrable connection on X. We give a formula for the cohomology sheaves of the De Rham complex of (E,∇) in terms of a Higgs complex constructed from the p-curvature of (E,∇). This formula generalizes the classical Cartier isomorphism, with which it agrees when (E,∇) is the constant connection.

1. Introduction

1.1 Let $X/\mathbf{C}$ be a smooth scheme over the complex numbers with associated analytic space X^{an}, and let (E,∇) be a coherent sheaf with integrable connection on $X/\mathbf{C}$. Then $E^\nabla := (\operatorname{Ker}\nabla^{an})$ is a locally constant sheaf of finite dimensional $\mathbf{C}$-vector spaces on X^{an}, and the canonical map

$$E^\nabla \otimes_{\mathbf{C}} \mathcal{O}_{X^{an}} \to E^{an}$$

is an isomorphism. Moreover, the De Rham complex $E\otimes\Omega^{\cdot}_{X/\mathbf{C}}$ of E is a resolution of E^∇ on X^{an}; that is,

$$\mathcal{H}^i(E^{an}\otimes\Omega^{\cdot}_{X/\mathbf{C}}) \cong \begin{cases} E^\nabla & \text{if } i=0,\\ 0 & \text{if } i>0.\end{cases} \tag{1.1.1}$$

If $X/\mathbf{C}$ is proper, or more generally if (E,∇) has regular singularities at infinity [Del70], the above formula has an algebraic analog which computes the De Rham cohomology sheaves of (E,∇) in the Zariski topology. Let α be the canonical map from X^{an} to X. The comparison theorems of Grothendieck [Gro66] and Deligne [Del70] provide canonical isomorphisms

$$\mathcal{H}^i(E\otimes\Omega^{\cdot}_{X/\mathbf{C}}) \cong R^i\alpha_*(E^\nabla) \tag{1.1.2}$$

for all i. In fact, if U is any open subset of X, there is a canonical isomorphism

$$H^i(U, E\otimes\Omega^{\cdot}_{X/\mathbf{C}}) \cong H^i(U^{an}, E^\nabla).$$

Here the term on the left is the hypercohomology of the De Rham complex of E; if U is affine this is computed simply by taking the cohomology of the complex of global sections of $\Omega^{\cdot}_{X/\mathbf{C}}$.

1.2 Our goal in this paper is to explain an analog of equation (1.1.2) in positive characteristics. Let $f\colon X\to S$ be a smooth morphism of schemes in characteristic p, and let (E,∇) be a sheaf of

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$\mathcal{O}_X$-modules with integrable connection on X/S. Consider the usual Frobenius diagram,

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/S}} & X' & \xrightarrow{\pi} & X \\ & \searrow & \downarrow & & \downarrow \\ & & S & \xrightarrow{F_S} & S \end{array} \tag{1.2.1}$$

in which the square is Cartesian and $\pi \circ F_{X/S}$ is the absolute Frobenius endomorphism F_X of X. The relative Frobenius morphism $F_{X/S}$ in this diagram is a homeomorphism, and the boundary maps of the complex $F_{X/S*}(E \otimes \Omega^{\cdot}_{X/S})$ are $\mathcal{O}_{X'}$-linear. Thus

$$F_{X/S*}\mathcal{H}^i(E \otimes \Omega^{\cdot}_{X/S}) \cong \mathcal{H}^i(F_{X/S*}(E \otimes \Omega^{\cdot}_{X/S})),$$

and these sheaves, which we identify and denote simply by $\mathcal{H}^i_{DR}(E, \nabla)$, are naturally $\mathcal{O}_{X'}$-modules. When (E, ∇) is the constant connection $(\mathcal{O}_X, d)$, they are calculated by the Cartier isomorphism [Kat70, 7.2]:

$$C^{-1}_{X/S} \colon \Omega^i_{X'/S} \cong \mathcal{H}^i_{DR}(\mathcal{O}_X, d) \tag{1.2.2}$$

A similar formula holds if the p-curvature of (E, ∇) (whose definition we shall recall below) is zero. The p-curvature vanishes if and only if the natural map $F^*_{X/S}E' \to E$ is an isomorphism [Kat70, §5]. Then the De Rham complex of (E, ∇) can be identified with $E' \otimes F_{X/S*}\Omega^{\cdot}_{X/S}$, and since $F_{X/S*}\Omega^{\cdot}_{X/S}$ is a complex of flat $\mathcal{O}_{X'}$ modules whose cohomology sheaves are also flat, the natural map

$$E' \otimes \Omega^i_{X'/S} \cong E' \otimes \mathcal{H}^i(F_{X/S*}\Omega^{\cdot}_{X/S}) \to \mathcal{H}^i(E' \otimes F_{X/S*}\Omega^{\cdot}_{X/S})$$

is an isomorphism.

We shall give a formula for the De Rham cohomology sheaves of an arbitrary module with integrable connection which generalizes the Cartier isomorphisms above. The formula in the general case depends, of course, on the p-curvature [Kat70, 7.2]. Recall that if D is a derivation of $\mathcal{O}_X$ over S, so is its pth iterate $D^{(p)}$, and the p-curvature ψ of ∇ is the $\mathcal{O}_X$-linear map

$$\psi \colon E \to E \otimes F^*(\Omega^1_{X'/S})$$

characterized by the formula

$$\langle \psi(e), 1 \otimes D \rangle = \nabla^p_D(e) - \nabla_{D^{(p)}}(e)$$

for every section e of E and every derivation D. It turns out that this map satisfies the integrability condition in the theory of Higgs fields [Sim92], and so iteration of ψ forms a complex

$$K^{\cdot}(E, \psi) := E \xrightarrow{\psi} E \otimes F^*_{X/S}(\Omega^1_{X'/S}) \xrightarrow{\psi} F^*_{X/S}(\Omega^2_{X'/S}) \longrightarrow \cdots . \tag{1.2.3}$$

Endow each term in this complex with the tensor product connection, using the given connection ∇ on E and the Frobenius descent connection $\mathrm{id} \otimes d$ on $F^*_{X/S}(\Omega^i_{X'/S})$. We shall see that the boundary maps in the $K^{\cdot}(E, \psi)$ are compatible with these connections, and hence that the cohomology sheaves $\mathcal{H}^i_{\psi}(E, \nabla)$ inherit a connection as well. Our generalization of Cartier's result asserts that for every i there is a natural isomorphism

$$\mathcal{H}^i_{\psi}(E, \nabla)^{\nabla} \cong \mathcal{H}^i_{DR}(E, \nabla). \tag{1.2.4}$$

This is evident when $i = 0$, and the general case follows easily when properly formulated.

Let $MIC(X/S)$ denote the abelian category whose objects are the sheaves of $\mathcal{O}_X$-modules endowed with an integrable connection ∇ and whose morphisms are morphisms of $\mathcal{O}_X$-modules compatible with the connections. For each i, $\mathcal{H}^i_{DR}$ and $\mathcal{H}^i_\psi$ are functors from the category $MIC(X/S)$ to the category of $\mathcal{O}_{X'}$-modules. As it turns out, it will be a little more convenient to work with $-\psi$ instead of ψ; of course, the functors $\mathcal{H}^i_{-\psi}$ and $\mathcal{H}^i_\psi$ are isomorphic.

Theorem 1.2.1. *The sequences $\{\mathcal{H}^i_{DR} : i \in \mathbf{N}\}$ and $\{\mathcal{H}^{i\nabla}_{-\psi} : i \in \mathbf{N}\}$ form cohomological ∂-functors [Gro57] from the abelian category $MIC(X/S)$ to the category of $\mathcal{O}_{X'}$-modules. Moreover, there is a unique isomorphism of cohomological ∂-functors*

$$C^{-1}_{X/S} \colon \mathcal{H}^{\cdot\nabla}_{-\psi} \to \mathcal{H}^{\cdot}_{DR}$$

which agrees with (1.2.4) in degree 0. In particular, if (E, ∇) is an object of $MIC(X/S)$, there is a canonical isomorphism of $\mathcal{O}_{X'}$-modules

$$C^{-1}_{X/S} \colon \mathcal{H}^i_{-\psi}(E,\nabla)^\nabla \cong \mathcal{H}^i_{DR}(E,\nabla).$$

To compare the isomorphism in Theorem (1.2.1) with the Cartier isomorphism (1.2.2), take (E,∇) to be $(\mathcal{O}_X, d)$. Since $\psi = 0$, $\mathcal{H}^i_{-\psi}(E,\nabla)$ is just $F^*_{X/S}\Omega^i_{X'/S}$, and so $\mathcal{H}^i_{-\psi}(E,\nabla)^\nabla \cong \Omega^i_{X'/S}$. We shall verify that, thanks to the choice of the sign, the isomorphism we construct in the above theorem agrees with the usual Cartier isomorphism $C^{-1}_{X/S}$ in this case. Note that there is a natural isomorphism of complexes:

$$\begin{array}{ccccccc}
E & \xrightarrow{\psi} & E \otimes F^*_{X/S}(\Omega^1_{X'/S}) & \xrightarrow{\psi} & F^*_{X/S}(\Omega^2_{X'/S}) & \longrightarrow & \cdots \\
\downarrow{\scriptstyle \mathrm{id}} & & \downarrow{\scriptstyle -\mathrm{id}} & & \downarrow{\scriptstyle \mathrm{id}} & & \\
E & \xrightarrow{-\psi} & E \otimes F^*_{X/S}(\Omega^1_{X'/S}) & \xrightarrow{-\psi} & F^*_{X/S}(\Omega^2_{X'/S}) & \longrightarrow & \cdots
\end{array}$$

Thus, had we used ψ in place of $-\psi$, we would have obtained $(-1)^i$ times the usual Cartier isomorphism in degree i.

1.3 Theorem (1.2.1) is very suggestive of Simpson's nonabelian Hodge theory [Sim92] for a smooth projective variety X over the complex numbers. Simpson associates to each irreducible object in $MIC(X/\mathbf{C})$ a new holomorphic sheaf E' with a Higgs field $\theta \colon E' \to E' \otimes \Omega^1_{X/\mathbf{C}}$ and constructs a (transcendental) quasi-isomorphism between the De Rham complex of (E,∇) and the Higgs complex of (E',θ). Our result can be viewed as a sheaf-theoretic analog of this quasi-isomorphism.

Theorem (1.2.1) is also related to the following striking formula of Barannikov and Kontsevich [Sab99]. Let $X/\mathbf{C}$ be quasi-projective and smooth, suppose $f \in \mathcal{O}_X(X)$ defines a proper morphism to the affine line, and let ∇ be the connection on $\mathcal{O}_X$ sending 1 to df. Then the hypercohomologies of the Higgs complex $(K^{\cdot}(\mathcal{O}_X), df)$ and of the De Rham complex $(\mathcal{O}_X, \nabla)$ have the same finite dimension in every degree. Recall that in characteristic p, the p-curvature of the connection on $\mathcal{O}_X$ sending 1 to a closed 1-form ω is given by $\psi(1) = F^*_{X/S}(\pi^*\omega - C_{X/S}(\omega))$ [Kat72, 7.22]. In particular, if $\omega = df$, where $f \in \mathcal{O}_X(X)$, then $C(\omega) = 0$, $\psi(1) = F^*_{X/S}\pi^*(df)$, and the p-curvature complex is the pullback by F_X of the complex

$$K^{\cdot}(\mathcal{O}_X, df) := \mathcal{O}_X \xrightarrow{df} \Omega^1_{X/S} \xrightarrow{\wedge df} \cdots,$$

If f has isolated critical points, one can deduce that the cohomologies of the Higgs complex $K^{\cdot}(\mathcal{O}_X, df)$ and the De Rham complex of $(\mathcal{O}_X, \nabla)$ have the same dimension.

1.4 Our proof of theorem (1.2.1) is quite simple. We show that for $i > 0$, $\mathcal{H}^{i,\nabla}_{-\psi}$ and $\mathcal{H}^i_{DR}$ are effaceable. Then the theorem follows from the universality of effaceable cohomological ∂-functors [Gro57].

In fact, since the category of $\mathcal{O}_X$-modules with connection has enough injectives, both families can be viewed as the right derived functors of the same functor. We actually present two versions of the proof, first in the classical case, and then in the context of connections with log poles, or, more generally, connections on log schemes. This presents some technical and conceptual difficulties, since the theory of Frobenius descent is not so straightforward in this case. Finally we give an explicit formula for the isomorphism in terms of local coordinates, as well as a few consequences. The appendix gives a formula for the action of p-curvature on divided powers.

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2. Higgs complexes and effaceability

2.1 For the convenience of the reader, we recall from [Sim92] some basic facts about Higgs fields and Higgs cohomology. Throughout this paper we will allow ourselves the following abuse of notation: if E is a sheaf on a space X, we write $e \in E$ to mean that e is a section of E over some open subset of X.

Let Ω be a locally free sheaf of finite rank on a scheme X, let T be its dual, and let E be any sheaf of $\mathcal{O}_X$-modules. Then the natural map

$$\mathrm{Hom}_{\mathcal{O}_X}(E, E\otimes\Omega) \to \mathrm{Hom}_{\mathcal{O}_X}(T, \mathrm{End}_{\mathcal{O}_X}(E))$$

is an isomorphism. If $\phi\colon E \to E\otimes\Omega$ and t is a local section of T, we denote by ϕ_t the corresponding section of $\mathrm{End}_{\mathcal{O}_X}(E)$. For each $i > 0$, let $\Omega^i := \Lambda^i\Omega$, and define ϕ^i by the diagram

$$\begin{array}{ccc} E\otimes\Omega^i & \xrightarrow{\phi\otimes\mathrm{id}} & E\otimes\Omega\otimes\Omega^i \\ & \searrow^{\phi^i} & \downarrow{\scriptstyle \mathrm{id}\otimes\wedge} \\ & & E\otimes\Omega^{i+1}. \end{array} \qquad (2.1.1)$$

Definition 2.1.1. Let Ω be a locally free sheaf on a scheme X, let T be its dual, and let E be a sheaf of $\mathcal{O}_X$-modules. A *T-Higgs field on* E is an $\mathcal{O}_X$-linear map $\phi\colon E \to E\otimes\Omega$ such that the composite of ϕ with the map $\phi^1\colon E\otimes\Omega \to E\otimes\Lambda^2\Omega$ induced by ϕ vanishes. If this is the case, $\phi^i\circ\phi^{i-1} = 0$ for all i, and the sequence of maps is the complex:

$$K^{\cdot}(E,\phi) := E \xrightarrow{\phi} E\otimes\Omega \xrightarrow{\phi^1} E\otimes\Lambda^2\Omega \to \cdots,$$

defines a complex, called the *Higgs complex* of (E,ϕ). The *Higgs cohomology of* (E,ϕ) is the cohomology of this Higgs complex.

Giving a T-Higgs field ϕ on E is equivalent to giving an action of the symmetric algebra $S^{\cdot}T$ of T on E, compatibly with its given structure of an $\mathcal{O}_X$-module. Let E_ϕ denote this $S^{\cdot}T$-module. If $t. := (t_1, \ldots t_n)$ is a basis for T, then the Higgs complex of (E,ϕ) can be identified with the Koszul complex $K^{\cdot}(\phi., E_\phi)$ of E with respect to the corresponding sequence $\phi.$ of endomorphisms of E, viewed as a module over $S^{\cdot}T$.

Remark 2.1.2. When E is quasi-coherent, one can give a geometric interpretation of the Higgs complex of a T-Higgs sheaf (E,ϕ) in the following way. Let $\mathbf{V}T^* := \mathrm{Spec}\, S^{\cdot}T$, let $\pi\colon \mathbf{V}T^* \to X$

be the canonical map, and let $i\colon X \to \mathbf{V}T^*$ be the zero section. Let $\tilde{E}_\phi$ denote the sheaf of $\mathcal{O}_{\mathbf{V}T^*}$-modules corresponding to (E,ϕ). Then there is a canonical isomorphism in the derived category of sheaves of $\mathcal{O}_{\mathbf{V}T^*}$-modules:

$$K^\cdot(E,\phi) \cong Ri^! \tilde{E}_\phi \cong Li^* \tilde{E}_\phi[-n] \otimes \Omega^n \tag{2.1.2}$$

To see this, let us recall the coordinate free construction of the Koszul complex from [Ber71]. The canonical map $u\colon \pi^*T \to \mathcal{O}_{\mathbf{V}T^*}$ can be viewed as a section of $\pi^*\Omega$, and multiplication by $(-1)^n$ times this section in the exterior algebra defines a complex of $\mathcal{O}_{\mathbf{V}T^*}$-modules:

$$K^\cdot(u) := \mathcal{O}_{\mathbf{V}T^*} \to \pi^*\Omega \to \pi^*\Omega^2 \to \cdots \pi^*\Omega^n.$$

Here n is the rank of Ω, and the complex lives in degrees $[-n,0]$. This complex gives a locally free resolution of the sheaf $i_*\Omega^n$. Hence

$$K^\cdot(u) \otimes \tilde{E}_\phi \cong i_*\Omega^n \overset{\mathbf{L}}{\otimes} \tilde{E}_\phi \cong \pi^*\Omega^n \otimes i_*\mathcal{O}_X \overset{\mathbf{L}}{\otimes} \tilde{E}_\phi \cong$$

$$Li^*(\tilde{E}_\phi \otimes \pi^*\Omega^n) \cong Li^*(\tilde{E}_\phi) \otimes \Omega^n.$$

But the Higgs complex $K^\cdot(E,\phi) = K(u) \otimes \tilde{E}_\phi[-n]$. Similarly,

$$Ri^!(\tilde{E}_\phi) \cong RHom(i_*\mathcal{O}_X, \tilde{E}_\phi) \cong \mathcal{H}om(K^\cdot(u), \tilde{E}_\phi) \otimes \pi^*\Omega^n \cong K^\cdot(E,\phi),$$

where the last isomorphism is induced by the pairing $\Omega^q \otimes \Omega^{n-q} \to \Omega^n$.

2.2 The following result is a simple analog for Higgs fields of the construction of the Gauss-Manin connection by Katz and Oda [KO68]. We continue with the previous notation: X is a scheme, T a locally free sheaf of $\mathcal{O}_X$-modules of rank n, Ω is its dual, and E a sheaf of $\mathcal{O}_X$-modules.

Proposition 2.2.1. *Let ϕ be a T-Higgs field on E, and let $0 \to \Omega' \to \Omega \to \Omega'' \to 0$ be an exact sequence of locally free $\mathcal{O}_X$-modules of finite type. Let $\phi''\colon E \to E \otimes \Omega''$ be the composition of ϕ with the projection $E \otimes \Omega \to E \otimes \Omega''$. Then each Higgs cohomology sheaf $\mathcal{H}^i(E,\phi'')$ is equipped with a Higgs field ϕ' with values in Ω', and there is a spectral sequence*

$$E_2^{i,j} \cong \mathcal{H}^i(\mathcal{H}^j(E,\phi''),\phi') \Rightarrow \mathcal{H}^{i+j}(E,\phi)$$

Proof. Let F denote the usual Koszul filtration of $\Lambda^\cdot\Omega$ associated to the inclusion of Ω' in Ω. Thus, $F^i\Lambda^j\Omega$ is the image of the natural map

$$\Lambda^i\Omega' \otimes \Lambda^{j-i}\Omega \to \Lambda^j\Omega,$$

and $\mathrm{Gr}^i_F \Lambda^{i+j}\Omega \cong \Lambda^i\Omega' \otimes \Lambda^j\Omega''$. If we also denote by F the corresponding filtration of $E\otimes\Omega$, then F is compatible with the boundary maps of the Higgs complex of (E,ϕ), which thus can be regarded as a filtered complex. Let $E_r^{i,j}$ be the spectral sequence of this filtered complex. Then:

$$E_0^{i,j} \cong \mathrm{Gr}^i_F(E \otimes \Lambda^{i+j}\Omega) \cong E \otimes \Lambda^i\Omega' \otimes \Lambda^j\Omega''.$$

Furthermore, one verifies that the boundary map $E_0^{i,j} \to E_0^{i,j+1}$ in the spectral sequence can be identified with the identity of $\Lambda^i\Omega'$ times the boundary map of the Higgs complex of ϕ''. Thus

$$E_1^{i,j} \cong \mathcal{H}^j(E,\phi'') \otimes \Lambda^i\Omega',$$

and in particular $d_1^{0,j}\colon E_1^{0,j} \to E_1^{1,j}$ is a map

$$\mathcal{H}^j(E,\phi'') \to \mathcal{H}^j(E,\phi) \otimes \Omega'.$$

One checks that $d_1^{i,j}$ is obtained from $d_1^{0,j}$ as in the diagram (2.1.1). Hence $d_1^{0,j}$ defines a Higgs field on $\mathcal{H}^j(E,\phi'')$ with values in Ω', and $E_1^{\cdot,j}$ can be identified with its Higgs complex. Thus

$$E_2^{i,j} \cong \mathcal{H}^i(\mathcal{H}^j(E,\phi''),\phi') \Rightarrow \mathcal{H}^{i+j}(E,\phi).$$

□

Corollary 2.2.2. *Let ϕ be a T-Higgs field on an $\mathcal{O}_X$-module E. Suppose there exists a nowhere vanishing section t of T such that ϕ_t is an automorphism of E. Then $\mathcal{H}^q(E,\phi) = 0$ for all q.*

Proof. If, in addition to the hypothesis of the corollary, T has rank one, then t defines an isomorphism $\Omega \cong \mathcal{O}_X$, and the Higgs complex of (E,ϕ) can be identified with the complex $E \to E$ whose boundary map is the endomorphism ϕ_t. Since ϕ_t is an isomorphism, this complex is acyclic. In the general case, t defines an exact sequence $0 \to \Omega' \to \Omega \to \mathcal{O}_X \to 0$. Then the Higgs field ϕ'' on E with values in $\mathcal{O}_X$ defined by this exact sequence is just ϕ_t, so $\mathcal{H}^j(E,\phi'') = 0$ for all j. Thus the corollary follows from the spectral sequence in (2.2.1). ☐

Corollary 2.2.3. *Let ϕ be a T-Higgs field on an $\mathcal{O}_X$-module E, let $(t_1,\dots t_n)$ be a basis for T, and let $(\phi_1,\dots \phi_n)$ be the corresponding sequence of endomorphisms of E. Suppose that for every i, ϕ_i acts surjectively (resp. injectively) on the kernel (resp. cokernel) of the map*

$$(\phi_1,\dots \phi_{i-1})\colon E \to E^{i-1}.$$

Then $\mathcal{H}^q(E,\phi) = 0$ for all $q > 0$ (resp. $q < n$).

Proof. We shall give the proof in the surjective case, which is slightly less standard. If $n = 1$, the Higgs complex can be identified with the complex $\phi_1\colon E \to E$, and the statement is a tautology. Let T'' be the submodule of T generated by $(t_1,\cdots t_{n-1})$ and let T' be the quotient. The duals fit into an exact sequence $0 \to \Omega' \to \Omega \to \Omega'' \to 0$, and an induction assumption implies that the Higgs cohomology sheaves $\mathcal{H}^j(E,\phi'')$ vanish for $j > 0$. Thus in the spectral sequence of (2.2.1), $E_2^{i,j} = 0$ for $j > 0$. Then the spectral sequence degenerates, and $\mathcal{H}^i(E,\phi) \cong \mathcal{H}^i(\mathcal{H}^0(E,\phi''),\phi')$ for all i. But $\mathcal{H}^0(E,\phi')$ is the kernel of the map $(\phi_1,\dots,\phi_{n-1})$, and the Higgs field ϕ' can be identified with the endomorphism of this kernel induced by ϕ_n. The Higgs complex of ϕ' has length one, so its cohomology vanishes in degrees larger than one, and the cohomology vanishes in degree one if (and only if) ϕ' is surjective. ☐

2.3 Let E be a sheaf of $\mathcal{O}_X$-modules on a scheme X with a T-Higgs field ϕ, where T is locally free of rank n. If $f\colon X' \to X$ is a morphism and $T' := f^*T$, then $E' := f^*E$ inherits a T'-Higgs field ϕ'. By the right exactness of f^*, there is a natural isomorphism $f^*\mathcal{H}^n(E,\phi) \to \mathcal{H}^n(E',\phi')$. For example, if x is a scheme-theoretic or geometric point of X and $f\colon x := \operatorname{Spec} k(x) \to X$ is the corresponding morphism, we write $E(x)$ for f^*E.

Proposition 2.3.1. *Suppose X is noetherian and T is locally free of rank n. Let (E,ϕ) be a coherent sheaf with a T-Higgs field on X, let $\tilde{E}_\phi$ be the corresponding sheaf on $\mathbf{V}T^*$, and let x be a point of X. Then the following are equivalent.*

i) *The nth Higgs cohomology sheaf of $(E(x),\phi(x))$ on the scheme x vanishes.*
ii) *The stalk of $\tilde{E}_\phi$ at $i(x)$ vanishes.*
iii) *The stalk of the Higgs complex $K^{\cdot}(E,\phi)$ at x is acyclic.*

If $k(x)$ is infinite, these are also equivalent to the existence of a $t \in T_x$ which acts bijectively on E_x.

Proof. Suppose (1) holds. Since formation of the nth Higgs cohomology group commutes with base change, it follows that the fiber of $\mathcal{H}^n(E,\phi)$ at x vanishes. By Nakayama's lemma, the same is true of its stalk at x. By (2.1.2), $\mathcal{H}^n(E,\phi) \cong i^*\tilde{E}_\phi \otimes \Omega^n$, and hence by Nakayama's lemma (now on $\mathbf{V}T^*$), the stalk of $\tilde{E}_\phi$ at $i(x)$ vanishes. This shows that (1) implies (2). If (2) holds, (2.1.2) shows that the stalk of the Higgs complex at x is quasi-isomorphic to the zero complex, hence is acyclic, so (2) implies (3). The implication of (1) by (3) follows from the fact that formation of $\mathcal{H}^n_\phi$ commutes with base change. Condition (2) implies that the maximal ideal $\mathfrak{p}_0$ of $S^{\cdot}T(x)$ corresponding to $i(x)$ does not belong to the support of $\tilde{E}_\phi(x)$. Then $\mathfrak{p}_0$ does not contain any associated prime $\mathfrak{p}$ of $E_\phi(x)$, *i.e.*, for each such $\mathfrak{p}$, the map $\mathfrak{p}_0 \to k(\mathfrak{p})$ is not zero. Since $\mathfrak{p}_0$ is generated by $T(x)$, the map $T(x) \to k(\mathfrak{p})$ is not zero, so its kernel is a proper $k(x)$-linear subspace of $T(x)$. Since $k(x)$ is infinite and there are only finitely many associated primes, there exists an element of $T(x)$ which is not in any of these kernels, and which then acts injectively, hence bijectively, on $E(x)$. Since the map $T_x \to T(x)$ is surjective, there exists a $t \in T_x$ which acts bijectively on $E(x)$. Then by the implication (1) implies (3) for the Higgs field defined by ϕ_t (with $n = 1$), it follows that ϕ_t is bijective on E_x. □

DEFINITION 2.3.2. Let (E, ϕ) be a T-Higgs module on X and let x be a point of X. Then x is said to be a *critical point* for ϕ if the stalk of the Higgs complex of (E, ϕ) at x is not acyclic. If this is not the case, x is said to be *noncritical*.

For example, if X/S is smooth and f is a global section of $\mathcal{O}_X$, then df is a global section of $\Omega^1_{X/S}$ and defines a $T_{X/S}$-Higgs field on $\mathcal{O}_X$. A point is a critical point for this Higgs field if and only if it is a critical point of the function f.

2.4 Let X/S be a smooth morphism of schemes in characteristic $p > 0$ and let $P_{X/S}$ denote the PD-envelope of the diagonal $X \to X \times_S X$ [BO78, 3.19]. Recall that $P_{X/S}$ has the same underlying topological space as X and that its structure sheaf $\mathcal{P}_{X/S}$, viewed as an $\mathcal{O}_X$-module through the right, carries a canonical HPD stratification, induced by the map $\delta\colon \mathcal{P}_{X/S} \to \mathcal{P}_{X/S} \otimes \mathcal{P}_{X/S}$ (see [BO78, p. 2.18]). Let $\nabla_{\mathcal{P}}$ denote the integrable connection on $\mathcal{P}_{X/S}$ corresponding to this stratification, and let $J_{X/S}$ be the ideal of X in $P_{X/S}$. Then $\nabla_{\mathcal{P}}$ maps $J^{[n]}_{X/S}$ to $J^{[n-1]}_{X/S} \otimes \Omega^1_{X/S}$ for all n, and hence induces a connection on the completion $\hat{\mathcal{P}}_{X/S}$ of $\mathcal{P}_{X/S}$ with respect to the system $\{J^{[n]}_{X/S} : n \in \mathbf{N}\}$. If E is any sheaf of $\mathcal{O}_X$-modules, let $R(E) := E \otimes \mathcal{P}_{X/S}$ and $\hat{R}(E) := E \hat{\otimes} \mathcal{P}_{X/S}$, the completion of $R(E)$ with respect to the system $\{J^{[n]}_{X/S}\}$. Note that we are computing the tensor product using the *left* module structure of $\mathcal{P}_{X/S}$, so that the connection of $\mathcal{P}_{X/S}$, coming from the right module structure, passes over to $R(E)$ and $\hat{R}(E)$: $\nabla(e \otimes z) := e \otimes \nabla(z)$ for $z \in \mathcal{P}_{X/S}$ and $e \in E$.

The construction R can also be interpreted in terms of differential operators. Let $\mathcal{D}_{X/S}$ denote the ring of PD-differential operators [BO78, §4] of X/S. Recall that as a sheaf of $\mathcal{O}_X$-modules,

$$\mathcal{D}_{X/S} \cong \varinjlim \mathcal{H}om(\mathcal{P}_{X/S}/J^{[n+1]}_{X/S}, \mathcal{O}_X),$$

where the $\mathcal{H}om$ is computed using the left $\mathcal{O}_X$-module structure on $\mathcal{P}_{X/S}$. Then the category of $\mathcal{O}_X$-modules with connection can be identified with the category of left $\mathcal{D}_{X/S}$-modules. If E is any $\mathcal{O}_X$-module, then $\hat{R}(E) \cong \mathcal{H}om_{\mathcal{O}_X}(\mathcal{D}_{X/S}, E)$. Here we are using the left $\mathcal{O}_X$-module structure on $\mathcal{D}_{X/S}$, and the connection on $\hat{R}(E)$ corresponds to the left $\mathcal{D}_{X/S}$-action coming from the action of $\mathcal{D}_{X/S}$ on itself on the right. If E is injective in the category of $\mathcal{O}_X$-modules, $\hat{R}(E)$ is injective in the category of $\mathcal{D}_{X/S}$-modules, since the functor $\hat{R}$ has an exact left adjoint which consists of forgetting the $\mathcal{D}_{X/S}$-module structure.

PROPOSITION 2.4.1. *If E is any sheaf of $\mathcal{O}_X$-modules, then for any $i > 0$,*

$$\mathcal{H}^i_{DR}(R(E)) \cong \mathcal{H}^i_{DR}(\hat{R}(E)) \cong 0,$$

and

$$\mathcal{H}^i_{\psi}(R(E), \nabla) \cong \mathcal{H}^i_{\psi}(\hat{R}(E), \nabla) \cong 0,$$

Proof. We may verify this proposition locally, with the aid of a system of coordinates $(x_1, \ldots x_n)$ for some open subset U of X/S. Let $\xi_i := 1 \otimes x_i - x_i \otimes 1$, and let $\partial_i := \partial/\partial x_i \in T_{X/S}$. For each multi-index $I := (I_1, \cdots, I_n)$, let $\xi^{[I]} := \xi_1^{[I_1]} \cdots \xi_n^{[I_n]}$. Then the set of divided power monomials $\xi^{[I]}$ forms a basis for $\mathcal{P}_{X/S}$ as an $\mathcal{O}_X$-module (using either the left or right structure), and the dual basis for $\mathcal{D}_{X/S}$ is the set of monomials $\partial^I := \partial_1^{I_1} \cdots \partial_n^{I_n}$. Recall that the stratification on $\mathcal{P}_{X/S}$ is induced from the map δ^*, which sends $\xi^{[K]}$ to $\sum_{I+J=K} \xi^{[I]} \otimes \xi^{[J]}$. Hence the connection $\nabla_{\mathcal{P}}$ sends $\xi^{[I]}$ to $\sum_i \xi^{[I-\epsilon_i]} \otimes dx_i$, where $(\epsilon_1, \dots \epsilon_n)$ is the standard basis for $\mathbf{Z}^n$ and where $\xi^{[J]} := 0$ if any $J_i < 0$.

A section of $R(E)$ (resp. of $\hat{R}(E)$) over U can be written uniquely as a sum (resp. formal sum) $\sum e_I \otimes \xi^{[I]}$, where each e_I is a section of E over U. The connection on $E \otimes \mathcal{P}_{X/S}$ on such a (formal) sum is given by

$$\nabla_{\mathcal{P}} \left(\sum_I e_I \otimes \xi^{[I]} \right) = \sum_i \sum_I e_I \otimes \xi^{[I-\epsilon_i]} \otimes dx_i. \tag{2.4.1}$$

Then the vanishing of $\mathcal{H}^i_{DR}(\hat{R}(E))$ and $\mathcal{H}^i_{DR}(R(E))$ follows from the standard calculation used to prove the crystalline Poincaré Lemma [BO78, 6.12].

Let $\partial'_i := \pi^*(\partial_i)$ and let $\psi_i := \psi(\partial'_i)$ be the endomorphism of $R(E)$ (resp. $\hat{R}(E)$) induced by the p-curvature of $\nabla_{\mathcal{P}}$. Since $\partial_i^{(p)} = 0$, $\psi_i = \nabla(\partial_i)^p$. Thus

$$\psi_i(\xi^{[I]}) = \xi^{[I-p\epsilon_i]}$$

Then the annihilator of $(\psi_1, \dots \psi_i)$ on $R(E)$ (resp. $\hat{R}(E)$) consists of the set of (formal) sums $\sum e_I \otimes \xi^{[I]}$ such that $e_{I_j} = 0$ whenever $I_j \geqslant p$ and $j \leqslant i$. The action of ψ_{i+1} on such sums is evidently surjective, so the vanishing follows from (2.2.3). □

2.5 An integrable connection on E is equivalent to the structure of a left $\mathcal{D}_{X/S}$-module on E. Such a connection then defines a natural horizontal map

$$\epsilon \colon E \to \hat{R}(E), \quad \text{where} \quad \epsilon(e)(D) := D(e)$$

for each section of e of E and D of $\mathcal{D}_{X/S}$.

Proof of Theorem (1.2.1). To see that $(E \otimes F^*_{X/S}\Omega^{\cdot}_{X'/S}, \psi)$ is a complex, we check that ψ defines an $F^*T_{X'/S}$-Higgs field on E, as explained in (2.1.1). This can be checked locally. Let $(\partial_1, \dots \partial_n)$ be the basis for $T_{X/S}$ dual to the differentials $(dx_1, \dots dx_n)$ of a system of coordinates, and let $\psi_i := \psi(\pi^*(\partial_i))$. Since ∇ is integrable and $[\partial_i, \partial_j] = 0$, $[\nabla(\partial_i), \nabla(\partial_j)] = 0$. Since $\partial_i^{(p)} = 0$, $\psi_i = \nabla(\partial_i)^p$, so $[\psi_i, \psi_j] = 0$. It follows that ψ satisfies the integrability condition required in the definition of a Higgs field (2.1.1) and that $\psi^{i+1}\psi^i = 0$. It also follows that each ψ_i commutes with each $\nabla(\partial_i)$, which implies that the boundary maps of the complex $(E^{\cdot}, \psi)$ commute with the connection on each term.

We claim next that the p-curvature of the induced connection on the Higgs cohomology sheaves vanishes. Again, it suffices to work locally, with the aid of a system of coordinates. Since the p-curvature of $F^*_{X/S}(\Omega^i_{X'/S})$ is zero, the p-curvature of the connection on $E \otimes F^*_{X/S}(\Omega^i_{X'/S})$ is the map induced by p-curvature of E. Thus it suffices to show that for each i and each j, the endomorphism of $\mathcal{H}^i(E, \psi)$ induced by $\psi_j \otimes \mathrm{id}$ is zero. But the Higgs cohomology can be identified with the Koszul cohomology of E with respect to $(\psi_1, \dots \psi_n)$, which is by construction annihilated by each ψ_j.

Associated to a short exact sequence $0 \to E' \to E \to E'' \to 0$ of $\mathcal{O}_X$-modules with integrable connection there is short exact sequence of De Rham complexes:

$$0 \to E' \otimes \Omega^{\cdot}_{X/S} \to E \otimes \Omega^{\cdot}_{X/S} \to E'' \otimes \Omega^{\cdot}_{X/S} \to 0,$$

and hence a long exact sequence of De Rham cohomology sheaves:

$$\cdots \to \mathcal{H}^i_{DR}(E', \nabla) \to \mathcal{H}^i_{DR}(E, \nabla) \to \mathcal{H}^i_{DR}(E'', \nabla) \to \mathcal{H}^{i+1}_{DR}(E', \nabla) \to \cdots .$$

Similarly there is a short exact sequence of p-curvature complexes

$$0 \to E' \otimes F^*_{X/S}\Omega^{\cdot}_{X'/S} \to E \otimes F^*_{X/S}\Omega^{\cdot}_{X'/S} \to E'' \otimes F^*_{X/S}\Omega^{\cdot}_{X'/S} \to 0,$$

and consequently a long exact sequence:

$$\cdots \to \mathcal{H}^i_\psi(E', \nabla) \to \mathcal{H}^i_\psi(E, \nabla) \to \mathcal{H}^i_\psi(E'', \nabla) \to \mathcal{H}^{i+1}_\psi(E', \nabla) \to \cdots .$$

As we observed above, each term in this sequence admits an integrable connection with p-curvature zero. It then follows from the theory of Cartier descent [Kat70, 5.1] that the sequence of horizontal sections:

$$\cdots \to \mathcal{H}^i_\psi(E', \nabla)^\nabla \to \mathcal{H}^i_\psi(E, \nabla)^\nabla \to \mathcal{H}^i_\psi(E'', \nabla)^\nabla \to \mathcal{H}^{i+1}_\psi(E', \nabla)^\nabla \to \cdots$$

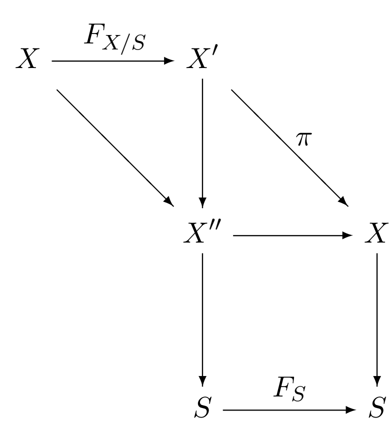
is still exact. (We should point out that the statement in *op. cit.* is for quasi-coherent sheaves, but the proof also applies to arbitrary $\mathcal{O}_X$-modules. See also (3.2.2).) Thus, each of $\{\mathcal{H}^i_{DR} : i \geqslant 0\}$, $\{\mathcal{H}^{i,\nabla}_\psi : i \geqslant 0\}$, and $\{\mathcal{H}^{i,\nabla}_{-\psi} : i \geqslant 0\}$ forms a cohomological ∂-functor. If (E, ∇) is a sheaf with an integrable connection, there is a horizontal injection $(E, \nabla) \to (\hat{R}(E), \nabla)$, so by (2.4.1), both sequences of functors are effaceable. Consequently there is a unique isomorphism of cohomological ∂-functors $\mathcal{H}^{\cdot,\nabla}_{-\psi} \to \mathcal{H}^\cdot_{DR}$ extending the obvious one in degree zero. □

Remark 2.5.1. A connection ∇ on E is quasi-nilpotent if locally on X, every local section e of E is annihilated by some power of the p-curvature. If this is the case, then the canonical map $\epsilon \colon E \to \hat{R}(E)$ factors through $R(E)$. This statement can be verified locally with the aid of a system of coordinates as above. By definition, $\epsilon(e)$ is the formal sum $\sum \partial^I(e) \otimes \xi^{[I]}$, and by [BO78, 4.10], this sum is in fact a finite sum (locally on X). The standard connection $\nabla_{\mathcal{P}}$ on $\mathcal{P}_{X/S}$ is quasi-nilpotent, and the category $MIC^{qn}(X/S)$ of modules with quasi-nilpotent connection is a thick abelian subcategory of $MIC(X/S)$. Thus Theorem (1.2.1) is also true in this subcategory.

3. Log schemes and log connections

3.1 In this section we extend Theorem (1.2.1) to the case of connections with log poles. We shall do this using the language of log schemes, which in fact considerably increases its generality and scope. Readers who are unfamiliar with this language can restrict to the case of the log structures arising from a smooth scheme X with a divisor D with normal crossings. In this case, the log structure is just the inclusion $M_X \to \mathcal{O}_X$, where M_X is the sheaf of sections of $\mathcal{O}_X$ which are invertible outside D.

The log case introduces several technical and conceptual difficulties (some of which do not arise in the case of divisors with relative normal crossing or in the case of semistable reduction). Let X/S be a smooth morphism of fine log schemes in characteristic $p > 0$. We denote simply by $\Omega^1_{X/S}$ the sheaf of relative log differentials and $T_{X/S}$ its dual; these are locally free sheaves of finite rank on X. First of all, in this generality it is not true that $\mathcal{O}_{X'}$ is the kernel of $d \colon F_{X/S*}(\mathcal{O}_X) \to F_{X/S*}(\Omega^1_{X/S})$. As Kato points out in [Kat89, 4.12], one must replace the relative Frobenius diagram with a refinement:



Here X'' is the fiber product of X with S over F_X, computed in the category of fine log schemes, and $X \to X' \to X''$ is the canonical factorization of the weakly purely inseparable map $X \to X''$ into an exact morphism followed by an étale morphism. (Our notation is not the same as Kato's). In the case of morphisms of Cartier type (for example, in the case of log structures arising from divisors with normal crossings or a semistable degeneration) $X' \to X''$ is the identity map and the underlying scheme of X'' is the usual fiber product, so that this subtlety can safely be ignored. Since the square is Cartesian and $X' \to X''$ is (log) étale, it is still true that the map $\pi^*\Omega^1_{X/S} \to \Omega^1_{X'/S}$ is an isomorphism. In this context, Kato constructs a canonical isomorphism

$$C^{-1}_{X/S}\colon \Omega^i_{X'/S} \to F_{X/S*}(\mathcal{H}^i(\Omega^{\cdot}_{X/S})) \tag{3.1.1}$$

generalizing the classical Cartier isomorphism.

Another difficulty is that Cartier descent is considerably more complicated for log schemes. Even if S is the spectrum of a field with the trivial log structure and X/S is smooth, X might not be regular, so $F_{X/S}$ need not be flat. And even in the classical DNC case, $F^*_{X/S}$ does not induce an equivalence between the category of $\mathcal{O}_{X'}$-modules and the category of $\mathcal{O}_X$-modules with integrable and p-integrable connection. For a thorough discussion, see [Lor00].

Nevertheless we shall see that there is indeed a log version of Theorem (1.2.1). In fact, with proper formulation, the statement is almost the same. Let (E,∇) be a module with (logarithmic) integrable connection on X/S. Again, the p-curvature of (E,∇) can be interpreted as an $\mathcal{O}_X$-linear map

$$\psi\colon E \to E \otimes F^*_{X/S}\Omega^1_{X'/S};$$

see [Ogu94, §1]. One verifies easily as before that ψ defines an $F^*_{X/S}T_{X/S}$-Higgs field on E and that it is horizontal for the natural connections on its source and target. This allows us to define functors $\{\mathcal{H}^i_{-\psi} : i \in \mathbf{N}\}$ from the category $MIC(X/S)$ to the category of $\mathcal{O}_{X'}$-modules.

Theorem 3.1.1. *Let X/S be a smooth morphism of fine log schemes in characteristic $p > 0$, let $F_{X/S}\colon X \to X'$ be the exact relative Frobenius map, and let $MIC(X/S)$ be the category of sheaves of $\mathcal{O}_X$-modules equipped with an integrable connection. The sequences $\{\mathcal{H}^i_{DR} : i \in \mathbf{N}\}$ and $\{\mathcal{H}^{i\nabla}_{-\psi} : i \in \mathbf{N}\}$ form cohomological ∂-functors from the abelian category $MIC(X/S)$ to the category of $\mathcal{O}_{X'}$-modules. Moreover, there is a unique isomorphism of cohomological ∂-functors*

$$C^{-1}_{X/S}\colon \mathcal{H}^{\cdot\nabla}_{-\psi} \to \mathcal{H}^{\cdot}_{DR}$$

which agrees with the obvious one in degree 0. In particular, if (E,∇) is an object of $MIC(X/S)$, there is a canonical isomorphism of $\mathcal{O}_{X'}$-modules

$$C^{-1}_{X/S}\colon \mathcal{H}^i_{-\psi}(E,\nabla)^{\nabla} \cong \mathcal{H}^i_{DR}(E,\nabla).$$

3.2 The first step is to show that $\{\mathcal{H}^i_{-\psi} : i \in \mathbf{N}\}$ forms a cohomological ∂-functor. Recall [Ogu94, §1] that if X/S is a smooth morphism of fine log schemes, then locally on X there exists a sequence of sections $(m_1, \dots m_n)$ of M_X defining an étale map from X to the logarithmic affine space $A^{n\times}_S$. Here $A^{n\times}$ is the log scheme corresponding to the prelog structure $\mathbf{N}^n \to \mathbf{Z}[\mathbf{N}^n]$ and $A^{n\times}_S := A^{n\times} \times S$. In particular, $(\mathrm{dlog}\, m_1, \dots \mathrm{dlog}\, m_n)$ forms a basis of $\Omega^1_{X/S}$; let $(\partial_1, \dots \partial_n)$ be the dual basis of $T_{X/S}$. (For example, in the case of the trivial log structure, each m_i is a section of $\mathcal{O}^*_X$, $\mathrm{dlog}\, m_i = dm_i/m_i$, and $\partial_i = m_i\partial/\partial m_i$.) Thus each ∂_i is a (log) derivation $\mathcal{O}_X \to \mathcal{O}_X$ relative to S, and hence defines a (log) PD-differential operator of order 1. Since the connection ∇ is integrable, it defines an action of $\mathcal{D}_{X/S}$ on E, and if $D \in \mathcal{D}_{X/S}$ we denote by ∇_D or $\nabla(D)$ the corresponding $\mathcal{O}_{X'}$-linear endomorphism of E. For each i, let $\partial'_i := \pi^*(\partial_i) \in \mathcal{D}_{X'/S} \cong \pi^*\mathcal{D}_{X/S}$.

The following result already appears in the proof of [Ogu94, 1.3.4]. (The formulas for h_i given there look different, but reduce to the simpler ones here when $p\mathcal{O}_X = 0$.)

Proposition 3.2.1. *Suppose X/S is equipped with a system of log coordinates $(m_1, \dots m_n)$, and let (E, ∇) be an $\mathcal{O}_X$-module with integrable connection. For each $i \in \{1, \dots n\}$, let $\psi_i := \psi(\partial_i') \in \mathrm{End}_{\mathcal{O}_X}(E)$, and let $h_i := \mathrm{id} - \partial_i^{p-1} \in \mathcal{D}_{X/S}$.*

i) $\nabla(\partial_i)\nabla(h_i) = \nabla(h_i)\nabla(\partial_i) = -\psi_i$.

ii) *Let h be the PD-differential operator $\prod_i h_i$. Then if the p-curvature of E is zero, $\nabla(h)\colon E \to E$ is a projection operator with image E^∇.*

Proof. In the ring $\mathcal{D}_{X/S}$, the pth power D^p of a derivation D is a differential operator of order p and needs to be distinguished from the derivation $D^{(p)}$ used in the definition of p-curvature. If ∂_i is one of the log derivations coming from a system of log coordinates as above, then $\partial_i^{(p)} = \partial_i$ [Ogu94, 1.2.2]. Then

$$\begin{aligned}\psi(\partial_i') &= \nabla(\partial_i)^p - \nabla(\partial_i^{(p)}) \\ &= \nabla(\partial_i^p) - \nabla(\partial_i) \\ &= \nabla(\partial_i)\nabla(\partial^{p-1} - 1) \\ &= -\nabla(\partial_i)\nabla(h_i)\end{aligned}$$

This proves the first statement. It follows that if $\psi = 0$, $\nabla\nabla_h(e) = 0$ for every $e \in E$. On the other hand, if $\nabla(e) = 0$, then $\nabla_{\partial_i}(e) = 0$ for every i, hence $\nabla_{h_i}(e) = e$ for every i and $\nabla_h(e) = e$. □

Corollary 3.2.2. *Let X/S be a smooth morphism of fine log schemes. Then if*

$$0 \to (E', \nabla') \to (E, \nabla) \to (E'', \nabla'') \to 0$$

is an exact sequence of $\mathcal{O}_X$-modules with integrable connection whose p-curvature vanishes, the sequence

$$0 \to E'^{\nabla'} \to E^\nabla \to E''^{\nabla''} \to 0$$

is also exact.

Proof. The statement is local on X, so we may assume that there exists a set of log coordinates. Then the exactness follows from the existence of the projection operator $\nabla(h)$ of (3.2.1). □

Proof of Theorem (3.1.1). The fact that $\{\mathcal{H}^i_{\psi,\nabla} : i \in \mathbf{N}\}$ and $\{\mathcal{H}^i_{DR} : i \in \mathbf{N}\}$ form cohomological ∂-functors follows as in the proof of Theorem (1.2.1), thanks to Corollary (3.2.2). Thus to prove the theorem it suffices to show that both are effaceable. We use the logarithmic version of the construction $\hat{R}$. In terms of the ring $\mathcal{D}_{X/S}$ of (logarithmic) differential operators, $\hat{R}(E) := \mathcal{H}om(\mathcal{D}_{X/S}, E)$. Suppose that X/S admits be a sequence of logarithmic coordinates $(m_1, \dots m_n)$. Then $\mathcal{D}_{X/S}$ is a free left $\mathcal{O}_X$-module [Ogu94, 1.1]. In fact there are two useful bases that we shall want to use. Let $\mathcal{P}$ denote the structure sheaf of the (exact) divided power envelope P of the diagonal $X \to X \times_s X$. If m is a section of M_X, $p_2^\flat(m)$ and $p_1^\flat(m)$ are two sections of M_P with the same on the diagonal, so there exists a section η of the ideal $J_{X/S}$ of the diagonal such that $p_2^\flat(m) = p_1^\flat(m)(1+\eta)$. If η_i the section thus constructed for each m_i, then $(\eta_1, \dots \eta_n)$ is a sequence of divided power generators for $J_{X/S}$, and the set of divided power monomials $\eta^{[I]} := \eta_1^{[I_1]} \dots \eta_n^{[I_n]}$ is a basis for $\mathcal{P}$ over $\mathcal{O}_X$ (as either a left or right module). Let

$$\zeta_i := \log(1+\eta_i) := \eta_i - \eta_i^{[2]} + 2!\eta_i^{[3]} + \cdots .$$

Then the divided power monomials in the ζ's also furnish a basis for $\mathcal{P}$. Let $\{\partial_I\}$ be the basis for $\mathcal{D}_{X/S}$ dual to $\{\eta^{[I]}\}$ and let $\{D_I\}$ be the basis dual to $\{\zeta^{[I]}\}$. It follows from the definitions and the cocycle condition that

$$\delta^*(\zeta_i) = 1 \otimes \zeta_i + \zeta_i \otimes 1$$

$$\delta^*(\zeta_K) = \sum_{I+J=K} \zeta^{[I]} \otimes \zeta^{[J]}$$

Hence $D_I D_J = D_{I+J}$ and

$$\nabla\zeta^{[I]} = \sum_i \zeta^{[I-\epsilon_i]} \otimes dlogm_i.$$

Then the fact that $\mathcal{H}^q_{DR}(\hat{R}(E) = 0$ for $q > 0$ again reduces to the divided power Poincaré lemma. For the effaceability of $\{\mathcal{H}^q_{-\psi} : q > 0\}$, one can either calculate directly or use the explicit local comparison (4.1.1) in the next section. For the direct calculation, note that the p-curvature operator ψ_i corresponding to D_i is $D_i^p - D_i$. Hence

$$\psi_i(\zeta^{[I]}) = \zeta^{[I-p\epsilon_i]} - \zeta^{[I-\epsilon_i]}.$$

Again we verify that $\mathcal{H}^q_{-\psi}(\hat{R}(E)) = 0$ for $q > 0$ using (2.2.3). Let e be a local section of $\hat{R}(E)$ annihilated by $(\psi_1, \ldots \psi_{i-1})$, and write e as a formal sum $\sum_i e_I \otimes \zeta^{[I]}$. Define e'_I inductively by setting

$$e'_I := \begin{cases} 0 & \text{if } I_i < p \\ e_{I-p\epsilon_i} + e'_{I-p\epsilon_i+1} & \text{if } I_i \geqslant p. \end{cases}$$

Then $e' := \sum e'_I \otimes \zeta^{[I]}$ is annihilated by $(\psi_1, \ldots \psi_{i-1})$ and $\psi_i(e') = e$. □

Remark 3.2.3. This is perhaps a good place to point out that, as Berthelot has observed, the equation on page 17 of [Ogu94] should read:

$$\Delta^*(\eta_i^{[n]}) = \sum_{a+b+c=n} c! \binom{a+c}{c} \binom{b+c}{c} \eta_i^{[a+c]} \otimes \eta_i^{[b+c]}.$$

4. Explicit formulas

4.1 Our first goal in this section is to give an explicit formula for the isomorphism $C_{X/S}$ of Theorem (3.1.1) in terms of local coordinates. For this purpose, let X/S be a smooth morphism of fine log schemes, and suppose that $(m_1, \ldots m_n)$ is a system of log coordinates for X/S. Then $(dlogm_1, \ldots dlogm_n)$ is a basis for $\Omega^1_{X/S}$. Let $(\partial_1, \ldots \partial_n)$ be the dual basis for $T_{X/S}$. Let $\omega_i := \operatorname{dlog} m_i \in \Omega^1_{X/S}$ and $\omega'_i := F^*_{X/S}\pi^*\omega_i \in F^*_{X/S}\Omega^1_{X'/S}$. If I is a subset of $\{1, \cdots n\}$ and $q := |I|$, we also denote by I the unique increasing function $\{1, \ldots q\} \to \{1, \ldots n\}$ whose image is I. Let $\omega_I := \omega_{I_1} \wedge \cdots \omega_{I_q} \in \Omega^q_{X/S}$, with similar notation for ω'_I. Note that each ω_i is closed, and that Kato's inverse Cartier operator (3.1.1) takes ω'_I to ω_I. The set of ω_I with $|I| = q$ forms a basis for $\Omega^q_{X/S}$, and similarly for $\Omega^q_{X'/S}$.

Let (E, ∇) be a sheaf of $\mathcal{O}_X$-modules with integrable connection on X/S. For each q we construct additive maps

$$\alpha^q \colon E \otimes \Omega^q_{X/S} \to E \otimes F^*_{X/S}\Omega^q_{X'/S}$$

$$\beta^q \colon E \otimes F^*_{X/S}\Omega^q_{X'/S} \to E \otimes \Omega^q_{X/S}$$

as follows. For each i, let $h_i := \mathrm{id} - \partial_i^{p-1}$ in the ring of log differential operators. For each $I \subseteq \{1, \ldots n\}$, define PD-differential operators $\alpha_I := \prod\{h_i : i \in I\}$ and $\beta_I := \prod\{h_i : i \notin I\}$. Then define

$$\alpha(\sum e_I \otimes \omega_I) = \sum \nabla_{\alpha_I}(e_I) \otimes \omega'_I$$
$$\beta(\sum e_I \otimes \omega'_I) = \sum \nabla_{\beta_I}(e_I) \otimes \omega_I$$

PROPOSITION 4.1.1. *With the above notation, $\alpha := \oplus\alpha^q$ and $\beta := \oplus\beta^q$ define morphisms of complexes:*

$$\begin{array}{ccccc}
(E \otimes \Omega^{\cdot}_{X/S}, \nabla) & \xrightarrow{\alpha} & (E \otimes F^*_{X/S}\Omega^{\cdot}_{X'/S}, -\psi) & & \\
& \searrow^{\beta\alpha} & \downarrow{\beta} & \searrow^{\alpha\beta} & \\
& & (E \otimes \Omega^{\cdot}_{X/S}, \nabla) & \xrightarrow{\alpha} & (E \otimes F^*_{X/S}\Omega^{\cdot}_{X'/S}, -\psi)
\end{array}$$

with the following properties:

i) *$\beta\alpha$ is homotopic to the identity.*

ii) *$\mathcal{H}^{\cdot}(\alpha\beta)\colon \mathcal{H}^{\cdot}_{-\psi}(E) \to \mathcal{H}^{\cdot}_{-\psi}(E)$ is a projection onto $\mathcal{H}^{\cdot}(E)^{\nabla}$.*

iii) *α induces an isomorphism $\alpha'\colon \mathcal{H}^{\cdot}_{DR}(E) \to \mathcal{H}^{\cdot}_{-\psi}(E)^{\nabla}$, β induces an isomorphism $\beta'\colon \mathcal{H}^{\cdot}_{-\psi}(E)^{\nabla} \to \mathcal{H}^{\cdot}_{DR}(E)$, and α' and β' are inverses of each other.*

Proof. In order to save space, we write De instead of $\nabla_D(e)$ if D is a PD-differential operator and e is a section of E. Let $\phi := -\psi$ and write $E^{\cdot}_{DR}$ for the De Rham complex of (E, ∇) and $E^{\cdot}_{\phi}$ for the Higgs complex of $(E, -\psi)$. If $i \in \{1, \ldots n\}$ and $I \subseteq \{1, \ldots, n\}$, let $\epsilon_{i,I} := (-1)^{m(i,I)}$, where m is the cardinality of the set of elements of I which are less than i. Thus

$$\omega_i \wedge \omega_I = \begin{cases} \epsilon_{i,I}\omega_{I\cup i} & \text{if } i \notin I, \\ 0 & \text{if } i \in I. \end{cases}$$

Any local section e of $E \otimes \Omega^q_{X/S}$ can be written uniquely as a sum $e = \sum e_I \otimes \omega_I$ with $e_I \in E$. Then the differential d of the complex $E^{\cdot}_{DR}$ is given explicitly as follows:

$$d(e_I \otimes \omega_I) = \sum_{i \notin I} \partial_i e_I \otimes \epsilon_{i,I}\omega_{I\cup i}.$$

Similarly, any element e of $E \otimes F^*_{X/S}(\Omega^q_{X'/S})$ can be written uniquely as a sum $e = \sum e_I \otimes \omega'_I$, and the differential ϕ of the complex $E^{\cdot}_{\phi}$ is given explicitly as:

$$\phi(e_I \otimes \omega'_I) = \sum_{i \notin I} \phi_i e_I \otimes \epsilon_{i,I}\omega'_{I\cup i}$$

To check that these form morphisms of complexes, we use (3.2.1):

$$\begin{aligned}
\alpha d(e_I \otimes \omega_I) &= \alpha\Big(\sum_{i \notin I} \partial_i e_I \otimes \epsilon_{i,I}\omega_{I\cup i}\Big) \\
&= \sum_{i \notin I} \alpha_{I\cup i}(\partial_i e_I) \otimes \epsilon_{i,I}\omega'_{I\cup i} \\
&= \sum_{i \notin I} \alpha_I(h_i\partial_i e_I) \otimes \epsilon_{i,I}\omega'_{I\cup i} \\
&= \sum_{i \notin I} \alpha_I(\phi_i e_I) \otimes \epsilon_{i,I}\omega'_{I\cup i} \\
&= \sum_{i \notin I} \phi_i(\alpha_I e_I) \otimes \epsilon_{i,I}\omega'_{I\cup i} \\
&= \phi(\alpha_I e_I \otimes \omega_I) \\
&= \phi\alpha(e_I \otimes \omega_I).
\end{aligned}$$

This shows that α is a morphism of complexes; the proof for β is similar:

$$\begin{aligned}
\beta\phi(e_I \otimes \omega'_I) &= \beta\big(\sum_{i\notin I} \phi_i e_I \otimes \epsilon_{i,I}\omega'_{I\cup i}\big) \\
&= \sum_{i\notin I} \beta_{I\setminus i}\phi_i e_I \otimes \epsilon_{i,I}\omega_{I\cup i} \\
&= \sum_{i\notin I} \beta_{I\setminus i}h_i\partial_i e_I \otimes \epsilon_{i,I}\omega_{I\cup i} \\
&= \sum_{i\notin I} \beta_I\partial_i e_I \otimes \epsilon_{i,I}\omega_{I\cup i} \\
&= \sum_{i\notin I} \partial_i\beta_I e_I \otimes \epsilon_{i,I}\omega_{I\cup i} \\
&= d(\beta_I e_I \otimes \omega_I) \\
&= d\beta(e_I \otimes \omega'_I).
\end{aligned}$$

Evidently $\beta\alpha$ is given by $\beta\alpha(e_I \otimes \omega_I) = h(e_I) \otimes \omega_I$, where $h := \prod\{h_i : 1 \leqslant i \leqslant n\}$ is the operator of (3.2.1). Similarly, $\alpha\beta(e_I \otimes \omega'_I) = h(e_I) \otimes \omega'_I$. Since $h_i := \mathrm{id} - \partial_i^{p-1}$, there exist differential operators r_i such that

$$h = \mathrm{id} + \sum_{i=1}^n \partial_i r_i.$$

Define R: $E \otimes \Omega^q_{X/S} \to E \otimes \Omega^{q-1}_{X/S}$ by

$$R(e_I \otimes \omega_I) := \sum_{j\in I} \epsilon_{j,I} r_j e_I \otimes \omega_{I\setminus j}.$$

We compute:

$$\begin{aligned}
(dR + Rd)(e_I \otimes \omega_I) &= \sum_{j\in I} d(\epsilon_{j,I} r_j e_I \otimes \omega_{I\setminus j}) + \sum_{i\notin I} R(\epsilon_{i,I}\partial_i e_I \otimes \omega_{I\cup i}) \\
&= \sum_{j\in I}\sum_{i\notin I\setminus j} \epsilon_{i,I\setminus j}\epsilon_{j,I}\partial_i r_j e_I \otimes \omega_{I\setminus j\cup i} + \sum_{i\notin I}\sum_{j\in I\cup i} \epsilon_{j,I\cup i}\epsilon_{i,I} r_j\partial_i e_I \otimes \omega_{I\cup i} \\
&= \sum_{j\in I} \epsilon_{j,I\setminus j}\epsilon_{j,I}\partial_j r_j e_I \otimes \omega_I + \sum_{j\in I}\sum_{i\notin I} \epsilon_{i,I\setminus j}\epsilon_{j,I}\partial_i r_j e_I \otimes \omega_{I\setminus j\cup i} + \\
&\quad \sum_{i\notin I} \epsilon_{i,I\cup i}\epsilon_{i,I} r_i\partial_i e_I \otimes \omega_I + \sum_{i\notin I}\sum_{j\in I} \epsilon_{j,I\cup i}\epsilon_{i,I} r_j\partial_i e_I \otimes \omega_{I\cup i\setminus j}
\end{aligned}$$

Since $\epsilon_{j,I} = \epsilon_{j,I\setminus j}$ if $j \in I$ and $\epsilon_{i,I} = \epsilon_{i,I\cup i}$ if $i \notin I$ and ∂_i and $r_i$commute,

$$\sum_{j\in I} \epsilon_{j,I}\epsilon_{j,I\setminus j}\partial_j r_j e_I \otimes \omega_I + \sum_{i\notin I} \epsilon_{i,I}\epsilon_{i,I\cup i} r_i\partial_i e_I \otimes \omega_I = \sum_{i=1}^n \partial_i r_i e_I \otimes \omega_I.$$

On the other hand, if $j \in I$ and $i \notin I$, then $i \neq j$. If $i < j$, then $m(j, I \cup i) = m(j, I) + 1$ and $m(i, I) = m(i, I \setminus j)$. If $j < i$, then $m(j, I \cup i) = m(j, I)$ and $m(i, I) = m(I, I \setminus j) + 1$. In any case, $\epsilon_{i,I\setminus j}\epsilon_{j,I} + \epsilon_{j,I\cup i}\epsilon_{i,I} = 0$. Thus the expression above reduces to

$$(dR + Rd)(e_I \otimes \omega_I) = \sum_{i=1}^n \partial_i r_i e_I \otimes \omega_I = (h - \mathrm{id})(e_I \otimes \omega_I).$$

This proves that h is homotopic to the identity.

It follows that the map on cohomology

$$\mathcal{H}(\alpha)\colon \mathcal{H}(E^{\cdot}_{DR}) \to \mathcal{H}(E^{\cdot}_{\phi})$$

is injective and split, the map

$$\mathcal{H}(\beta)\colon \mathcal{H}(E^{\cdot}_{\phi}) \to \mathcal{H}(E^{\cdot}_{DR})$$

is surjective and split, and the image of α coincides with the image of $\mathcal{H}(h)$. Recall that $E^{\cdot}_{\phi}$ is a complex of modules with integrable connection, and that the corresponding cohomology groups inherit a connection whose p-curvature is zero. Thus by (3.2.1) the action of $\mathcal{H}(h)$ on these cohomology groups is a projection onto the horizontal part. In other words, $\mathcal{H}(\alpha)$ identifies $\mathcal{H}^{\cdot}_{DR}(E)$ with $\mathcal{H}^{\cdot}_{\phi}(E)^{\nabla}$, as claimed in the theorem. □

Remark 4.1.2. Let us continue to suppose that X/S admits a set of log coordinates as in (4.1.1). Then for any E, the morphism of complexes

$$\beta_E\colon K^{\cdot}(E, -\psi) \to (E \otimes \Omega^{\cdot}, \nabla)$$

induces morphisms of cohomology sheaves:

$$\mathcal{H}^{\cdot}(\beta_E)^{\nabla}\colon \mathcal{H}^{\cdot}_{-\psi}(E)^{\nabla} \to H^{\cdot}(E, \nabla).$$

As E varies, these maps form a morphism of cohomological ∂-functors which agrees with the tautological one on $\mathcal{H}^0$. It follows from the uniqueness theorem for such morphisms that $H^{\cdot}(\beta)^{\nabla}$ is in fact the isomorphism labeled $C^{-1}_{X/S}$ in Theorem (3.1.1). Let us apply this remark when $(E, \nabla) = (\mathcal{O}_X, d)$. Taking into account the explicit formula for β, we see that $C^{-1}_{X/S}$ takes ω'_I to ω_I for any I with $|I| = q$. Thus the isomorphism of (3.1.1) agrees with the one defined by Kato, and, when the log structure is trivial, with the classical Cartier operator.

4.2 The classical Cartier operator is often described as a mapping from the set of closed i-forms on X to the set of i-forms on X'. At least for $i = 1$, this is possible in our generalized context as well: there is a map from the closed one-forms the De Rham complex of E to the closed forms in its p-curvature complex which lifts $C_{X/S}$. We show this with the aid of a construction that essentially appeared already in [Ogu77, 2.11]. This gives a "physical interpretation" of the generalized Cartier isomorphism $C_{X/S}$ (3.1.1) in this case.

Proposition 4.2.1. *Let X/S be a smooth morphism of fine log schemes in characteristic $p > 0$, let $\omega \in \Gamma(X, E \otimes \Omega^1_{X/S})$ be closed, and let $[\omega]$ be its class in $\Gamma(X, \mathcal{H}^1_{DR}(E, \nabla))$. Define an integrable connection ∇' on $E' := E \oplus \mathcal{O}_X$ by: $\nabla'(e \oplus a) := \nabla(e) + a\omega \oplus da$.*

i) *The p-curvature ψ' of ∇' is given by $\psi'(e \oplus a) = (\psi(e) - a\omega', 0)$, where ψ is the p-curvature of E and $\omega' \in E \otimes F^*_{X/S}\Omega^1_{X/S}$ satisfies*

$$\langle \pi^* D, \omega' \rangle = \langle D^{(p)}, \omega \rangle - \nabla_D^{p-1} \langle D, \omega \rangle.$$

for every $D \in T_{X/S}$.

ii) *The element ω' defined above is a closed cycle in the Higgs complex $(E \otimes F^*_{X/S}\Omega^{\cdot}_{X/S}, -\psi)$, and its image in $\Gamma(X, \mathcal{H}^1_{-\psi}(E, \nabla))$ is $C_{X/S}([\omega])$.*

Proof. One checks easily that ∇' is integrable. In fact, there is an exact sequence of modules with integrable connection

$$0 \to (E, \nabla) \to (E', \nabla') \to (\mathcal{O}_X, d) \to 0$$

whose class in $\Gamma(X, \mathcal{E}xt^1(\mathcal{O}_X, E)) \cong H^0(X, \mathcal{H}^1_{DR}(E))$ corresponds to $[\omega]$. According to (3.1.1), this sequence produces a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^0_{DR}(\mathcal{O}_X, d) & \longrightarrow & \mathcal{H}^1_{DR}(E, \nabla) \\ \downarrow C_{X/S} & & \downarrow C_{X/S} \\ \mathcal{H}^{0,\nabla}_{-\psi}(\mathcal{O}_X, d) & \longrightarrow & \mathcal{H}^{1,\nabla}_{-\psi}(E, \nabla) \end{array}$$

The image of 1 along the top row is, by construction, the class of ω. The image of 1 along the bottom row is, by definition $-\psi'(0 \oplus 1)$, where ψ' is the p-curvature of ∇'. It is clear *a priori* that $\psi'(e \oplus a) = \psi(e) - a\omega'$ for some $\omega' \in E \otimes F^*_{X/S}\Omega^1_{X'/S}$. In particular, $-\psi'(0 \oplus 1) = \omega'$, so the diagram shows that $C_{X/S}[\omega]$ is the class of ω'.

To compute ω', first check by induction on k that

$$\nabla'^k_D(e \oplus a) = (\nabla^k_D e + \sum_{i=0}^{k-1} \binom{k}{i} D^i(a) \nabla^{k-i-1}_D \langle D, \omega \rangle \oplus D^k a)$$

Hence

$$\nabla'^p_D(e \oplus a) = (\nabla^p_D e + a\nabla^{p-1}_D \langle D, \omega \rangle \oplus D^p a),$$

and

$$\begin{aligned} \psi'_{\pi^* D}(e \oplus a) &= \nabla'^p_D(e \oplus a) - \nabla'_{D^{(p)}}(e \oplus a) \\ &= (\psi_D(e) + a\nabla^{p-1}_D \langle D, \omega \rangle - a\langle D^{(p)}, \omega \rangle \oplus D^p a - D^{(p)} a) \end{aligned}$$

Since $D^p a = D^{(p)} a$, we see that $\psi'(e \oplus a) = (\psi(e) - a\omega' \oplus 0)$, where $\omega' \in F^*_{X/S}(\Omega^1_{X'/S})$ satisfies $\langle \pi^* D, \omega' \rangle = \langle D^{(p)}, \omega \rangle - \nabla^{p-1}_D \langle D, \omega \rangle$. □

5. Consequences and remarks

5.1 Throughout this section we let X/S be a smooth morphism of fine log schemes in characteristic $p > 0$.

A special case of the following result appeared already in [Ogu77], where it was one of the key ingredients in the proof of Katz's formula [Kat72] relating p-curvature to the Kodaira-Spencer mapping.

COROLLARY 5.1.1. *Let (E, ∇) be a module with integrable connection on X/S endowed with a horizontal filtration N such that the p-curvature of $\mathrm{Gr}_N E$ vanishes. Let (E, d) be the spectral sequence of the filtered complex $(E \otimes \Omega^{\cdot}_{X/S}, N)$. Then there is a commutative diagram whose horizontal arrows are isomorphisms:*

$$\begin{array}{ccccc} E_1^{i,j} & \xrightarrow{\cong} & \mathcal{H}^{i+j}_{DR}(\mathrm{Gr}^i_N E \otimes \Omega^{\cdot}_{X/S}) & \xleftarrow{C^{-1}_{X/S}} & \mathrm{Gr}^i_N(E)^\nabla \otimes \Omega^{i+j}_{X'/S} \\ \downarrow d_1^{i,j} & & \downarrow & & \downarrow -\overline{\psi} \\ E_1^{i+1,j} & \xrightarrow{\cong} & \mathcal{H}^{i+j+1}_{DR}(\mathrm{Gr}^{i+1}_N E \otimes \Omega^{\cdot}_{X/S}) & \xleftarrow{C^{-1}_{X/S}} & \mathrm{Gr}^{i+1}_N(E)^\nabla \otimes \Omega^{i+j+1}_{X'/S} \end{array}$$

Proof. Consider the short exact sequence in the category $MIC(X/S)$;

$$0 \to \mathrm{Gr}^{i+1}_N E \to N^i E / N^{i+2} E \to \mathrm{Gr}^i_N E \to 0.$$

The arrow $d_1^{i,j}$ of the spectral sequence is the boundary map in the long exact sequence of cohomology associated to this sequence by the cohomology ∂-functor $\mathcal{H}_{DR}$. By Theorem (3.1.1), there is a commutative diagram

$$\begin{array}{ccc} \mathcal{H}^{i+j}_{-\psi}(\mathrm{Gr}^i_N E) & \xrightarrow{\partial} & \mathcal{H}^{i+j+1}_{-\psi}(\mathrm{Gr}^{i+1}_N E) \\ C^{-1}_{X/S}\downarrow & & C^{-1}_{X/S}\downarrow \\ \mathcal{H}^{i+j}_{DR}(\mathrm{Gr}^i_N E) & \xrightarrow{\partial} & \mathcal{H}^{i+j+1}_{DR}(\mathrm{Gr}^{i+1}_N E) \end{array}$$

But $-\psi$ acting on $\mathrm{Gr}^i_N E$ is zero, so $\mathcal{H}_{-\psi}(\mathrm{Gr}^i_N E) \cong \mathrm{Gr}^i_N E$, and the boundary map is just the map induced by $-\psi$. □

5.2 For the remainder of this section, we suppose that X/S is a smooth morphism of fine log schemes, with S noetherian of characteristic $p > 0$. The following result is immediate consequence of (3.1.1) and (2.3.1).

Corollary 5.2.1. *Let (E, ∇) be a coherent sheaf with integrable connection on X/S. If $x \in X$ is noncritical (2.3.2) for the p-curvature of ∇, then the De Rham complex $(E \otimes \Omega^{\cdot}_{X/S}, \nabla)$ is acyclic in some neighborhood of x.*

Proposition 5.2.2. *Let (E, ∇) be a sheaf with integrable connection on X/S, where $S = \mathrm{Spec}\, A$ is affine. Then the E_2 term of the "second spectral sequence" of hypercohomology:*

$$E_2^{i,j} = H^i(X, \mathcal{H}^j(E \otimes \Omega^{\cdot}_{X/S})) \Rightarrow H^{i+j}_{DR}(E, \nabla)$$

can be rewritten:

$$E_2^{i,j} \cong H^i(X', \mathcal{H}^j_{-\psi}(E)^{\nabla}).$$

Consequently, if E is coherent and the critical locus (2.3.2) of ψ is proper over S, then the global hypercohomology $H^i_{DR}(E, \nabla)$ is finitely generated over A.

Proof. Since $F \colon X \to X'$ is a homeomorphism, Theorem (3.1.1) implies that we can rewrite $E_2^{i,j}$ as $H^i(X', \mathcal{H}^j_{-\psi}(E)^{\nabla}$. This gives the reinterpretation of the spectral sequence. The sheaves $\mathcal{H}^j_{-\psi}(E)^{\nabla}$ are coherent sheaves of $\mathcal{O}_{X'}$-modules. If their support is proper, it follows that each $E_2^{i,j}$ is finitely generated over A, and hence so is the abutment. □

If (E, ∇) is an $\mathcal{O}_X$-module with integrable connection and ω is a one-form on X, then the *ω-twist* of (E, ∇) is the connection ∇^{ω} on E defined by $\nabla^{\omega}(e) := \nabla(e) + e \otimes \omega$. If ω is closed, then ∇^{ω} is again integrable. If (E, ∇) is the constant connection on $\mathcal{O}_X$, this connection is denoted by L_{ω}, and its p-curvature is the map $1 \mapsto F^*_{X/S}(\pi^*\omega - C_{X/S}[\omega]) \otimes e$ (see [Kat72, 7.1.2]). In general, the ω-twist of (E, ∇) is the tensor product of (E, ∇) with L_{ω}, and hence its p-curvature ψ^{ω} is the map $e \mapsto \psi(e) + F^*_{X/S}(\pi^*\omega - C_{X/S}[\omega]) \otimes e$, where $C_{X/S}[\omega]$ is the Cartier operator applied to the cohomology class of ω. In particular, if $f \in \mathcal{O}_X(X)$ and $\omega = df$, then $\psi' = \psi + F^*_X(df) \otimes \mathrm{id}$. Note that if (E, ∇) is quasi-nilpotent (2.5.1), then the critical locus of ψ' coincides with the critical locus of f.

Corollary 5.2.3. *Suppose that $S = \mathrm{Spec}\, A$, let (E, ∇) be a coherent sheaf on X/S with integrable connection, and let f be a global section of $\mathcal{O}_X$. If the p-curvature of (E, ∇) is nilpotent and the critical locus of f is proper over S, then the De Rham cohomology of the df-twist of (E, ∇) is finitely generated over A.*

6. Appendix: p-curvature and divided powers

6.1 The key to the effaceability of the functors $\mathcal{H}^q_\psi$ for $q > 0$ was the computation of the p-curvature of the divided power envelope of the diagonal embedding. For this purpose, a local calculation, involving a choice of coordinates, was sufficient. In some circumstances it can be desirable to have an intrinsic formulation, and this is the purpose of the following proposition.

Let X/S be a smooth morphism of schemes in characteristic $p > 0$, and let (A, ∇) be a sheaf of $\mathcal{O}_X$-modules with integrable connection on X/S. Suppose that A is endowed with the structure of an $\mathcal{O}_X$-algebra which is compatible with the connection, and that there exists an ideal I of A endowed with a divided power structure $\{\gamma_m \colon I \to I : m \geqslant 1\}$ such that

$$\nabla_D(\gamma_m(a)) = \gamma_{m-1}(a)\nabla_D(a)$$

for all $m \geqslant 1$, $a \in I$, and $D \in T_{X/S}$. Observe that $a^p = 0$ for any $a \in I$. Hence if $f \in \mathcal{O}_X$, $a \in I$, and $D \in T_{X/S}$,

$$(\nabla_D(fa))^p = (D(f)a + f\nabla_D(a))^p = f^p(\nabla_D(a))^p.$$

Thus the map ϕ_D sending a to $(\nabla_D(a))^p$ is a linear map $I \to F_{X*}(A)$. Since ∇_D sends $I^{[2]}$ to I, ϕ_D annihilates $I^{[2]}$. It is also F_X-linear in D, so that $D \mapsto \phi_D$ defines an $\mathcal{O}_X$-linear map

$$\phi\colon F_X^* I \to A \otimes F^*_{X/S}\Omega^1_{X'/S}$$

which factors through $F_X^*(I/I^{[2]})$. We write $a^{[m]}$ for $\gamma_m(a)$ and use the conventions $\gamma_0(a) := 1$ for all a and $\gamma_m(a) := 0$ for $m < 0$.

Proposition 6.1.1. *Let (A, I, γ, ∇) be as above and let ψ be the p-curvature of (A, ∇). Then for any section a of I and any $m > 0$,*

$$\psi(a^{[m]}) = a^{[m-1]}\psi(a) + a^{[m-p]}\phi(1 \otimes a).$$

Proof. We need to calculate the action of differential operators on the divided powers of elements of I. The formula which follows does not use the fact that $pA = 0$, and is expressed in terms of partitions. If T is a set, by a *partition* of T we mean a set π of disjoint nonempty subsets of T whose union is all of T. Thus if π is such a partition, an element s of π is a subset of T. We let $|s|$ denote the cardinality of $|s|$ and $|\pi|$ the cardinality of π.

Lemma 6.1.2. *For any $n \geqslant 1$,*

$$\nabla^n_D(a^{[m]}) = \sum_\pi a^{[m-|\pi|]} \prod_{s\in\pi} \nabla^{|s|}_D(a),$$

where the sum is taken over the set P_n of all partitions of $\{1, \dots n\}$.

Proof. If $n = 1$, the formula just says that $\nabla_D(a^{[m]}) = a^{[m-1]}\nabla_D(a)$, which is certainly true for all $m \in \mathbf{Z}$. We proceed by induction on n. Assume the formula is true for n. If π is a partition of $\{1, \dots, n\}$ and s is an element of π, let π_s be the partition of $\{1, \dots, n+1\}$ obtained by adding $n+1$ to s and leaving the other elements of π unchanged: $\pi_s := \pi \setminus \{s\} \cup \{s \cup \{n+1\}\}$. Similarly, denote by $\pi_\emptyset$ the partition obtained by adjoining the singleton $\{n+1\}$ to π: $\pi_\emptyset := \pi \cup \{\{n+1\}\}$. In this way we obtain all the partitions of $\{1, \dots n+1\}$. Note that $|\pi_s| = |\pi|$ and $\pi_\emptyset = |\pi| + 1$.

From the Leibniz rule and the induction hypothesis, we obtain:

$$\begin{aligned}
\nabla_D^{n+1} a^{[m]} &= \nabla_D \left(\sum_{\pi \in P_n} a^{[m-|\pi|]} \prod_{s \in \pi} \nabla_D^{|s|}(a) \right) \\
&= \sum_{\pi \in P_n} \nabla_D (a^{[m-|\pi|]}) \prod_{s \in \pi} \nabla_D^{|s|}(a) + \sum_{\pi \in P_n} a^{[m-|\pi|]} \nabla_D \prod_{s \in \pi} \nabla_D^{|s|}(a) \\
&= \sum_{\pi \in P_n} a^{[m-|\pi|-1]} \nabla_D(a) \prod_{s \in \pi} \nabla_D^{|s|}(a) \\
&\quad + \sum_{\pi \in P_n} a^{[m-|\pi|]} \sum_{s \in \pi} \nabla_D^{|s|+1}(a) \prod_{s' \in \pi \setminus \{s\}} \nabla_D^{|s'|}(a) \\
&= \sum_{\pi \in P_n} a^{[m-|\pi_\emptyset|]} \prod_{t \in \pi_\emptyset} \nabla_D^{|t|}(a) + \sum_{s \in \pi \in P_n} a^{[m-|\pi_s|]} \prod_{t \in \pi_s} \nabla_D^{|t|}(a) \\
&= \sum_{\pi \in P_{n+1}} \prod_{t \in \pi} a^{[m-|\pi|]} \nabla_D^{|t|}(a)
\end{aligned}$$

□

If $\pi \in P_n$ and $|\pi| = r$, write $\pi = (s_1, \ldots, s_r)$ with $|s_1| \leqslant |s_2| \cdots \leqslant |s_r|$, and let $c(\pi) := (|s_1|, \ldots, |s_r|)$. Then $c(\pi)$ does not depend on the choice of the ordering of π, and c is a function from P_n to the set of finite nondecreasing subsequences of the positive integers. In fact, the fibers of c are exactly the orbits of P_n under the natural action of the symmetric group S_n. For each subsequence $I = (I_1, \ldots I_r)$, let $\nu_n(I)$ denote the cardinality of the corresponding orbit, and let $r(I)$ be the length of I. Then Lemma (6.1.2) implies that

$$\nabla_D^n(a^{[m]}) = \sum_I \nu_n(I) a^{[m-r(I)]} \prod_j \nabla_D^{I_j}(a) \tag{6.1.1}$$

The cyclic group $\mathbf{Z}/n\mathbf{Z}$ operates on P_n through its inclusion in the symmetric group S_n, and it is evident that the only partitions fixed under this action are the partitions $\pi_1 := \{1, \ldots n\}$ and $\pi_n := \{\{1\} \ldots \{n\}\}$. In particular when $n = p$, all the other orbits have cardinality p, and hence all the other orbits under the full action of S_n have cardinality divisible by p. Thus, if $n = p$, $\nu_n(I)$ is divisible by p unless $I = (p)$ or $I = (1, 1, \ldots 1)$. Hence if $pA = 0$, (6.1.1) when $p = n$ reduces to:

$$\nabla_D^p(a^{[m]}) = a^{[m-1]} \nabla_D^p(a) + a^{[m-p]} \nabla_D(a)^p.$$

By the definition of p-curvature,

$$\begin{aligned}
\psi_D(a^{[m]}) &= \nabla_D^p(a^{[m]}) - \nabla_{D^{(p)}}(a^{[m]}) \\
&= a^{[m-1]} \nabla_D^p(a) + a^{[m-p]} \nabla_D(a)^p - \nabla_{D^{(p)}}(a^{[m]}) \\
&= a^{[m-1]} \psi_D(a) + a^{[m-p]} \phi_D(1 \otimes a).
\end{aligned}$$

This completes the proof of (6.1.1) □

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