SINGULARITIES OF THE HEIGHT STRATA IN THE MODULI OF K3 SURFACES

ARTHUR OGUS

Abstract. Moduli spaces of varieties in characteristic p are stratified by Newton polygons. For example, the moduli space of polarized K3 surfaces of degree d admits a stratification by the height h. Using the classification of supersingular K3 crystals, we determine the dimension and singular locus of each stratum. For example, if d is prime to p, the singular locus of the set where the height is at least h coincides with the set of points where the Artin invariant σ_0 is $\leq h-1$.

Let k be a perfect field of characteristic $p > 0$, let W be its Witt ring, and let F_k (resp. F_W) denote the absolute Frobenius automorphism of k (resp. of W). Fix a positive integer d. A *polarized K3 surface of degree* d over a k-scheme S is a smooth projective morphism $f\colon \mathcal{X} \to S$ and a relatively ample invertible sheaf $\mathcal{L}$ on $\mathcal{X}$ such that for every $s \in S$, the fiber $\mathcal{X}_s$ is a K3 surface and the self-intersection of $\mathcal{L}_s$ on X_s is $2d$. We shall also assume that $\mathcal{L}$ is *p-primitive*, *i.e.*, that $\mathcal{L}_s$ is not divisible by p for any $s \in S$. In his seminal paper [1], Artin used the formal Brauer group to define a height stratification S_h on the base S of a versal family $\mathcal{X}/S$ of polarized K3 surfaces. He found a sequence of closed subschemes

$$S = S_1 \supseteq S_2 \supseteq \cdots S_i \supseteq \cdots$$

and showed that each $S_{i+1} \subseteq S_i$ is defined by a single equation. For example, the "nonordinary locus" S_2 is defined by the vanishing of the action of Frobenius on $R^2 f_*(\mathcal{O}_X)$. For $i \geq 11$, all the S_i have the same reduced structure, and their intersection is the "supersingular locus S_∞," which (at least set-theoretically) admits a further stratification

$$S_{\infty,1} \subseteq S_{\infty,2} \subseteq \cdots S_{\infty,10} = S_\infty$$

by the "Artin invariant" σ_0. Artin proved that, in a neighborhood of a certain point of $S_{\infty,1}$, S_i has codimension $i-1$ if $i \leq 11$. Recently Katsura and van der Geer have looked again at these strata, and proved, among other things, that each S_i is equidimensional and that its singular locus is contained in the supersingular stratum S_{11} [6].

Katsura and van der Geer use the formal group methods of Artin, so to continue the propaganda campaign of [4], in this paper we use crystalline methods to define and analyze the strata S_h and their singularities in more detail. In our attempt to convince the reader that it is not only possible, but even easy, to extract geometric information from crystalline periods, we will try to make our argument self-contained, without reference to the theory of formal groups. This attempt is not meant to hide our gratitude to the aforementioned authors for their inspiration to return to the beautiful subject of K3 surfaces. It is also a pleasure to thank the referee for a very thorough job which resulted in many improvements to our exposition.

Our main new result is easiest to state when the polarization degree is prime to p. In this case the singular locus of S_i is, set-theoretically, $S_{\infty,i-1}$. It follows that S_i is normal if $i \leq 10$ and is generically nonreduced if $i \geq 11$. We also show that $S_{11} = S_\infty$ scheme-theoretically in this case.

The crystalline cohomology of a K3 surface X/k over k is a free W-module $H := H_{cris}(X/W)$ of rank 22. The absolute Frobenius endomorphism F_X of X induces an F_W-linear endomorphism Φ of H, which we sometimes view as a W-linear map $F_W^* H \to H$. The cup-product pairing and trace map endow H with a symmetric bilinear form

$$(\ |\ )\colon Sym^2 H \to W$$

which defines an isomorphism from H to its dual $H^\vee$. It is known (and trivial if p is odd) that this form is *even*: $(x|x)$ is divisible by 2 for every $x \in H$ [5]. Writing $Q(x) := 1/2(x|x)$, we find that Q is a *quadratic form* on H, *i.e.*, $Q \in Sym^2(H^\vee)$, and $(x|y) = Q(x+y) - Q(x) - Q(y)$ for $x, y \in H$. In fact, the triple (H, Φ, Q) forms a *K3-crystal*, as defined below.

Definition 1. *A K3 crystal over k of rank r is a finitely generated free module over W of rank r equipped with a nondegenerate quadratic form Q and an F_W-linear endomorphism $\Phi\colon H \to H$ satisfying the following conditions:*

1. *The reduction of Φ modulo p has rank one.*
2. *$Q(\Phi(x)) = p^2 F_W^*(Q(x))$ for any $x \in H$.*

The nondegeneracy of Q means that the associated bilinear form $(x, y) := Q(x+y) - Q(x) - Q(y)$ defines an isomorphism from H to its dual $H^\vee$.

If L is a line bundle on X, its crystalline Chern class $\xi := c_{cris}(L) \in H^2_{cris}(X/W)$ lies in $T_H := \{x \in H : \Phi(x) = px\}$, and it is p-primitive if and only if ξ is not divisible by p in T_H [4, 1.5]. We therefore define a *polarized K3 crystal* to be a pair (H, ξ), where H is a K3-crystal and ξ is an element of T_H which is not divisible by p. The degree of (H, ξ) is by definition $Q(\xi) \in \mathbf{Z}_p$. If the degree is prime to p and p is odd, then the orthogonal complement of ξ in H is again a K3-crystal. Many of the arguments we shall give can be simplified somewhat in this case by working with this orthogonal complement, and we invite the reader to adopt this point of view.

The structure of K3 crystals over an algebraically closed field k can be described fairly explicitly. Suppose that H is a K3 crystal of rank 22 and that k is algebraically closed. We refer to [3] for the definition of the Hodge and Newton polygons of an F-crystal. The fundamental theorem of Mazur tells us that the Newton polygon of (H, Φ) lies on or above the Hodge polygon, has integral break-points, and, because of the duality coming from Q, is symmetric. We shall call the x-coordinate of its first breakpoint the *Newton height* (or just *height*) of H, and we say that H is *supersingular* if all its Newton slopes are 1; in this case we say that the height is ∞. Mazur's theorem implies that $\mathrm{ht}(H) \geq 1$, and the symmetry implies that $22 - 2\mathrm{ht}(H)$ is the horizontal length of the Newton segment of slope 1. Since $\Phi(\xi) = p\xi$, this length is at least 1, so $\mathrm{ht}(H) \in \{1, 2 \ldots 10, \infty\}$. If $\mathrm{ht}(H) < \infty$, then according to a theorem of Katz [3], H can be expressed as a direct sum $H = H_h \bigoplus H_1 \bigoplus H'_h$, where H_h is an F-crystal with Hodge numbers $(1, h-1, 0)$, H'_h is the twisted dual of H_h, with Hodge numbers $(0, h-1, 2)$, and $H_1 \cong W \otimes T_H$ has rank $22 - 2h$ and Hodge numbers $(0, 22-h, 0)$. If $h(H) = \infty$, T has rank 22 and the p-adic ordinal of the discriminant of the restriction of Q to T is $p^{2\sigma_0}$, where σ_0 is an integer with $1 \leq \sigma_0 \leq 11$, called the *Artin invariant* of H. Furthermore, it is shown in [4] that H admits an orthogonal direct sum decomposition $H \cong H_0 \oplus H_1$, where H_0 is a supersingular K3 crystal of rank $2\sigma_0$ and H_1 is a twist of a unit root crystal. The quadratic form restricted to H_1 is nondegenerate, and its restriction to H_0 is p times a nondegenerate form Q'. It is apparent that $\sigma_0 \leq 10$ if H admits a polarization of degree prime to p, and in fact it is always true for the K3 crystals arising from K3 surfaces, as we shall see later.

Suppose that $H := (H, \Phi, Q)$ is a K3 crystal over an algebraically closed field k and $h \geq 1$ is an integer such that Φ^h is divisible by p^{h-1}; write $\Phi^h = p^{1-h}\Phi_h$. Note that $Q(\Phi_h(x)) = p^2 F_W^* Q(x)$ for $x \in H$. The abstract *Hodge and conjugate filtrations* M_h and N_h of (H, Φ_h) are defined as follows:

$$\begin{aligned} M_h^i H &:= \Phi_h^{-1}(p^i H) \\ N_h^i H &:= p^{i-2}\Phi_h(M_h^{2-i} H) \end{aligned}$$

We shall especially consider the filtrations induced by M_h and N_h on H_{DR}, the reduction modulo p of H. It follows from the compatibility of Φ_h and Q that $M_h^1 H_{DR}$ (resp. $N_h^1 H_{DR}$) is the annihilator of $M_h^2 H_{DR}$ (resp. $N_h^2 H_{DR}$). Mazur's theorem implies that, if X/k is a K3 surface, the filtration M_1 on $H_{DR} \cong H_{DR}(X/k)$ is just the Hodge filtration F_{Hdg}, and the filtration N_1 on $H_{DR}(X/k)$ is the conjugate filtration F_{con}. Consequently we shall write F_{Hdg} and F_{con} for the filtrations induced by M_1 and N_1 on H_{DR} in the abstract case as well. It follows that Φ induces an injective map

$$\psi \colon F_k^* \operatorname{Gr}^0_{Hdg} H_{DR} \to H_{DR}$$

whose image is exactly $F^2_{con} H_{DR}$, and an isomorphism

$$\gamma \colon F_k^* \operatorname{Gr}^1_{Hdg} H_{DR} \to \operatorname{Gr}^1_{con} H_{DR}.$$

Note that $F^2_{Hdg} H_{DR}$ and $F^2_{con} H_{DR}$ are isotropic, so that Q induces quadratic forms on $\operatorname{Gr}^1_{Hdg} H_{DR}$ and $\operatorname{Gr}^1_{con} H_{DR}$, and γ is compatible with these forms.

It is possible to compute the Newton slopes of a K3 crystal from the purely mod p data consisting of H_{DR}, F_{Hdg}, F_{con} and γ. We shall define, by induction on h, a class $\mathbf{K3}_h$ of K3 crystals over k, and for each $H \in \mathbf{K3}_h$ an increasing sequence of submodules

$$E_0 \subseteq E_1 \subseteq \cdots E_h$$

of H_{DR}, as follows. For $h = 1$, $\mathbf{K3}_1$ is the class of all K3-crystals over k, and E_1 is the submodule $N_1^2 H_{DR} = F_{con}^2 H_{DR}$. If $\mathbf{K3}_h$ has been defined and $H \in \mathbf{K3}_h$, then by definition $H \in \mathbf{K3}_{h+1}$ if and only if $E_h \subseteq F_{Hdg}^1 H_{DR}$. Then there is a map:

$$F_S^* E_h \to F_S^* F_{Hdg}^1 H_{DR} \to F_S^* \operatorname{Gr}_{Hdg}^1 H_{DR} \xrightarrow{\gamma} \operatorname{Gr}_{con}^1 H_{DR}$$

and E_{h+1} is defined to be the inverse image in $F_{con}^1 H_{DR} \subseteq H_{DR}$ of the image of this map. It follows by induction that each E_{h+1}/E_h has dimension at most one, and, by the compatibility of Q with γ, that each E_h is totally isotropic.

Lemma 2. *Let H be a K3 crystal over k and h a positive integer such that Φ^i is divisible by p^{i-1} for all $i \leq h$. Then $H \in \mathbf{K3}_h$, and*

$$E_h = N_h^2 + E_{h-1} = N_h^2 + N_{h-1}^2 + \cdots N_1^2 \subseteq H_{DR}.$$

Proof: It is convenient to set $E_0 := 0$; then the lemma is true for $h = 1$ by the definition of E_1. Suppose that $h > 1$ and that Φ^i is divisible by p^{i-1} for all $i \leq h$. By the induction assumption, $H \in \mathbf{K3}_{h-1}$, $E_{h-1} = E_{h-2} + N_{h-1}^2 H_{DR}$, and $E_{h-2} \subseteq F_{Hdg}^1 H_{DR}$. By definition, E_{h-1} is the inverse image of $\gamma F_k^*(E_{h-2})$ under the projection $\pi \colon F_{con}^1 H_{DR} \to \operatorname{Gr}_{con}^1 H_{DR}$. Since $\Phi\Phi_{h-1}$ is divisible by p, $N_{h-1}^2 H \subseteq M_1^1 H$, so $E_{h-1} \subseteq F_{Hdg}^1 H_{DR}$ and $H \in \mathbf{K3}_h$. Furthermore, $N_h^2 H$ is the image of $p^{-1}\Phi\Phi_{h-1} H$, *i.e.*, the image of $N_{h-1}^2 H$ under $p^{-1}\Phi$. Modulo $F_{con}^2 H_{DR}$, this is the image of $N_{h-1}^2 H$ under the map

$$N_{h-1}^2 H_{DR} \to F_k^* F_{Hdg}^1 H_{DR} \to \operatorname{Gr}_{Hdg}^1 H_{DR} \xrightarrow{\gamma} \operatorname{Gr}_{con}^1 H_{DR}.$$

which we just denote by $\gamma F_k^*(N_{h-1}^2 H_{DR})$. We conclude that

$$\pi(E_h) := \gamma F_k^*(E_{h-1}) = \gamma F_k^*(E_{h-2} + N_{h-1}^2) = \pi(E_{h-1}) + \pi(N_h^2).$$

Since E_h and $E_{h-1} + N_h^2$ both contain F_{con}^2 and have the same image modulo F_{con}^2, they must be equal. □

Proposition 3. *Let H be a K3 crystal over k and let h be a positive integer. Then the following are equivalent:*

1. *H has Newton height at least h.*
2. *For all $i \leq h$, Φ^i is divisible by p^{i-1}.*
3. *$H \in \mathbf{K3}_h$.*
4. *For all $i < h$, Φ^i is divisible by p^{i-1} and $N_i^2 H_{DR} \subseteq F_{Hdg}^1 H_{DR}$.*

Proof: The proof is by induction on h. The case $h = 1$ is trivial, and we assume that the proposition is true for h and that conditions (1)–(4) all hold, and we will prove their equivalence with $h+1$ in place of h. In particular, E_i is defined for $i \leq h$ and contained in $F^1_{Hdg}H_{DR}$ for $i < h$. It follows immediately from Lemma 2 that $E_h \subseteq F^1_{Hdg}H_{DR}$ if and only if $N^2_h H_{DR} \subseteq F^1_{Hdg}H_{DR}$, so conditions (3) and (4) are equivalent. Since $\Phi^{h+1} = p^{h-1}\Phi\Phi_h$, Φ^{h+1} is divisible by p^h if and only if $N^2_h H_{DR} \subseteq F^1_{Hdg}H_{DR}$. Thus (3) and (4) are also equivalent to (2). If Φ^{h+1} is divisible by p^h, then its Newton slopes are at least h, and hence the Newton slopes of Φ are at least $h/(h+1)$ and Φ has Newton height at least $h+1$. Conversely, suppose that H has Newton height at least $h+1$. The Newton slopes of Φ are at least $h/(h+1)$, and so the Newton slopes of $\Phi_h = p^{1-h}\Phi^h$ are at least $h^2/(h+1) - (h-1) = 1/(h+1) > 0$. Then Φ_h has no unit root part, and its action on H_{DR} is nilpotent. Since Φ_h and Φ commute, the image $F^2_{con}H_{DR}$ of Φ on H_{DR} is invariant under Φ_h, and since it is one-dimensional, Φ_h acts as zero on $F^2_{con}H_{DR}$. This says that $\Phi_h\Phi$ is divisible by p, *i.e.*, that Φ^{h+1} is divisible by p^h. □

Lemma 4. *Suppose that* $H \in \mathbf{K3}_h$ *and* $N^2_h H_{DR} \subseteq E_{h-1} + F^2_{Hdg}H_{DR}$. *Then* H *is supersingular.*

Proof: It follows from the hypothesis and Lemma 2 that for any $x \in H$, $\Phi_h(x) \in N^2_{h-1}H + \cdots N^2_1 H + pH + M^2 H$. Thus, there exist elements $x_1, \ldots x_h \in H$ and $z \in M^2 H$ such that

$$\Phi_h(x) = \Phi_{h-1}(x_1) + \Phi_{h-2}(x_2) + \cdots \Phi(x_{h-1}) + px_h + z.$$

Multiplying by p^{h-1} and applying Φ, we find that

$$\Phi^{h+1}(x) = p\Phi^h(x_1) + p^2\Phi^{h-1}(x_2) + \cdots p^h\Phi(x_h) + p^{h+1}x_{h+1},$$

where $x_{h+1} := p^{-2}\Phi(z)$. Then it follows by induction on m that for any $m \geq 1$ and $x \in H$, there exist $x_i \in H$ such that

$$\Phi^{h+m}(x) = p^m\Phi^h(x_1) + p^{m+1}\Phi^{h-1}(x_2) + \cdots p^{m+h}\Phi(x_h) + p^{h+1+m}x_{h+1}.$$

In particular Φ^{h+m} is divisible by p^m, and all its slopes are at least m. Then all the slopes of Φ are at least $m/(h+m)$, and taking the limit with m, we see that the slopes of Φ are all 1, *i.e.*, H is supersingular. □

The following is the key semi-linear algebra result that enables us to determine the singularities of the height strata.

Lemma 5. *Let* H *be a K3 crystal over* k.

1. *If* $\operatorname{ht}(H)$ *is finite, and* $i \leq \operatorname{ht}(H)$, *then* $E_i = N^2_i H_{DR}$ *and has dimension* i.
2. *If* H *is supersingular and* $i \leq \sigma_0(H)$, *then* $E_i = N^2_i H_{DR}$ *and has dimension* i. *Moreover,* $E_{\sigma_0} = E_{\sigma_0 - 1} + F^2_{Hdg}H_{DR}$, *and for* $i \geq \sigma_0$, $E_i = E_{\sigma_0}$ *and is the image of the annihilator of* Q *on* $k \otimes T_H$ *and also the annihilator of the image of* $k \otimes T_H \to H_{DR}$.

Proof: Suppose that $\mathrm{ht}(H)$ is finite and $i \leq \mathrm{ht}(H)$. Then by Lemmas 2 and 4, E_{i-1} is strictly contained in E_i. On the other hand, $\dim E_i \leq i$ for all i, so we can conclude that the dimension of E_i is i. Consider the Katz decomposition $H \cong H_h \oplus H_1 \oplus H'_h$, and restrict attention to the factor H_h. Since H_h has only 0 and 1 as Hodge slopes, there is an endomorphism V of H_h such that $\Phi V = p$. Then $\Phi_i V^i := p^{1-i}\Phi^i V^i = p$, so that Φ_i also has only 0 and 1 as Hodge slopes. Moreover, Φ has $(h-1)/h$ as its unique Newton slope, and so Φ_i has $i(h-1)/h-(i-1) = (h-i)/h$ as its unique Newton slope. Since the Hodge and Newton polygons of Φ_i have the same endpoint, the Hodge numbers of Φ_i must be $(i, h-i)$, and this proves that N_i^2 has dimension i. Up to this point we have been considering only the factor H_h, but we can conclude that N_i^2 of the entire crystal has dimension at least i. Lemma 2 implies that $N_i^2 H_{DR} \subseteq E_i$ for all i, and it follows that $N_i^2 H_{DR} = E_i$.

The proof of (2) uses the classification of supersingular K3 crystals from [4], which we briefly recall. Let

$$\begin{aligned} E &:= \bigcap_{h=1}^{\infty} M_h^1 H = \bigcap_{h=1}^{\infty} (\Phi^h)^{-1}(p^h H) \\ E'' &:= \sum_{h=1}^{\infty} N_h^2 H = \sum_{h=1}^{\infty} p^{1-h}\Phi^h(H) \end{aligned}$$

Then E is the largest submodule of H which is invariant under $\phi := p^{-1}\Phi$, $E'' \subseteq E$, and E'' is the set of all $h \in H$ such that $(h|e) \in pW$ for all $e \in E$. Better, let $E' := p^{-1}E \subseteq \mathbf{Q}\otimes H$, which is then the smallest submodule of $\mathbf{Q}\otimes H$ containing H which is invariant under ϕ. Then E' is the set of $h \in \mathbf{Q}\otimes H$ such that $(h|e) \in W$ for all $e \in E$, and $pE' \subseteq E \subseteq H \subseteq E'$. The k-vector spaces H/E and E'/H are naturally dual, and so have the same dimension σ_0. Then $2\sigma_0$ is the dimension of E'/E, and is also the p-adic ordinal of the discriminant of Q on E. Let $T := \{e \in E : \phi(e) = e\}$, and $T' := \{e' \in E' : \phi(e') = e'\}$. Then $W \otimes_{\mathbf{Z_p}} T \cong E$, $W \otimes_{\mathbf{Z_p}} T' \cong E'$, and these isomorphisms identify ϕ with $F_W \otimes \mathrm{id}$. The image of H in $E'/E \cong k \otimes T'/T$ is a k-linear subspace not contained in any proper ϕ-stable subspace. If e'_1 and e'_2 are elements of E', then $pe'_1 \in E$, hence $(e'_1|e'_2)' := (pe'_1|e'_2) \in W$ and $(\ |\ )'$ is a symmetric bilinear form on E' which is compatible with ϕ and hence descends to T'. The annihilator of this form modulo p is just the image of E in E', and so we can view it as a symmetric bilinear form on E'/E. When $p = 2$, the set of all $v' \in E'/E$ such that $(v'|v')' = 0$ is a linear subspace containing the image of H. Since this space is ϕ-invariant, it must be all of E'/E, so in fact $(\ |\ )'$ is even and we can define $Q'(e') := 1/2(e'|e')$ for $e' \in E'$. Then Q' is a nondegenerate quadratic form on T'/T, and the image K' of H in $k \otimes T'/T$ is a *strictly characteristic subspace* [4, 3.19]: it is totally isotropic of dimension σ_0, $(K' + \phi(K'))/K'$ has dimension one, and K' is not contained in any proper $\mathbf{F}_p$-rational subspace. The automorphism group of k over $\mathbf{F}_p$ acts on the set of all strictly characteristic subspaces, and it is convenient (elsewhere) to set $K := \phi^{-1}(K')$. As explained on page 33 of [4], there is a (unique) line ℓ in V such that

$$\phi(K) = \ell \oplus \phi(\ell) \oplus \cdots \phi^{\sigma_0 - 1}(\ell),$$

and in fact

$$V = \ell \oplus \phi(\ell) \oplus \cdots \phi^{2\sigma_0 - 1}(\ell).$$

There is an exact sequence

$$k \otimes T \to H_{DR} \to K' \to 0.$$

The subspaces $F^1_{Hdg}H_{DR}$ and $N^1_h H_{DR}$ contain the image of $k \otimes T$, and hence are determined by their images $F^1_{Hdg}K'$ and $N^1_h K'$ in K', and it is easy to check that $F^1_{Hdg}K' = K \cap \phi(K)$ and $N^1_h K' = \phi(K) \cap \phi^{h+1}(K)$. Suppose $h \leq \sigma_0$. Then because the spaces $\{\phi^i(\ell) : 0 \leq i \leq 2\sigma_0 - 1\}$ form a direct sum decomposition of V,

$$\begin{aligned} F^1_{Hdg}K' &= K \cap \phi(K) = \ell \oplus \cdots \phi^{\sigma_0 - 2}(\ell) \\ N^1_h K' &= \phi(K) \cap \phi^{h+1}(K) = \phi^h(\ell) \oplus \phi^{h+1}(\ell) \oplus \cdots \phi^{\sigma_0 - 1}(\ell) \end{aligned}$$

Thus in fact $0 = N^1_{\sigma_0}K' \subseteq N^1_{\sigma_0 - 1}K' \subseteq K'$, and each $N^1_i K'$ has codimension i if $i \leq \sigma_0$. Hence $N^2_i H_{DR}$ has dimension i if $i \leq \sigma_0$, and $N^2_{\sigma_0}H_{DR}$ is exactly the image of $k \otimes T \to H_{DR}$. By Lemma 2, E_i has dimension i if $i \leq \sigma_0$, $E_i = E_{\sigma_0}$ if $i \geq \sigma_0$, and E_{σ_0} is the annihilator of the image of $k \otimes T \to H_{DR}$. This annihilator is $pE^\vee/E$, and hence is also the image of the annihilator of $k \otimes T$. Moreover,

$$N^1_h K' \cap F^1_{Hdg}K' = \phi^h(\ell) \oplus \phi^{h+1}(\ell) \oplus \cdots \phi^{\sigma_0 - 2}(\ell),$$

so $N^1_{\sigma_0}K' = N^1_{\sigma_0 - 1}K' \cap F^1_{Hdg}K' = 0$. Again by duality,

$$E_{\sigma_0} = E_{\sigma_0 - 1} + F^2_{Hdg}H_{DR}.$$

□

To summarize:

Proposition 6. *Let H be a K3 crystal over an algebraically closed field k and let $h \leq \mathrm{ht}(H)$ be a positive integer. Then the following are equivalent:*

1. $E_h^\perp \cap F^1_{Hdg}H_{DR} = E_{h-1}^\perp \cap F^1_{Hdg}H_{DR}$.
2. $E_h + F^2_{Hdg}H_{DR} = E_{h-1} + F^2_{Hdg}H_{DR}$.
3. *H is supersingular, and $\sigma_0(H) \leq h$.*
4. $F^2_{Hdg}H_{DR} \subseteq E_h$.
5. *$E_{h-1} + F^2_{Hdg}H_{DR}$ contains the null space of Q on $\mathbf{F}_p \otimes T$.*
6. *$E_{h-1} + F^2_{Hdg}H_{DR}$ contains a nonzero point of $\mathbf{F}_p \otimes T$.*

Proof: Since $F^1_{Hdg}H_{DR}$ is the annihilator of $F^2_{Hdg}H_{DR}$, conditions (1) and (2) are equivalent. If (2) holds, then by Lemma 2,

$$N^2_h H_{DR} \subseteq E_{h-1}H_{DR} + F^2_{Hdg}H_{DR},$$

and by Lemma 4, H is supersingular. Then Lemma 5 shows that (2) implies (3) and that (3) implies (4) and (5). Suppose that H has height $h \leq 10$. Consider the Katz decomposition $H \cong H_h \oplus H_1 \oplus H'_h$. Since Φ is divisible by p on $H_1 \oplus H'_h$, the same is true of Φ_i for all $i \leq h$, hence $N_i^2 H_{DR}$ and E_i are contained in H_{hDR} for $i \leq h$. On the other hand, $F^2 H_{DR} \subseteq H'_{hDR}$. Thus condition (4) implies that H is supersingular, and by Lemma 5 $h \leq \sigma_0$. Evidently (5) implies (6). Similarly if $h < \infty$, then in the Katz decomposition the image of $T \to H$ is contained in H_1, and hence modulo p cannot intersect $E_{h-1} + F^2_{Hdg} H_{DR}$. If (6) holds, it follows that H is supersingular, and $E_{h-1} + F^2_{Hdg} H_{DR}$ contains a ϕ-invariant line. Then by duality $E^{\perp}_{h-1} K \cap F^1_{Hdg} K$ is contained in a proper ϕ-invariant subspace, so in fact $h \geq \sigma_0$. □

If H is the K3 crystal of a K3 surface, then the rank of H is 22 and its crystalline discriminant is $-1 \in \mathbf{Z}_p/\mathbf{Z}_p^2$ [5, 4.9, 4.10]. We can use this fact to prove that the Artin invariant of a supersingular K3 crystal coming from a surface is at most 10, without assuming the Tate conjecture.

Proposition 7. *The Artin invariant σ_0 of a K3 crystal associated to a supersingular K3 surface X/k is at most* 10.

Proof: This is a consequence of the Tate conjecture, as explained by Artin [1]. To prove it unconditionally, use the direct sum decomposition $H \cong H_0 \oplus H_1$. As shown in [4, 3.15], the crystalline discriminant $(\frac{H_0}{p}) \in \{\pm 1\} \cong \mathbf{F}_p^*/\mathbf{F}_p^{*2}$ is $-(\frac{-1}{p})^{\sigma_0}$ if p is odd. Since the discriminant of H is $(\frac{-1}{p})$, we cannot have $\sigma_0 = 11$. If $p = 2$, it is easy to see that T_{H_0} has a dual basis (x_i, y_i), $i = 1, \ldots \sigma_0$ with respect to which the quadratic form is $x_1^2 + y_1^2 + x_1 y_1 + \sum_{i>1} x_i y_i$, and that its discriminant is $3(-1)^{\sigma_0} \in \mathbf{Q}_2/\mathbf{Q}_2^{*2}$. Since the discriminant is -1, it cannot be the case that $\sigma_0 = 11$, since in that case H_1 would be zero. □

It is convenient to introduce a variation of the above construction for polarized K3 crystals. If (H, ξ) is a polarized K3 crystal, it follows from the fact that $\Phi(\xi) = p\xi$ that the image ξ_{DR} of ξ in H_{DR} lies in $F^1_{Hdg} \cap F^1_{con} H_{DR}$ and is fixed by γF_k^*. If $H \in \mathbf{K3}_h(k)$ and $i \leq h$ we define $E_{i,\xi}$ to be the span in H_{DR} of E_i and the image ξ_{DR} of ξ in H_{DR}. If H is supersingular we define $\tau(H, L)$ to be $\sigma_0(H) - 1$ if $\xi_{DR} \in Ann(T/pT)$ and to be $\sigma_0(H)$ otherwise. For example, if the degree of ξ is prime to p, $\tau(H, L) = \sigma_0(H)$. It is also convenient to define $\tau(H, L)$ and $\sigma_0(H)$ to be ∞ if H is not supersingular. If (X, L) is a polarized K3 surface over k, we write $\tau(X, L)$ and $\sigma_0(X)$ for $\tau(H, \xi)$ and $\sigma_0(H)$, where H is the crystalline cohomology of X/W and ξ is the crystalline Chern class of L.

Corollary 8. *Let (H, ξ) be a polarized K3 crystal with $H \in \mathbf{K3}_h(k)$, let $i \leq h - 1$, and let $\overline{E}_{i,\xi}$ be the image of $E_{i,\xi}$ in $\mathrm{Gr}^1_{Hdg} H_{DR}$, i.e.,*

$$\overline{E}_{i,\xi} = (E_{i,\xi} + F^2_{Hdg})/F^2_{Hdg}.$$

Then

$$\dim(\overline{E}_{i,\xi}) = \min(i+1, \tau(H,L)).$$

Furthermore, if H is supersingular, $E_{\tau,\xi} = E_{\tau-1,\xi} + F^2_{Hdg}H_{DR}$.

Proof: Evidently $\dim(\overline{E}_{i,\xi}) \leq \dim(E_{i,\xi}) \leq \dim(E_i) + 1 \leq i + 1$. If H is supersingular with Artin invariant σ_0, by Lemma 5 $\overline{E}_i \subseteq \overline{E}_{\sigma_0} = \overline{E}_{\sigma_0-1}$, so $\dim(\overline{E}_{i,\xi}) \leq \sigma_0$. If $\xi \in Ann(T/pT)$, then in fact $E_{\sigma_0,\xi} = E_{\sigma_0}$, and $\dim(\overline{E}_{i,\xi}) \leq \sigma_0 - 1 = \tau$. Thus in any case

$$\dim(\overline{E}_{i,\xi}) \leq \min(i+1, \tau(H,L)).$$

Suppose that H is not supersingular. Then by Lemma 5, E_i has dimension i and by condition (4) of Proposition 6, E_i does not meet $F^2_{Hdg}H_{DR}$. Furthermore, the Katz decomposition shows that $E_i + F^2_{Hdg}H_{DR}$ does not meet the image of T/pT. Hence $\dim(\overline{E}_{i,\xi}) = i+1$. Suppose that H is supersingular. By Lemma 5, the dimension of the image $\overline{E}_i$ of E_i in $\mathrm{Gr}^1_{Hdg} H_{DR}(X/k)$ is $\min(i, \sigma_0 - 1)$. Hence if ξ_{DR} does not belong to $E_i + F^2_{Hdg}H_{DR}$, the dimension of $\overline{E}_{i,\xi}$ is $\min(i+1, \sigma_0) \geq \min(i+1, \tau)$. On the other hand, if $\xi_{DR} \in E_i + F^2_{Hdg}H_{DR}$, then condition (6) of Proposition 6 holds with $h = i+1$, so that $\sigma_0 \leq i+1$, and by Lemma 5, ξ_{DR} belongs to the annihilator of T/pT. Then $\overline{E}_{i,\tau} = \overline{E}_i = \overline{E}_{\sigma_0-1}$ has dimension $\sigma_0 - 1 = \tau$. □

We are now ready to investigate the moduli of K3 surfaces. If $f\colon \mathcal{X} \to S$ is a family of K3 surfaces over a k-scheme S with polarization $\mathcal{L}$ of degree d, the relative De Rham cohomology $H_{DR}(\mathcal{X}/S) := R^2 f_* \Omega^{\cdot}_{\mathcal{X}/S}$ is locally free of rank 22. It is equipped with its Hodge and conjugate filtrations F_{Hdg} and F_{con}, and the associated graded sheaves $\mathrm{Gr}_{Hdg} H_{DR}$ and $\mathrm{Gr}_{con} H_{DR}$ are also locally free. If $\mathcal{X}/S$ is obtained by pullback from a smooth k-scheme, $H_{DR}(\mathcal{X}/S)$ has a canonical Gauss-Manin connection $\nabla\colon H_{DR} \to \Omega^1_{X/S} \otimes H_{DR}$. The Hodge filtration is moved by at most one step by ∇, and the induced map

$$F^2_{Hdg}H_{DR}(\mathcal{X}/S) \to \mathrm{Gr}^1_{Hdg} H_{DR}(\mathcal{X}/S) \otimes \Omega^1_{S/k}$$

is $\mathcal{O}_S$-linear and is equivalent to a map

$$\kappa'\colon F^2_{Hdg}H_{DR}(\mathcal{X}/S) \otimes \mathrm{Gr}^1_{Hdg} H_{DR}(\mathcal{X}/S)^\vee \to \Omega^1_{S/k}.$$

Since $\Omega^2_{X/S}$ is trivial along the fibers and the Hodge spectral sequence of $\mathcal{X}/S$ degenerates, there is a natural isomorphism

$$F^2_{Hdg} \otimes \mathrm{Gr}^1_{Hdg} H_{DR}(\mathcal{X}/S)^\vee \cong f_*(\Omega^2_{X/S}) \otimes R^1 f_*(\Omega^1_{\mathcal{X}/S})^\vee \cong R^1 f_*(T_{X/S})^\vee.$$

Thus the map κ' can be interpreted as the differential of the map from S into the moduli stack. These facts, as well as the basic deformation theory of K3 surfaces, are very clearly explained in [2].

The absolute Frobenius morphism of X induces an $\mathcal{O}_S$-linear map

$$\phi\colon F^*_S H_{DR}(\mathcal{X}/S) \to H_{DR}(\mathcal{X}/S),$$

whose kernel is exactly $F_S^*(F^1_{Hdg}H_{DR})$ and whose image is exactly $F^2_{con}H_{DR}$, and the inverse Cartier isomorphism induces an isomorphism:

$$\gamma\colon F_S^* \operatorname{Gr}^1_{Hdg} H_{DR} \to \operatorname{Gr}^1_{con} H_{DR}. \tag{1}$$

The filtration F_{con} is stable under ∇, and if $F_S^* \operatorname{Gr}^1_{Hdg} H_{DR}$ is endowed with the connection killing the elements of $\operatorname{Gr}^1_{Hdg} H_{DR}$, the map γ is horizontal. The quadratic form Q induces nondegenerate forms on $\operatorname{Gr}^1_{Hdg} H_{DR}$ and $\operatorname{Gr}^1_{con} H_{DR}$ compatible with γ. The Chern class $\xi_{DR} := c_{DR}(\mathcal{L})$ of $\mathcal{L}$ is a global horizontal section of $H_{DR}(\mathcal{X}/S)$ contained in $F^1_{Hdg} \cap F^1_{con}H_{DR}(\mathcal{X}/S)$.

If S is a k-scheme, let $\mathcal{F}(S)$ denote the category of polarized K3 surfaces of degree d, with morphisms the isomorphisms preserving the polarizations. A morphism $S' \to S$ defines a functor $\mathcal{F}(S) \to \mathcal{F}(S')$, and $\mathcal{F}$ forms a fibered category with effective descent. It should be easy to verify that $\mathcal{F}$ is in fact a Deligne-Mumford stack, but since we shall be only interested in local properties of families of K3 surfaces, we shall not attempt to carry this out here. If d is prime to p, $\mathcal{F}$ is formally smooth of dimension 19 over k, and there exist formally étale morphisms $S \to \mathcal{F}$ with S a smooth k-scheme of dimension 19. Formation of H_{DR} is compatible with base change $S' \to S$, and the same is true of its filtrations and their associated graded sheaves.

We define a sequence of stacks and closed immersions $\mathcal{F}_{h+1} \subseteq \mathcal{F}_h \subseteq \cdots \mathcal{F}_1 = \mathcal{F}$. That is, each $\mathcal{F}_i(S)$ will be a strictly full subcategory of $\mathcal{F}_{i-1}(S)$, and for each object $(\mathcal{X}/S, \mathcal{L})$ of $\mathcal{F}(S)$, the map $S_i := \mathcal{F}_i \times_{\mathcal{F}} S \to S$ will be a closed immersion. Furthermore, associated to each object $(\mathcal{X}/S, \mathcal{L})$ of $\mathcal{F}_i(S)$, will be a quasi-coherent subsheaf $E_i(\mathcal{X}/S) \subseteq H_{DR}(\mathcal{X}/S)$. These objects are defined inductively as follows: For any object $(\mathcal{X}/S, \mathcal{L})$ of $\mathcal{F}_1(S) = \mathcal{F}(S)$, let

$$E_1(\mathcal{X}/S) := F^2_{con}H_{DR}(\mathcal{X}/S) \subseteq H_{DR}(\mathcal{X}/S).$$

If $h \geq 1$ and $(\mathcal{X}/S, \mathcal{L})$ is an object of $\mathcal{F}_h(\mathcal{X}/S)$, then $E_h(\mathcal{X}/S) \subseteq H_{DR}(\mathcal{X}/S)$ is defined, and $\mathcal{F}_{h+1}(S)$ is defined to be the full subcategory of $\mathcal{F}_h(S)$ whose objects are the elements $(\mathcal{X}/S, \mathcal{L})$ such that $E_h(\mathcal{X}/S) \subseteq F^1_{Hdg}H_{DR}(\mathcal{X}/S)$. If $(\mathcal{X}/S, \mathcal{L})$ is an object of $\mathcal{F}_{h+1}(S)$, then $E_{h+1}(\mathcal{X}/S)$ is by definition the inverse image in $F^1_{con}H_{DR}(\mathcal{X}/S) \subseteq H_{DR}(\mathcal{X}/S)$ of the image of $F_S^*E_h$ under the map

$$F_S^*F^1_{Hdg}H_{DR}(X/S) \longrightarrow F_S^* \operatorname{Gr}^1_{Hdg} H_{DR}(X/S) \xrightarrow{\gamma} \operatorname{Gr}^1_{con} H_{DR}(X/S). \tag{2}$$

Since γ is horizontal, it follows that E_h is stable under ∇. Let $E_{h,\mathcal{L}}(\mathcal{X}/S, \mathcal{L})$ denote the $\mathcal{O}_S$-submodule of H_{DR} generated by $E_h(\mathcal{X}/S)$ and $c_{DR}(\mathcal{L})$, which again is contained in $F^1_{Hdg} \cap F^1_{con}H_{DR}(\mathcal{X}/S)$ and invariant under ∇ and γF_S^*. We also define $E_0(\mathcal{X}/S) := 0$, so that $E_{0,\mathcal{L}}(\mathcal{X}/S)$ is the $\mathcal{O}_S$-submodule of S generated by $c_{DR}(\mathcal{L})$. Note that $E_{h,\mathcal{L}}$ can be defined inductively in the same way as E_h, and that an object $(\mathcal{X}/S, \mathcal{L})$ of $\mathcal{F}_h(S)$ lies in $\mathcal{F}_{h+1}(S)$ if and only if $E_{h,\mathcal{L}}(\mathcal{X}/S, \mathcal{L})$ is contained in $F^1_{Hdg}H_{DR}(\mathcal{X}/S)$.

It follows from Proposition 3 that $\mathcal{F}_h(k)$ is the category of polarized K3 surfaces of degree d with Newton height at least h.

Lemma 9. *Let $(\mathcal{X}/S, \mathcal{L})$ be an object of $\mathcal{F}_h(S)$ and let $i \leq h$ be a positive integer.*

1. *Each $E_i(\mathcal{X}/S)$ is a coherent totally isotropic subsheaf of $H_{DR}(\mathcal{X}/S)$ locally generated by i elements, and*
$$E_0(\mathcal{X}/S) \subseteq E_1(\mathcal{X}/S) \subseteq \cdots E_h(\mathcal{X}/S) \subseteq H_{DR}(\mathcal{X}/S).$$
2. *If $g\colon S' \to S$ is a morphism of k-schemes and $(\mathcal{X}'/S', \mathcal{L})$ is obtained from $(\mathcal{X}/S, \mathcal{L})$ by pullback, then the natural isomorphism*
$$g^* H_{DR}(\mathcal{X}/S) \to H_{DR}(\mathcal{X}'/S')$$
induces a surjection $g^*E_i(\mathcal{X}/S) \to E_i(\mathcal{X}'/S')$ and an isomorphism
$$g^*(Q_i(\mathcal{X}/S) \to Q_i(X'/S),$$
where $Q_i(\mathcal{X}/S) := H_{DR}(\mathcal{X}/S)/E_i(\mathcal{X}/S)$.
3. *Suppose that S is of finite type over k and that no closed fiber of $\mathcal{X}/S$ is supersingular with Artin invariant less than i. Then $E_i(\mathcal{X}/S)$ is a local direct factor of $H_{DR}(\mathcal{X}/S)$ of rank i, and its formation commutes with any base change $S' \to S$.*

Proof: The statements are true if $h = 1$, and, proceeding by induction on h, we assume that the lemma holds for h and that $h \geq 1$. Furthermore, it suffices to treat the case $i = h$. Then $E_{h-1} \subseteq E_h$ and the quotient is locally monogenic. Let $\tilde{E}_i \subseteq \mathrm{Gr}^1_{con} H_{DR}(\mathcal{X}/S)$ denote the image of of $F^*_S(E_i)$ under the map (2). Then there is an exact sequence
$$0 \to \tilde{E}_{h-1} \to \tilde{E}_h \to \tilde{E}_h/\tilde{E}_{h-1} \to 0,$$
and the quotient is locally monogenic. If π is the projection $F^1_{con} H_{DR} \to \mathrm{Gr}^1_{con} H_{DR}(X/S)$, E_{i+1} is by definition $\pi^{-1}(E_i)$, and so $E_h \subseteq E_{h+1}$ and $E_{h+1}/E_h \cong \pi(E_{h+1})/\pi(E_h) \cong \tilde{E}_h/\tilde{E}_{h-1}$ is locally monogenic. Since E_h is totally isotropic, the same is true of $\gamma F^*_k(E_h)$ and E_{h+1}. Statements (2) and (3) can be verified locally on S, so we may assume that there exist splittings of F_{Hdg} and F_{con}, and hence that there exists a map $\tilde{\gamma}\colon F^*_S H_{DR} \to H_{DR}$ extending the map γ defined in (1). Then E_h is the sum of the images of $(\tilde{\gamma} F_S *)^i F^2_{con} H_{DR}$ for $i = 0 \cdots h-1$, and is independent of the choice of $\tilde{\gamma}$, and since g is compatible with Frobenius, the base change map sends $g^* E_h(\mathcal{X}/S)$ onto $E_h(\mathcal{X}/S')$. It follows that formation of the quotient $Q_i(\mathcal{X}/S)$ is compatible with any base change $S' \to S$.

Suppose now that S is of finite type and that no closed fiber is supersingular with Artin invariant less than h. For any closed point s of S, the natural map from the fiber $E_h(s)$ of E_h at s to $E_h(X_s)$ is a surjective map of k-vector spaces. Its source $E_h(s)$ is generated by h elements, and by Lemma 5, the target $E_h(X_s)$ has dimension h. It follows that the map
$$E_h(s) \to E_h(X_s)$$

is an isomorphism. Since H_{DR} is locally free, there is an exact sequence

$$0 \to \mathrm{Tor}_1(k(s), Q_h) \to E_h(s) \to H_{DR}(s) \to Q_h(s) \to 0,$$

and since the map $E_h(s) \to H_{DR}(s)$ is injective, $\mathrm{Tor}_1(k(s), Q_h) = 0$ and Q_h is locally free in a neighborhood of s. Hence E_h is a local direct summand and its formation commutes with base change. □

Corollary 10. *If $(\mathcal{X}/S, \mathcal{L})$ is an object of $\mathcal{F}_{11}(S)$ and $j \in \mathbf{N}$, then the set of closed points of S for which $\sigma_0(X_s) \leq j$ is a Zariski closed subset of S_{red}, and similarly for $\tau(X_s, \mathcal{L}_s)$.*

Proof: By Lemma 5 and Proposition 7, $\sigma_0(X_s) \leq j$ if and only if the dimension of $E_{10}(X_s) \leq j$, *i.e.*, if and only if the dimension of $Q_{10}(X_s) \geq 22 - j$. Since $Q_{10}(\mathcal{X}/S)$ is a coherent sheaf on S whose formation commutes with base change, the corollary follows from the semi-continuity theorem. The same proof works with τ in place of σ, using Corollary 8. □

Proposition 11. *Each $\mathcal{F}_{h+1} \subseteq \mathcal{F}_h$ is a closed immersion, locally defined by a single equation, and the ideal of $\mathcal{F}_{h+1}$ in $\mathcal{F}_h$ is the image of the pairing $(\ |\ )\colon F^2_{Hdg} \otimes E_h/E_{h-1} \to \mathcal{O}_{\mathcal{F}_h}$.*

Proof: We argue by induction on h, and, assuming the proposition true for $h-1$, we may work locally on the base S of a polarized family $(\mathcal{X}/S, \mathcal{L})$ defining an object of $\mathcal{F}_{h-1}(S)$. Then $E_{h-1} \subseteq F^1_{Hdg}H_{DR}$, and hence is orthogonal to F^2_{Hdg}, and the map $F^2_{Hdg} \otimes E_h \to \mathcal{O}_{\mathcal{F}_h}$ factors through $F^2_{Hdg} \otimes E_h/E_{h-1}$. Its image is an ideal I of $\mathcal{O}_{\mathcal{F}_h}$, and since E_h/E_{h-1} is locally monogenic, so is I. If $g\colon S' \to S$ is any morphism, the pullback of $\mathcal{X}/S$ to S' lies in $\mathcal{F}_{h+1}$ if and only if $g^*E_h \to g^* \mathrm{Gr}^0_{Hdg} H_{DR}$ is zero, *i.e.*, if and only if g^*E_h is orthogonal to $g^*F^2_{Hdg}$. Thus the morphism $S' \to S \to \mathcal{F}$ factors through $\mathcal{F}_{h+1}$ if and only if $S' \to S$ factors through the closed subscheme of S defined by I. □

If $(\mathcal{X}/S, \mathcal{L})$ is a polarized K3 surface over S, there is a commutative diagram

$$\begin{array}{ccc}
F^2_{Hdg}H_{DR}(\mathcal{X}/S) \otimes \mathrm{Gr}^1_{Hdg} H_{DR}(\mathcal{X}/S) & & \\
\downarrow{\scriptstyle \delta} & \searrow{\scriptstyle \kappa} & \\
F^2_{Hdg}H_{DR}(\mathcal{X}/S) \otimes \mathrm{Gr}^1_{Hdg} H_{DR}(\mathcal{X}/S)^\vee & \xrightarrow{-\kappa'} & \Omega^1_{S/k}
\end{array}$$

Here δ comes from the canonical pairing and κ and κ' are the (Kodaira-Spencer) maps induced by the Gauss-Manin connection. That is, if ω is a section of $F^2_{Hdg}H_{DR}$ and η (resp. ϕ) is a section of $\mathrm{Gr}^1_{Hdg} H_{DR}$ (resp. of its dual), then $\kappa(\omega \otimes \eta) = (\omega|\nabla(\eta'))$ for any $\eta' \in F^1_{Hdg}H_{DR}$ lifting η, $\kappa'(\omega \otimes \phi) = \phi(\nabla(\omega))$, and $\kappa'(\omega \otimes \delta(\eta)) = (\nabla\omega|\eta')$.

Suppose now that $(\mathcal{X}/S, \mathcal{L})$ is a polarized K3 surface over S defining an object of $\mathcal{F}_h(S)$, so that $E_i \subseteq H_{DR}(\mathcal{X}/S)$ is defined for $i \leq h$. As we saw in the construction, these spaces are invariant under the Gauss-Manin connection. Furthermore, $E_{h-1} \subseteq F^1_{Hdg} H_{DR}$, and since it is horizontal, it is annihilated by the Kodaira-Spencer mapping κ of $\mathcal{X}/S$. Let S' be the closed subscheme $\mathcal{F}_{h+1} \times_{\mathcal{F}_h} S$ of S. Note that E_h does not lie in $F^1_{Hdg} H_{DR}$ until we restrict to S'. In fact the following proposition shows that the conormal to S' in S is precisely measured by this fact.

Proposition 12. *Let $(\mathcal{X}/S, \mathcal{L})$ be an object of $\mathcal{F}_h(S)$ and let $S' := \mathcal{F}_{h+1} \times_{\mathcal{F}_h} S$. Then there is a commutative diagram:*

$$\begin{array}{ccccc}
F^2_{Hdg} \otimes E_h & \longrightarrow & F^2_{Hdg} \otimes \mathrm{Gr}^1_{Hdg} H_{DR}(\mathcal{X}/S)_{|_{S'}} & & \\
\Big\downarrow {\scriptstyle (\,|\,)} & & \Big\downarrow {\scriptstyle -\kappa} & \searrow & \\
I_{S'}/I^2_{S'} & \xrightarrow{\;d\;} & \Omega^1_{S/k_{|_{S'}}} & \longrightarrow & \Omega^1_{S'/k}
\end{array}$$

In particular, the image of the conormal to S' by the differential d coincides with the image of $F^2_{Hdg} \otimes E_h/E_{h-1}$ under the Kodaira-Spencer mapping κ.

Proof: Choose local sections ω of $F^2_{Hdg} H_{DR}$ and η of E_h over S. Then by Proposition 11, $f := (\omega|\eta)$ is a local generator for the ideal $I_{S'}$ of S' in S. Since E_h is horizontal, there exists $\theta \in \Omega^1_{S/k}$ such that $\nabla(\eta) = \theta \otimes \eta$ modulo E_{h-1}. Hence

$$df = (\nabla(\omega)|\eta) + (\omega|\nabla(\eta)) = (\nabla(\omega)|\eta) + \theta(\omega|\eta)$$

Taking the images in $\Omega^1_{S/k_{|_{S'}}}$, we see that $df = \kappa'(\omega \otimes \delta(\eta)) = -\kappa(\omega \otimes \eta)$. □

If $(X/k, L)$ is a polarized K3 surface, by standard deformation theory (and the fact that X admits no vector fields!), the unpolarized surface X/k admits a versal formal k-deformation $\mathcal{X}/\mathcal{S}$, with $\mathcal{S} \cong \operatorname{Spec} k[[t_1, \ldots t_{20}]]$. As explained in [2], the versal formal k-deformation $(\mathcal{X}/\mathcal{S}_L)$ embeds as a closed formal subscheme of $\mathcal{S}$, and its ideal I_L is monogenic (and nonzero). If $\mathfrak{m}$ is the (maximal) ideal of definition of $\mathcal{S}$, the Kodaira-Spencer mapping defines an isomorphism. Using this fact and Proposition 12 we can easily calculate the following first-order description of $\mathcal{F}_h$.

Corollary 13. *Let $(\mathcal{X}/\mathcal{S})$ be the unpolarized versal formal deformation of a polarized K3 surface $(X/k, L)$, let i be an integer with $i \leq \mathrm{ht}(X)$, and let $I_{i,L} \subseteq \mathfrak{m}$ be the ideal of $\mathcal{S}_{i,L}$ in $\mathcal{S}$.*

1. *The ideal $I_{i,L}$ of $\mathcal{S}_{i,L}$ in $\mathcal{S}$ can be generated by i elements, and its image in $\mathfrak{m}/\mathfrak{m}^2$ is the image of $F^2_{Hdg} \otimes \overline{E}_{i-1,L}$ under the Kodaira-Spencer isomorphism κ:*

$$F^2_{Hdg} \otimes \overline{E}_{i-1,L} \subseteq F^2_{Hdg} \otimes \mathrm{Gr}^1_{Hdg} H_{DR}(X/k) \xrightarrow{\;\kappa\;} \mathfrak{m}/\mathfrak{m}^2.$$

2. *The Zariski tangent space of* $\mathcal{F}_i$ *at the* k*-valued point corresponding to* $(X/k, L)$ *is naturally isomorphic to*
$$\mathrm{Gr}^0_{Hdg} \otimes \overline{E}^{\perp}_{i-1,L} \subseteq \mathrm{Gr}^0_{Hdg} \otimes H^1(X/k, \Omega^1_{X/k}).$$
In particular, its dimension is $\max(20-i, 20-\tau(X,L))$.

Proof: We have already seen in Proposition 11 that each $\mathcal{S}_{i,\mathcal{L}} \subseteq \mathcal{S}_{i-1,\mathcal{L}}$ is a closed immersion defined by a single equation. It follows by induction that $\mathcal{S}_{i,\mathcal{L}} \subseteq \mathcal{S}_{\mathcal{L}}$ is defined by $i-1$ equations and hence that $\mathcal{S}_{i,\mathcal{L}} \subseteq \mathcal{S}$ is defined by i equations. Let $T \subseteq \mathcal{S}$ be the closed subscheme defined by $\mathfrak{m}^2$. Then there is a surjective map $\Omega^1_{T/k} \to \mathfrak{m}/\mathfrak{m}^2$, which is in fact an isomorphism if p is odd. As is well-known, the ideal of $T_{\mathcal{L}} := T \cap \mathcal{S}_{\mathcal{L}}$ is the image of $F^2_{Hdg} \otimes c_{Hdg}(L) \in F^2_{Hdg} \otimes H^1(X/k, \Omega^1_{X/k})$ under the Kodaira-Spencer mapping [4, 1.14]. This says exactly that the ideal $I_{1,\mathcal{L}}$ of $S_{1,\mathcal{L}}$ in T is the image of $F^2_{Hdg} \otimes \overline{E}_{0,L}$ under the Kodaira-Spencer mapping, proving (1) when $i = 1$. On the closed subscheme $S := T_{i,\mathcal{L}}$ of T, $(\mathcal{X}, \mathcal{L})$ defines an object of $\mathcal{F}_i$. Proposition 12 says that the image of $I_{i+1,\mathcal{L}}/I_{i,\mathcal{L}}$ in $\Omega^1_{S/k}$ coincides with the image of $F^2_{Hdg} \otimes E_i$. Taking images in $\mathfrak{m}/\mathfrak{m}^2 + I_{i,\mathcal{L}}$, we see that the image of $I_{i+1,\mathcal{L}}$ coincides with the image of $F^2_{Hdg} \otimes E_i$. Assuming (1) for i, it follows that the image of $I_{i+1,\mathcal{L}}$ mod $\mathfrak{m}^2$ coincides with the image of $\overline{E}_i$, and so (1) follows by induction. The Zariski tangent space of $\mathcal{F}_i$ is therefore the dual of the quotient of $\mathfrak{m}/\mathfrak{m}^2$ by the image of $F^2_{Hdg} \otimes E_{i-1,L}$, which can be identified with $\mathrm{Gr}^0_{Hdg} \otimes \overline{E}^{\perp}_{i-1,L}$ using the duality map. Thus by Corollary 8, its dimension is $\max(20-i, 20-\tau(X,L))$. □

Proposition 14. *Let* $(\mathcal{X}/S, \mathcal{L})$ *be a family of polarized K3 surfaces over a finite type* k*-scheme* S *such that the resulting map* $S \to \mathcal{F}$ *is unramified.*

1. *If* i *is an integer such that every fiber* X_s *is supersingular with Artin invariant* $\leq i$*, then the dimension of* S *is at most* $i-1$.
2. *If every fiber* X_s *is supersingular, then the dimension of* S *is at most* 9.
3. *If* $h \leq 11$ *is an integer such that every fiber* X_s *has height at least* h*, then the dimension of* S *is at most* $20-h$.

Proof: Without loss of generality we may suppose that S is reduced, and even smooth, and that every fiber X_s has height at least i. Then $E_i(\mathcal{X}/S)$ is defined, and by Lemma 9, formation of
$$Q_i(\mathcal{X}/S) := H_{DR}(\mathcal{X}/S)/E_i(\mathcal{X}/S)$$
commutes with any base change $S' \to S$. Since S is reduced and Q_i is coherent, there is a dense open subset of S on which Q_i is locally free, and we may assume that this true on all of S. Then $E_i(\mathcal{X}/S)$ is locally free, and its formation commutes with any base change $S' \to S$.

Now suppose that every fiber has Artin invariant at most i. Then by Lemma (5), $F^2_{Hdg}(X_s) \subseteq E_i(s)$ for every s, and since F^2_{Hdg} and E_i are local direct factors and S is reduced and of finite type over k, $F^2_{Hdg}(\mathcal{X}/S)H_{DR} \subseteq E_i(\mathcal{X}/S)$. Since E_i is horizontal, the Kodaira-Spencer mapping factors:

$$\begin{array}{ccc} F^2_{Hdg}H_{DR} \otimes \mathrm{Gr}^1_{Hdg} H_{DR}(\mathcal{X}/S)^\vee & & \\ \downarrow & \searrow & \\ F^2_{Hdg}H_{DR} \otimes \overline{E}_i^\vee & \longrightarrow & \Omega^1_{S/k} \end{array}$$

Since $S \to \mathcal{F}$ is unramified, for each s in S the map $H^1(X_s, \Omega^1_{X_s/k}) \to \mathfrak{m}_s/\mathfrak{m}_s^2$ is surjective, and hence the dimension of $\mathfrak{m}_s/\mathfrak{m}_s^2$ is at most the dimension of $\overline{E}_i(X_s/k)$, which by Lemma 5 is at most $i-1$. This proves (1), and (2) follows, since the Artin invariant σ_0 is at most 10, by Proposition 7

To prove (3), suppose that $h \leq 11$ and that every fiber has height at least h. If $h = 11$, every fiber is supersingular, so by (2), the dimension of S is at most $9 = 20-11$. Suppose that $h \leq 10$, and that the dimension of S is at least $20-h$. Since $20-h \geq 10$, not all the points of S are supersingular, and in particular we may restrict to an open set on which every point has finite height. Then by Lemma 5, every $\overline{E}_{h-1,\mathcal{L}}(X_s/k)$ has dimension h, and we may assume that $\overline{E}_{h-1,\mathcal{L}}(\mathcal{X}/S, \mathcal{L})$ is locally free of rank h and that its formation commutes with base change. Since $E_{h-1,\mathcal{L}}$ is a horizontal subspace of $H_{DR}(\mathcal{X}/S)$, its image $\overline{E}_{h-1,\mathcal{L}}$ in $\mathrm{Gr}^1_{Hdg} H_{DR}(\mathcal{X}/S)$ is annihilated by the Kodaira-Spencer mapping κ. Then the image of κ has dimension at most $20-h$, and hence S has dimension at most $20-h$. □

We can assemble what we have proved as follows:

Theorem 15. *For $1 \leq h \leq 11$, $\mathcal{F}_h \subseteq \mathcal{F}$ is a local complete intersection of codimension $h-1$ and dimension $20-h$. For $11 \geq h \geq 2$, the ideal of $\mathcal{F}_h$ in $\mathcal{F}_{h-1}$ is invertible and isomorphic to $(F^2_{Hdg})^{1-p^h}$. If $(X/k, L)$ is a polarized K3 surface of height at least h, then $(X/k, L)$ defines a singular point of $\mathcal{F}_h$ if and only if $\tau(X/k, L) < h$.*

Proof: We saw above, and it has long been known, that a versal deformation of a polarized K3 surface $(X/k, L)$ is a local complete intersection of dimension 19 (and is even smooth unless $\tau(X/k, L) = 0$). Since $S_h \subseteq S_1$ is locally defined by $h-1$ equations its dimension is everywhere at at least $20-i$. By Proposition 14, the dimension is exactly $20-i$, so S_h is a local complete intersection. It follows that I_h/I_{h-1} is invertible, and by Proposition 12, it is isomorphic to E_h/E_{h-1}, which is locally generated by, and hence isomorphic to, $(F^2_{Hdg})^{1-p^h}$. Furthermore, if $(X/k, L)$ has height at least h, it defines a singular point of $\mathcal{F}_h$ if and only if the tangent space of $\mathcal{F}_h$ at $(X/k, L)$ has dimension larger than the dimension $20-h$. By Corollary 13, this is true if and only if $\tau < h$. □

Corollary 16. *Suppose d is prime to p and $1 \leq h \leq 11$ and $(\mathcal{X}/S, \mathcal{L})$ is an object of $\mathcal{F}(S)$ defining an étale morphism $S \to \mathcal{F}$. Then the singular locus of S_h coincides with the set $S_{\infty,h-1}$. Its codimension is everywhere at least $22-2h$ in S_h, and S_h is normal if $h \leq 10$. On the other hand, S_{11} is generically nonreduced.*

Proof: If d is prime to p, $\tau(X, L) = \sigma_0(X)$, and so by Theorem 15, the singular set of S_h consists exactly of $S_{\infty,h-1}$. If $1 \leq h \leq 10$, Proposition 14 says that this set has dimension at most $h-2$. Thus the codimension of the singular set is at least $20 - h - (h-2) = 22 - 2h \geq 2$. Since S_h is a local complete intersection, it is normal by Serre's criterion. The singular set of S_{11} is $S_{\infty,10}$, which by Proposition 7 is, set-theoretically, all of S_{11}. Thus S_{11} is generically singular, hence generically nonreduced, since the ground field k is algebraically closed. □

The following result gives a more precise description of the generic singularities of $\mathcal{F}_h$, which occur at those points at which $\tau(X, L) = \tau + 1$.

Theorem 17. *Let $(\mathcal{X}/S, \mathcal{L})$ be a family of polarized K3 surfaces over S and let s be a k-rational point of S at which the fiber X/k is supersingular and let $\tau := \tau(X, L)$. Let $\mathfrak{m}$ be the maximal ideal of s in S_τ, so that the ideal I of $\mathcal{F}_{\tau+1}$ in $\mathcal{F}_\tau$ is contained in $\mathfrak{m}^2$. Use any basis of $H^0(X/k, \Omega^2_{X/k})$ to identify $F^2_{Hdg}H_{DR}(X/k)$ with k and let*

$$\kappa' \colon H^1(X/k, \Omega^1_{X/k})^\vee \to \mathfrak{m}/\mathfrak{m}^2$$

be the resulting Kodaira-Spencer mapping. Then the image of Q via the map

$$\kappa^* \colon Sym^2(H^1(X/k, \Omega^1_{X/k})^\vee) \to Sym^2(\mathfrak{m}/\mathfrak{m}^2) \to \mathfrak{m}^2/\mathfrak{m}^3$$

generates I (modulo $\mathfrak{m}^3$).

Proof: We may assume without loss of generality that $(\mathcal{X}/S, \mathcal{L})$ is versal. Then s is a smooth point of S_τ and a singular point of $S_{\tau+1}$. We may work locally around s, and let S' denote an affine open neighborhood of s in S_τ, which we may assume is smooth.

Lemma 18. *Locally on S', if $i \leq \tau$, $E_i \subseteq H_{DR}(\mathcal{X}/S')$ has a basis annihilated by the Gauss-Manin connection.*

Proof: Let T be the p-adic completion of a smooth lifting of S' Then the crystalline cohomology of X/T is a lifting H_T of $H_{DR}(\mathcal{X}/S')$ to T, *i.e.*, a locally free sheaf of $\mathcal{O}_T$-modules whose reduction modulo p is canonically isomorphic to $H_{DR}(\mathcal{X}/S')$. Locally we may also choose a lifting F_T of the absolute Frobenius endomorphism of S'. Then the Frobenius endomorphism of F_X induces a map $\Phi_T \colon F_T^* H_T \to H_T$. Each point $s \in S'(k)$ admits a unique Teichmüller lifting $t \in T(W)$ (compatible with F_T), and $t^*(\Phi_T)$ becomes identified with the action of Frobenius Φ on the crystalline cohomology of $\mathcal{X}_s/W$.

For $i \geq 1$, the i^{th} iterate $\Phi_T^i \colon F_T^{i*} H_T \to H_T$ is defined in the obvious way. For every closed point s with corresponding Teichmüller point t, $t^*\Phi_T$ is divisible by p^{i-1}, by Proposition (3), and it follows easily by induction that the same is true of Φ_T^i. Write $\Phi_T^i := p^{1-i}\Phi_{i,T}$ and let $N_i^2 H_T$ denote the image of $\Phi_{i,T}$. For each Teichmüller point $t \in T(W)$, the image of $\tau^*(N_i^2 H_T)$ in $\tau^* H_T$ is the image $N_i^2(t)H_{T,t}$ of Φ_i acting on the crystalline cohomology of the corresponding $\mathcal{X}_s$. By Lemma 5, the fiber $E_i(s)$ identifies with the image of $N_i^2(t)$ in the de Rham cohomology at s. Since $E_i(\mathcal{X}/S')$ is a local direct factor of $H_{DR}(\mathcal{X}/S)$, it follows that in fact $E_i(\mathcal{X}/S') = N_i^2 H_{DR}(\mathcal{X}/S')$, *i.e.*, the image of the map $\Phi_{i,S'} \colon F_S^{*i} H_{DR} \to H_{DR}$. Since the map $\Phi_{i,T}$ is horizontal, the same is true of $\Phi_{i,S'}$. But $F_S^* H_{DR}$ has a basis of horizontal sections, and consequently so does its image $E_i(\mathcal{X}/S')$ under $\Phi_{i,S'}$. □

Localizing as necessary, we may choose a basis (x, y_i, z) for $H_{DR}(\mathcal{X}/S')$, compatible with the Hodge filtration. We may choose z so that $(x|z) = 1$ and $(y_i|z) = 0$ for all i. Then the expression for Q as an element of $Sym^2(H^\vee)$ is

$$Q = \sum_{i<j} (y_i|y_j) y_i' y_j' + \sum_i Q(y_i) {y_i'}^2 + x'z' + Q(z){z'}^2.$$

If $\eta \in E_{\tau,\mathcal{L}}$ lifts a basis for $E_{\tau,\mathcal{L}}/E_{\tau-1,\mathcal{L}}$, we can write $\eta = fx + \sum g_i y_i + hz$, and I is generated by $(x|\eta) = h$. By Lemma 5, x is a basis for $E_{\tau,L}(s)/E_{\tau-1,L}(s)$. Hence η can be chosen so that $\eta = x$ modulo $\mathfrak{m}$, *i.e.*, so that g_i and h belong to the maximal ideal $\mathfrak{m}$ and f is a unit. Since η and x are isotropic,

$$0 = Q(\eta) = \sum_{i<j} g_i g_j (y_i|y_j) + \sum_i g_i^2 Q(y_i) + fh + h^2 Q(z).$$

Then

$$-fh = \sum_{i<j} g_i g_j (y_i|y_j) + \sum_i g_i^2 Q(y_i) + h^2 Q(z) \in \mathfrak{m}^2$$

also generates I, and $h \in \mathfrak{m}^2$.

According to the previous lemma, we can choose η to be horizontal, so that

$$-\nabla(x) = df \otimes x + \sum dg_i \otimes y_i + g_i \nabla(y_i) + dh \otimes z + h\nabla(z).$$

But g_i and h annihilate $\Omega^1_{S'/k}(s)$, and since $h \in \mathfrak{m}^2$, dh maps to zero in $\Omega^1_{S'/k}(s)$. Thus $\kappa'(y_j') := y_j'(\nabla(x))$ maps to the class of $-dg_j$ in $\Omega^1_{S'/k}(s)$, that is, to the class of $-g_j$ in $\mathfrak{m}/\mathfrak{m}^2$. Consequently

$$\kappa^*(Q) = \sum_{i<j} g_i g_j (y_i|y_j) + \sum_i g_i^2 Q(y_i) = -fh \pmod{\mathfrak{m}^3}$$

□

Artin has asked if $\mathcal{F}_\infty := \cap \mathcal{F}_i$ is reduced. For simplicity we restrict to the case in which d is prime to p and p is odd.

Theorem 19. *Let $(\mathcal{X}/S, \mathcal{L})$ be a versal deformation of a polarized K3 surface $(X/k, L)$ of degree d. If d is prime to p and p is odd, then $S_\infty = S_{11}$, and in particular is generically nonreduced. If X is supersingular with $\sigma_0 = 10$, then étale locally in a neighborhood of the point of S corresponding to X, $\Sigma := S_{11,red}$ is smooth and $\mathcal{O}_{S_{11}} \cong \mathcal{O}_\Sigma[T]/(T^2)$.*

Proof: Let S' denote the open subset of S_{10} obtained by omitting the set of supersingular points with Artin invariant σ_0 at most 9. By Lemma 5, $E_{10}(\mathcal{X}/S')$ is a local direct factor of $H_{DR}(\mathcal{X}/S')$ of rank 10, and it is contained in the orthogonal complement H_{prim} of $c_{DR}(\mathcal{L})$. Consequently its orthogonal complement $E_{10}^\perp$ in H_{prim} is also a local direct summand. In fact E_{10} is, fiber by fiber, a maximal totally isotropic subspace of H_{prim}, hence $E_{10} = E_{10}^\perp$ fiber by fiber, and hence $E_{10} = E_{10}^\perp$. Now over S'_{11}, E_{11} is a totally isotropic subspace containing E_{10}, hence contained in $E_{10}^\perp$, and hence $E_{10} = E_{11}$. Then it follows from the definition of S'_{12} that $S'_{12} = S'_{11}$, and then that $S'_i = S'_{11}$ for every $i \geq 11$. To extend this result to all of S, we use the fact that S_{11} is a local complete intersection of dimension 9, and hence has depth 9. The ideal I of S_{12} in S_{11} vanishes outside the set of points with Artin invariant at most 9, and hence outside a closed set of dimension at most 8. Consequently $I = 0$. Continuing by induction, we conclude that $S_\infty = S_{11}$.

In fact, it is easy and instructive to calculate S'_{11} explicitly. Let us work locally around a point s corresponding to a supersingular K3 surface with $\sigma_0 = 10$. Then S'_{10} is smooth, and E_i is a local direct factor of $H_{DR}(\mathcal{X}/S_{10})$ for $i \leq 10$. On S_{10}, $E_9 \subseteq F^1_{Hdg}H_{DR}$, and E_9 is horizontal and totally isotropic. Furthermore, $E_9 \subseteq E_{10} \subseteq E^\perp_{9,\mathcal{L}}$, and all are local direct factors of H_{DR}. Let $V := E^\perp_{9,\mathcal{L}}/E_9$, a locally free sheaf of rank 3. The induced quadratic form on V is nondegenerate, and by Lemma 5, the image of $F^2_{Hdg}H_{DR}$ in V is a local direct factor of rank one and is isotropic. In some étale neighborhood of s there exists a basis (x, y, z) for V compatible with the Hodge filtration and such that $(x|x) = (x|y) = (z|z) = 0$ and $(x|z) = Q(y) = 1$. Let η be a basis for E_{10}/E_9, and write $\eta = fx + gy + hz$, with f, g, h sections of $\mathcal{O}_{S_{10}}$. At each point of S_{11}, $F^2_{Hdg}H_{DR} \subseteq E_{10}$, so that f is a unit in a neighborhood of s. Changing η if necessary, we may assume that $f = 1$. Then since η is isotropic, $h = -g^2$, and h generates the ideal of S_{11}. Since E_{10} is horizontal, we can write $\nabla(x) + dg \otimes y = \theta\eta = \theta x$ modulo g, so that $\nabla(x) = -dg \otimes y$ modulo $F^1_{Hdg}H_{DR}$ and g. Then $\kappa(y) = (\nabla(x)|y) = -dg$ modulo g. Since κ induces an isomorphism $H^1(X, \Omega^1_{X/k})/E_9 \cong \Omega^1_{S_{10}/k}(s)$, it follows that $dg(s)$ is not zero in $\Omega^1_{S_{10}/k}(s)$. Thus the closed subscheme S'' of S_{10} defined by g is smooth at s, and S_{11} is defined by g^2. In $\mathcal{O}_{S_{11}}$, $g^2 = 0$, and $\eta = x + gy$. Hence $F^*(\eta) = F^*(x) + g^p y = F^*(x)$, which maps to zero in Gr^1_{Hdg} and hence is killed by γ. This shows again that $S_{12} = S_{11}$. □

Remark 20. If $\mathcal{X}/S$ is a smooth proper morphism of schemes in characteristic p, in general it is not clear how to define the maximal subscheme of S over which $\mathcal{X}$ is supersingular. This difficulty is illuminated by the fact that S_∞ is not reduced. Especially, note that if S is a versal deformation of a supersingular K3 surface X with $\sigma_0 = 10$, then $NS(X)$ prolongs over $S_{11,red}$, not to all of S_{11}.

Remark 21. If $(\mathcal{X}/S, \mathcal{L})$ is a versal family of polarized K3 surfaces, the tangent and cotangent spaces of S inherit from the quadratic form on $H^1(\Omega^1_{\mathcal{X}/S})$ quadratic forms with values in $(F^2_{Hdg})^{\otimes 2}$; if d is prime to p and p is odd, these forms are nondegenerate. Furthermore, the conormal sheaves to each S_i are totally isotropic, as are the cotangent spaces to each S_{∞,σ_0}. It is this fact that underlies the proof of Theorem 17, and we wonder if it has further depth.

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