Crystalline prisms: Reflections on the present and the past

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Abstract

Let Y/S be a morphism of crystalline prisms, *i.e.*, a p-torsion free p-adic formal schemes endowed with a Frobenius lift, and let $\overline{Y}/\overline{S}$ denote its reduction modulo p. We show that the category of crystals on the prismatic site of $\overline{Y}/S$ is equivalent to the category of $\mathcal{O}_Y$-modules with integrable and quasi-nilpotent p-connection and that the cohomology of such a crystal is computed by the associated p-de Rham complex. If X is a closed subscheme of $\overline{Y}$, smooth over $\overline{S}$, then the prismatic envelope $\Delta_X(Y)$ of X in Y admits such a p-connection, and the category of prismatic crystals on X/S is equivalent to the category of $\mathcal{O}_{\Delta_X(Y)}$-modules with compatible p-connection, and cohomology is again computed by p-de Rham complexes. Our main tools are a detailed study of prismatic envelopes and a formal smoothness property for Y/S when working with prisms in the p-completely flat topology. We also explain how earlier work by several authors relating Higgs fields, p-connections, and connections can be placed in the prismatic context.

Introduction

This article was inspired by a conversation with Bhargav Bhatt in the fall of 2019 in which he explained that, if X/k is a smooth scheme over a perfect field k which admits a formal lifting Y/W along with its Frobenius endomorphism, then the prismatic cohomology of X relative to the prismatic base $(W, (p))$ can be computed as the hypercohomology of the "p-de Rham complex" $(\Omega^{\cdot}_{Y/W}, pd)$. In a further email exchange, Bhatt agreed with my speculation that, in this situation, the category of prismatic crystals on X/W should be equivalent to the category of $\mathcal{O}_Y$-modules with integrable and quasi-nilpotent p-connection. Unfortunately, the foundational manuscript on prismatic cohomology [7] neither mentions nor proves these statements, which resonate with previous work on the Cartier transform due to several authors, for example in [24], [26], [25], and [28]. Our purpose here is to confirm these statements and to elucidate the relations between the prismatic theory and the afore-mentioned previous work. We discuss only crystalline prisms, omitting what is undoubtedly the most interesting aspect of the theory.

Although striking, the comparison between prismatic and p-de Rham cohomology described above is not widely applicable, since Frobenius liftings rarely exist globally. However, one can often embed X/k as a closed subscheme of a smooth Y/W which does admit a Frobenius lift and then hope to endow the prismatic envelope of X in Y with a p-de Rham complex which calculates the prismatic cohomology of X/W. This is indeed possible and leads us to reconsider the old-fashioned idea that the geometry and cohomology of a general X can be understood by studying its tubular neighborhoods in a smooth Y. For example, Hartshorne [12] used this method to *define* the de Rham cohomology of a possibly singular variety X over a field k of characteristic zero by showing that de Rham cohomology of the formal completion of Y along X is independent of the choice of Y. Similarly, if X/k is a scheme over a perfect field k of characteristic p, one can compute (and even define) the crystalline cohomology of X/W by taking the de Rham complex of the divided power envelope of X in a smooth Y/W [3, 14, 6]. This approach works well only if X/k is smooth; the pathologies of crystalline cohomology for singular schemes can be avoided (at the expense of losing its integral structure) by replacing the divided power envelope by the open p-adic analytic tube (of radius < 1) of X in Y [22, 5]. A more sophisticated foundation for each of these theories is provided by the infinitesimal topos [11], [8], the crystalline topos [11, 3], and the convergent topos [23], respectively.

We shall discuss an analogous approach for prismatic cohomology. Let S be $\operatorname{Spf} W$, or, more generally a "formal ϕ-scheme," *i.e.*, a p-torsion free p-adic formal scheme equipped with a lift ϕ of the absolute Frobenius morphism of $\overline{S}$, its reduction modulo p. Let $X/\overline{S}$ be a smooth scheme, embedded in a p-completely smooth formal scheme Y/S which is also endowed with a Frobenius lift. We shall see in Proposition 3.6 that the structure sheaf of the "prismatic envelope" $\mathbb{\Delta}_X(Y)$ of X in Y, viewed as a sheaf of $\mathcal{O}_Y$-modules, is endowed with a canonical quasi-nilpotent p-connection, and in Theorem 3.11 that the derived image of the associated p-de Rham complex is functorial in X. These results are enough to establish the existence of a functorial cohomology theory based on p-de Rham cohomology, without relying on the formalism of Grothendieck topologies. Since that formalism does have important advantages, we go on in Theorem 6.15 to show that these complexes do calculate the cohomology of the structure sheaf of the prismatic topos of X/S. Similar results with coefficients: a crystal E on the prismatic site of X/S can be evaluated on $\mathbb{\Delta}_X(Y)$, the resulting sheaf of $\mathcal{O}_Y$-modules has a quasi-nilpotent p-connection which is compatible with the p-connection on $\mathbb{\Delta}_X(Y)$, and the resulting p-de Rham complex calculates the cohomology of E. When S is perfect these results can be deduced from Shiho's theory of (what we call) the "F-transform" and the usual crystalline theory.

Although this may all sound comfortable and reassuring to old-timers, there are new technical difficulties. First of all, an explicit description of prismatic envelopes seems difficult to come by, even locally with the aid of coordinates. In [7], these envelopes are constructed using universal constructions in the category of δ-rings, but, at least to this author, the geometric structure of these envelopes remains obscure. We try to address this difficulty by showing, in Theorem 2.19, that crystalline prismatic envelopes can be constructed as a limit of a sequence of p-adic dilatations, which were already used in [22], [25], and [28]. A second difficulty arises from the fact that morphisms in the prismatic site must be compatible with the chosen Frobenius lifting. This makes the task of understanding coverings of the final object of the prismatic topos more complicated, an issue we explore in §6.2. The coverings constructed in the foundational paper [7], which have the advantage of working in many topologies, seem to be too unwieldy to enable us to establish the form of the Poincaré lemma we need. It turns out that, if one works in the p-completely flat topology, such coverings can be constructed by simply taking prismatic envelopes of embeddings in formally smooth formal schemes endowed with a Frobenius lift. This fact, a generalization of a technique due to Morrow and Tsuji [20], is based on a (p-completely flat) local lifting property for such formal schemes explained in Theorem 1.12, and is perhaps the key geometric insight which enables our method to work. [1]

The use of extremely large "p-completely faithfully flat coverings" seems unavoidable. We will therefore of necessity be considering highly non-noetherian formal schemes, requiring some modifications of usual notions and techniques. We will follow the general methods suggested in [7] and [9], which we review, with some additional details and examples, in the appendix §7. Since we shall be working exclusively with p-adic formal schemes, the notion of p-complete flatness becomes considerably simpler than the more general notion of I-complete flatness, and we shall not need to consider derived completeness.

Our manuscript begins with a general discussion of Frobenius liftings, using standard deformation theory techniques, rather than the formalism of δ-rings. The main outtake is that the category of formal schemes endowed with a Frobenius lift is quite rigid, in that it is difficult to find morphisms which are compatible with the Frobenius lifts, even locally in the Zariski or étale topology. The most important positive result in this section is Theorem 1.12, which shows that this difficulty can be overcome by working in the p-completely flat topology.

The geometric heart of our work occurs in section §2, where we discuss various kinds of p-adic tubular neighborhoods. In decreasing order of size, these are formal completions, divided power envelopes, p-adic dilatations, and prismatic envelopes. We attempt to describe these, and especially the last two, as explicitly as possible. Let Y be a formal ϕ-scheme let $\overline{Y}$ its reduction modulo p. One key fact is that the scheme theoretic image $F_{\overline{Y}}(\overline{Y})$ of the Frobenius endomorphism of $\overline{Y}$ is the smallest subscheme of Y whose ideal has divided powers (2.4). Another is that if X is regularly immersed in $\overline{Y}$, then the map $\overline{\mathbb{A}}_X(Y) \to X$ is faithfully flat (but not of finite presentation) (see Theorem 2.19). We also discuss some useful generalities about the behaviour of prismatic envelopes. One phenomenon that remains somewhat mysterious is the extent to which prismatic neighborhoods, and morphisms between them, depend on the Frobenius lift; see for example, Proposition 2.40 and Example 2.29.

Section §3 is the "cohomological" heart of the paper. We begin with a review

[1] I learned in June 2021 at a conference in honor of Luc Illusie that this construction of coverings was independently discovered by Yichao Tian in a closely related work [27].

of connections and p-connections and their corresponding complexes. Key to crystalline cohomology is the fact that divided power envelopes carry natural connections, and we show that dilatations and prismatic envelopes carry natural p-connections. Then we formulate and prove Theorem 3.11, the prismatic Poincaré lemma, which shows that the p-de Rham complex of a prismatic envelope is independent of the choice of embedding, up to quasi-isomorphism. This also works for the p-de Rham complex of a module with p-connection, and we show that the category of these is, up to equivalence, also independent of the embedding. The key computation is already revealed by the case of a point, say $X = \operatorname{Spec} k$, with liftings $Y = \operatorname{Spf} W$ and $Z = \operatorname{Spf} W[x]\hat{}$, with $\phi(x) = x^p$. Then the prismatic neighborhood of X in Z is the formal spectrum of the completed PD-algebra $W\langle t\rangle\hat{}$, where $x = pt$. Then $d't = dx$, so p-de Rham complex of $\mathbb{\Delta}_X(Z)$ identified with the de Rham complex of the PD-algebra, and the Poincaré lemma follows.

Section §4 begins our effort to relate the prismatic theory to previous work. We emphasize what we call the "F-transform," which was used by many authors, beginning perhaps with Berthelot's ideas in [4], and then by others, including [10], [24], [25], [26], [28]. We focus on Shiho's work [26] showing how a relative Frobenius lift $F\colon Y \to Y'$ allows one to define an equivalence from the category of modules with quasi-nilpotent p-connection on Y' to the category of modules with quasi-nilpotent connection on Y. After reviewing Shiho's work, we prove Theorem 4.6, showing that the de Rham complex of the F-transform of a module with p-connection (E', ∇') is quasi-isomorphic to the p-de Rham complex of (E', ∇'). Theorem 4.8 show that that the F-transform takes prismatic envelopes to divided power envelopes. We explain in Theorem 4.11 a prismatic analog of the construction of the inverse Cartier transform [24, §2.4] [25, §1.5] and [28]: the prismatic envelope $\mathbb{\Delta}_\Gamma(Y \times_S Y')$ of the graph of F serves as a kernel for F-transform. In the prismatic context, this construction also works in the derived sense, which is not the case for the Cartier transform. Subsection § 4.3 discusses the relationship between the p-curvature ψ of the F-transform of a module with p-connection (E', ∇') and the Higgs field θ obtained by reducing ∇' modulo p. One might hope that ψ is the Frobenius pull-back of θ, but this is only approximately true, as the explicit formula given in Theorem 4.14 shows. The computation is somewhat complicated, and this result is not used elsewhere in this manuscript.

Section §5 prepares the way for the study of crystals on the prismatic topos. We adapt the traditional approach, originally due to Grothendieck, Berthelot, and Illusie. We explain and compare the groupoids, stratifications, and rings of differential operators that control the classical crystalline theory, the prismatic theory, and the intermediate case considered by Shiho. Finally we discuss linearization of prismatic differential operators. Our approach relies on a review and recasting of the theory of stratifications and groupoid actions in appendix §8.

The last section is devoted to the prismatic site, its crystals, and cohomology. The key point is that, if one works in the p-completely flat topology, the prismatic envelope of a scheme X in a (completely) smooth p-adic formal scheme with a Frobenius lift forms a covering of the final object in the prismatic topos of X. Using the prismatic Poincaré lemma, we show that the linearization of the p-de Rham complex of this envelope provides an acyclic resolution of the prismatic structure sheaf of X. A similar construction works for crystals. We should mention that technical difficulties seem to require the use of an auxilliary site, consisting of "small" prisms (see Definition 2.17). We also describe a variation of the prismatic site (based on what we call " ϕ-prisms" or "PD-prisms"), which was inspired by work of Oyama and Xu and which gives a geometric (site-theoretic) interpretation of the F-transform. It shows that, if S is a crystalline prism and $X/\overline{S}$ is smooth, then the prismatic topos of the Frobenius pullback X' of X is equivalent to the PD-prismatic topos of X/S, which is very closely related to the usual (big) crystalline topos. This gives a new "pure thought" proof of Shiho's theorem. [2] We finish with a few applications, which are mostly illustrations of how our results can be used to give simple and explicit proofs of results that are already stated in [7]. These include the comparison between prismatic and crystalline cohomology, the fact that prismatic Frobenius is an isogeny, and the relationship between the cotangent complex and prismatic cohomology.

Throughout this paper we fix a perfect field k and let W be its ring of Witt vectors. If Y/W is a scheme or formal scheme and n is a natural number, we denote by Y_n the subscheme (or formal subscheme) defined by p^n; we may also write $\overline{Y}$ for Y_1. We remind the reader again that we will only be studying "crystalline" prisms, in which the invertible ideal I in [7, Definition 3.2] is generated by p. Hence the prisms we consider will necessarily be p-torsion free. In this case the data of a δ-structure is equivalent to that of a Frobenius lift, and we shall therefore emphasize the latter.

We end with two appendices, which the reader is invited to consult as the need arises. The first addesses the technicalities we face when working with p-adic sheaves and modules without noetherian hypotheses. The point of view we adopt follows that of Drinfeld in [9], where we essentially work with inverse systems of p-torsion modules. For example, the category of p-adically separated and complete abelian groups is not abelian, but it does form an exact subcategory of the category of such systems, which is abelian. Localization presents another problem: for example, if A is a p-adically complete ring and a is an element of A then the p-adic completion of the localization of A by a may not be flat. This is overcome by the notion of "p-complete flatness," which for us just means p-torsion freeness and flatness modulo each power of p. There is a concomitant notion of "p-complete quasi-coherence," in the Zariski and p-completely flat topologies. We end with a discussion of "very regular" sequences, which we found to be useful in our description of dilatations and prismatic envelopes.

The second appendix revisits the notions of groupoids, stratifications, and crystals. We define the notion of the action of a groupoid (or even a category) on

[2] Daxin Xu has recently informed me that some similar constructions have been carried by Kimihiko Li in [19].

an object in a fibered category and explain the relationshp between groupoid actions, stratifications, differential operators, and crystals. We also explain some simple-looking tautologies, including linearization of differential operators, which are nevertheless useful in some of the calculations in our discussions of prismatic stratifications.

Enormous thanks go to Bhargav Bhatt, for the initial conversatin which inspired this work, for his encouragement to carry it out, and for his frequent patient technical advice that helped clarify some of my initial difficulties. Thanks also go to Matthew Morrow, to Atsuki Shiho, to Ahmed Abbes, and especially to Luc Illusie, for additional advice and enlightenment.

1 Frobenius liftings

We begin by reviewing some basic deformation theory of Frobenius lifts that will be useful in the sequel.

1.1 Definitions and preliminaries

Definition 1.1. *A formal ϕ-scheme is a p-torsion free p-adic formal scheme Y endowed with an endomorphism ϕ_Y lifting the absolute Frobenius endomorphisms of the subscheme of definition Y_1 defined by p. A morphism of formal ϕ-schemes is a morphism of formal schemes compatible with the Frobenius lifts.*

If $Y = \operatorname{Spf} B$ is an affine formal ϕ-scheme, we may write ϕ_Y or just ϕ instead of $\phi_Y^\sharp$ for the endomorphism of B corresponding to ϕ_Y. If $b \in B$, then necessarily $\phi(b) = b^p + p\delta(b)$ for some $\delta(b) \in B$, unique since B is p-torsion free. Thus one finds an endomorphism δ of B satisfying some simple axioms, which in turn imply that $b \mapsto b^p + p\delta(b)$ is an endomorphism of B, necessarily a Frobenius lift. In [7], the endomorphism δ plays the leading role, but here we shall work primarily with ϕ. In Definition 3.2 of [7], a *crystalline prism* is defined to be a p-torsion free p-adically complete ring endowed with a Frobenius lift, *i.e.*, an affine ϕ-scheme. If one allows prisms to be non-affine, our notion of ϕ-schemes is thus equivalent to their notion of crystalline prisms.

We shall not systematically investigate the category of ϕ-schemes, and in particular, if we say that a morphism of ϕ-schemes has some property typically associated to formal schemes, this will just mean that the underlying morphism of formal schemes has that property.

If $f : Y \to S$ is a morphism of ϕ-schemes, we can form the relative Frobenius diagram:

$$\begin{tikzcd} Y \arrow[r, "\phi_{Y/S}"] \arrow[dr, "f"'] & Y' \arrow[r, "\pi"] \arrow[d, "f'"] & Y \arrow[d, "f"] \\ & S \arrow[r, "\phi_S"] & S, \end{tikzcd} \tag{1.1}$$

where the square is Cartesian and the composition along the top is the given Frobenius lifting ϕ_Y of Y. Often we will work with a fixed ϕ-scheme S and the category of morphisms of ϕ-schemes $Y \to S$, which we shall call *S-ϕ-schemes.*

Example 1.2. If $(S, \phi_S) = \mathrm{Spec}(R, \phi)$ is an affine ϕ-scheme and I is an index set,

$$\begin{aligned} \mathbf{A}^I &:= \mathrm{Spf}\, R[\{x_i : i \in I\}]\hat{} \\ \mathbf{A}^I_\phi &:= \mathbf{A}^{I \times \mathbf{N}} = \mathrm{Spf}\, R[\{x_{i,j} : i \in I, j \in \mathbf{N}\}]\hat{}. \end{aligned}$$

Define morphisms:

$$\begin{aligned} \phi\colon \mathbf{A}^I_\phi \to \mathbf{A}^I_\phi &: \phi^\sharp(\sum r_{i,j} x_{i,j}) := \sum \phi_S^\sharp(r_{i,j})(x_{i,j}^p + p x_{i,j+1}) \\ r\colon \mathbf{A}^I_\phi \to \mathbf{A}^I &: r^\sharp(\sum r_i x_i) = \sum r_i x_{i,0} \ . \end{aligned}$$

The map r has the following universal property. If (T, ϕ_T) is any formal S-ϕ-scheme and $t\colon T \to \mathbf{A}^I$ is a morphism of formal schemes, there is a unique morphism of S-ϕ-schemes $t_\phi\colon (T, \phi) \to (\mathbf{A}^I_\phi, \phi)$, such that $r \circ t_\phi = t$. Namely, $t^\sharp_\phi\colon \{x_{i,j}\} \to \Gamma(T, \mathcal{O}_T)$ is given by the inductively defined formula

$$t^\sharp_\phi(x_{i,0}) := t^\sharp(x_i), \quad t^\sharp_\phi(x_{i,j+1}) := p^{-1}\left(\phi^\sharp_T(t^\sharp_\phi(x_{i,j})) - (t^\sharp_\phi(x_{i,j}))^p\right).$$

This construction can be localized and globalized. For example, suppose $I = \{0, \ldots, N\}$. Then for each $i \in I$, the special affine open subset $D(x_i)$ of $\mathrm{Spf}\, \mathbf{A}^N_\phi$ is invariant under ϕ, with

$$\phi(x_{i,0}^{-1}) = x_{i,0}^{-p}(1 - p x_{i,1} + p^2 x_{i,2} - \cdots).$$

These formal ϕ-schemes can be glued to form a formal ϕ-scheme $\mathbf{P}^N_\phi$ with a morphism to $\mathbf{P}^N$ satisfying the analogous universal property.

More generally, as observed in [7, 2.7], the forgetful functor $(T, \phi_T) \mapsto T$ from the category of formal ϕ-schemes to the category of p-torsion free formal schemes has a left adjoint $X \mapsto (X_\phi, \phi)$. For example, if X is affine, say $X = \mathrm{Spf}\, A$, write A as a quotient of some $B := R[x_i : i \in I]\hat{}$ with kernel J, let ϕ be the endomorphism of $B_\phi := R[x_{i,j} : i \in I, j \in J]\hat{}$ defined as above, and let J_ϕ be the smallest ideal of this ring containing the image of J and which is invariant under ϕ. Then B_ϕ/J_ϕ inherits a Frobenius lift. The set of p-torsion elements is a ϕ-invariant ideal, and dividing by the closure of this ideal yields the desired universal construction. Let us remark that if $\tilde{Y} \to Y$ is étale, then one checks easily that $\tilde{Y}_\phi \cong \tilde{Y} \times_Y Y_\phi$, and in particular, that $\tilde{Y}_\phi \to Y_\phi$ is again étale. (See also Joyal's interpretation [16] of the Witt vector functor.) We shall use this in Proposition 6.5 to construct coverings of the final object in the prismatic topos, generalizing a construction of Koshikawa (see [7, 4.18]).

Remark 1.3. The category of formal ϕ-schemes admits products and fiber products, but some care about p-torsion is required. First note that the inclusion functor from the category of p-adic formal schemes to its full subcategory of p-torsion free p-adic formal schemes admits a right adjoint $T \mapsto T_{\mathrm{tf}}$. Namely, the closure of the sheaf of p-torsion sections of $\mathcal{O}_T$ forms an ideal $\overline{I}_{\mathrm{tors}}$ whose formation is compatible with localization, and thus defines a closed formal subscheme T_{tf} of T which is in fact p-torsion free. (see Corollary 7.18), If T and T' are p-torsion free p-adic formal schemes, then their product $T \times T'$ (computed in the category of p-adic formal schemes) is p-torsion free, but if $T \to S$ and $T' \to S$ are morphisms of p-torsion free p-adic formal schemes, the fiber product $T \times_S T'$ might not be. However, $(T \times_S T')_{\mathrm{tf}}$ represents the fiber product in the category of torsion-free p-adic formal schemes. The computation of the p-torsion in such a product may not be obvious, but if $T \to S$ or $T' \to S$ is p-completely flat, then there is no such torsion. To make the analogous constructions in the category of ϕ-schemes, first observe that if $T \to S$ and $T' \to S$ are morphisms of formal ϕ-schemes, then the product $T \times T'$ and fiber product $T \times_S T'$ inherit Frobenius lifts, although the latter may not be p-torsion free. However, observe also that if T'' is a p-adic formal scheme endowed with a Frobenius lift ϕ, then the ideal of p-torsion elements and its closure are invariant under ϕ, and hence the closed formal subscheme T''_{tf} inherits a formal ϕ-scheme structure.

1.2 ϕ-schemes and ϕ-aligned subschemes

Let Y be a formal ϕ-scheme and let X be a closed subschema of Y_1. By a "lifting of X to Y_n we mean a closed subscheme X_n of Y_n which is flat over $\mathbf{Z}/p^n\mathbf{Z}$ such that $X_n \cap Y_1 = X_1$. Our first goal is to study the compatibility between liftings of X to subschemes of Y and the endomorphism ϕ.

Definition 1.4. *Let Y be a formal ϕ-scheme and let X be a closed subscheme of Y_1. We say that a lifting $X_n \subseteq Y_n$ is ϕ_n-aligned if ϕ maps X_n to itself, and we say that X is ϕ_n-alignable if it admits a ϕ_n-aligned lifting. We say X is ϕ-alignable if X admits a lifting to a formal ϕ-scheme contained in Y.*

We begin with a review of some elementary deformation theory. The first result is standard and we do not review its proof.

Proposition 1.5. *Let Y be a p-torsion free p-adic formal scheme and let $i_n \colon X_n \to Y_n$ be a closed immersion, with X_n flat over $\mathbf{Z}/p^n\mathbf{Z}$. Let I_n denote the ideal of X_n in Y_n and J_n the ideal of X_n in Y_{n+1}. Suppose that $i_{n+1} \colon X_{n+1} \to Y_{n+1}$ is a lifting of i_n, defined by an ideal $I_{n+1} \subseteq \mathcal{O}_{Y_{n+1}}$.*

1. *There is a natural isomorphism $\zeta_n \colon J_n/I_{n+1} \cong \mathcal{O}_X$, given by:*

$$\begin{aligned}\zeta_n : J_n/I_{n+1} =& (I_{n+1} + p^n\mathcal{O}_{Y_{n+1}})/I_{n+1} \cong p^n\mathcal{O}_{Y_{n+1}}/(p^n\mathcal{O}_{Y_{n+1}} \cap I_{n+1}) \\ \cong& p^n\mathcal{O}_{Y_{n+1}}/p^n I_{n+1} \xleftarrow[{[p^n]}]{\sim} \mathcal{O}_{X_1}\end{aligned}$$

2. *If $\tilde{i}_{n+1} \colon \tilde{X}_{n+1} \to Y_{n+1}$ is another lifting of i_n, with ideal $\tilde{I}_{n+1} \subseteq \mathcal{O}_{Y_{n+1}}$,*

let $\eta \in N_{X_1/Y_1} := \mathcal{H}om(I_1/I_1^2, \mathcal{O}_{X_1})$ *be defined by the following diagram:*

$$\begin{tikzcd} \tilde{I}_{n+1} \arrow[r] \arrow[d, "\tilde{\pi}"'] & J_n/I_{n+1} \arrow[d, "\zeta_n"] \\ I_1/I_1^2 \arrow[r, "\eta"] & \mathcal{O}_{X_1}. \end{tikzcd}$$

Then in fact

$$\tilde{I}_{n+1} = \{x + [p^n]\eta\pi(x) : x \in I_{n+1}\},$$

where $\pi\colon I_{n+1} \to I_1/I_1^2$ *is the projection.*

3. *If* $\eta \in N_{X_1/Y_1}$, *let* $\tilde{I}_{n+1} := \{x + [p^n]\eta\pi(x) : x \in I_{n+1}\}$. *Then* $\tilde{I}_{n+1}$ *defines a lifting of* i_{n+1}.

4. *The construction just described makes the set of liftings of* i_n *to* Y_{n+1} *into a pseudo-torsor under* N_{X_1/Y_1}. □

The next result is surely also standard, but we explain its proof nonetheless.

Proposition 1.6. *Let* $g\colon Y \to Y'$ *be a morphism of* p*-torsion free formal schemes, and for each* $n \in \mathbf{N}$, *let* $g_n\colon Y_n \to Y'_n$ *the induced morphism. Suppose we are given a commutative diagram*

$$\begin{tikzcd} X_n \arrow[r, "i_n"] \arrow[d, "f_n"'] & Y_n \arrow[d, "g_n"] \\ X'_n \arrow[r, "i'_n"] & Y'_n, \end{tikzcd}$$

where i_n *and* i'_n *are closed immersions, defined by ideals* I_n *and* I'_n *respectively, and where* X_n *and* X'_n *are flat over* $\mathbf{Z}/p^n\mathbf{Z}$. *Let* J_n *denote the ideal of* X_n *in* Y_{n+1}.

1. *If we are given lifts* X_{n+1} *and* X'_{n+1} *of* X_n *and* X'_n, *there is a unique homomorphism* ϵ *fitting in the commutative diagram below in which the rightmost vertical arrow is induced by* $g^\sharp$. *Furthermore,* g *maps* X_{n+1} *to* X'_{n+1} *if and only if* ϵ *vanishes.*

$$\begin{tikzcd} I'_1/{I'}_1^2 \arrow[d, dashed, "\epsilon"'] & I'_{n+1} \arrow[l, "\pi'"'] \arrow[r] \arrow[d, dashed] & J'_n \arrow[d] \\ f_*(\mathcal{O}_{X_1}) & g_*(J_n/I_{n+1}) \arrow[l, "\zeta"'] & g_*(J_n) \arrow[l, "\rho"'] \end{tikzcd}$$

2. *Assume that we are given lifts $X_{n+1}, \tilde{X}_{n+1}$ of X_n and $X'_{n+1}, \tilde{X}'_{n+1}$ of X'_n, differing by η and η' respectively, in the sense of Proposition 1.5. Let ϵ (resp. $\tilde{\epsilon}$) be the obstruction defined in (1) for the pair X_{n+1}, X'_{n+1} (resp. $\tilde{X}_{n+1}, \tilde{X}'_{n+1}$. Then*

$$\tilde{\epsilon} = \epsilon + f^\sharp \circ \eta' - f_*(\eta) \circ g^*,$$

where $g^*\colon I'_1/{I'_1}^2 \to I_1/I_1^2$ is the homomorphism induced by g.

3. *There exist liftings $\tilde{X}_{n+1}$ and $\tilde{X}'_{n+1}$ compatible with g if and only if the image of ϵ in the quotient of $\mathrm{Hom}(I'_1/{I'}^2_1, f_*(\mathcal{O}_{X_1})$ by the images of*

$$\eta' \mapsto f^\sharp \circ \eta' \colon \mathrm{Hom}(I'_1/{I'}^2_1, \mathcal{O}_{X'_1}) \to \mathrm{Hom}(I'_1/{I'}^2_1, f_*(\mathcal{O}_{X_1}))$$

and

$$\eta \mapsto f_*(\eta) \circ g^* \colon \mathrm{Hom}(I_1/I_1^2, \mathcal{O}_X)) \to \mathrm{Hom}(I'_1/{I'}^2_1, f_*(\mathcal{O}_{X_1}))$$

vanishes.

4. *Assume that we are given lifts $X_{n+1}, \tilde{X}_{n+1}$ of X_n and $X'_{n+1}, \tilde{X}'_{n+1}$ of X'_n and that g_{n+1} maps X_{n+1} to X'_{n+1}. Then g_{n+1} maps $\tilde{X}_{n+1}$ to $\tilde{X}'_{n+1}$ if and only if the diagram*

$$\begin{tikzcd} I'_1/{I'}^2_1 \arrow[r, "g^*"] \arrow[d, "\eta'"'] & f_*(I_1/I_1^2) \arrow[d, "f_*(\eta)"] \\ \mathcal{O}_{X'_1} \arrow[r, "f^\sharp"] & f_*(\mathcal{O}_{X_1}) \end{tikzcd}$$

commutes.

Proof. For (1), observe that $g^\sharp_{n+1}$ takes J'_n to J_n and hence induces the rightmost arrow in the diagram. The composed map $I'_{n+1} \to f_*(\mathcal{O}_{X_1}$ factors through the surjection π because $I'_{n+1} \otimes \mathcal{O}_X \cong I'_1 \otimes \mathcal{O}_X$, and thus we get the unique arrow ϵ making the diagram commute. Now g_{n+1} maps X_{n+1} to X'_{n+1} if and only if $g^\sharp$ takes I'_{n+1} to I_{n+1}, *i.e.*, if and only if $\epsilon = 0$. Note also that if $x' \in I'_{n+1}$, then $[p^n]\epsilon(\pi'(x')) \in J_n$ and its image under ζ agrees with that of $\pi(g^\sharp(x')$. Thus $g^\sharp(x') - [p^n]\epsilon(\pi'(x')) \in I_{n+1}$.

To prove (2), suppose that $\tilde{x}' \in \tilde{I}'_{n+1}$, and write $\tilde{x}' = x' + [p^n]\eta'\pi'(x')$ with $x' \in I'_{n+1}$. Let $x := g^\sharp(x') - [p^n]\epsilon(\pi'(x')) \in I_{n+1}$. Then

$$\begin{aligned} g^\sharp(\tilde{x}') &= g^\sharp(x') + [p^n]f^\sharp(\eta'\pi(x')) \\ &= x + [p^n]\epsilon(\pi'(x')) + [p^n]f^\sharp(\eta'\pi'(x')) \end{aligned}$$

Note that $\pi(x) = \pi(g^\sharp(x')) = f^*(\pi'(x') = f^*(\pi'(\tilde{x}'))$. Now let $\tilde{x} := x + [p^n]\eta\pi(x) \in \tilde{I}_{n+1}$ and rewrite the above equation as:

$$\begin{aligned} g^\sharp(\tilde{x}') &= \tilde{x} - [p^n]\eta\pi(x) + [p^n]\epsilon(\pi'(x')) + [p^n]f^\sharp(\eta'\pi'(x')) \\ &= \tilde{x} - [p^n]\eta f^*(\pi'(\tilde{x}')) + [p^n]\epsilon(\pi'(x')) + [p^n]f^\sharp(\eta'\pi'(x')) \end{aligned}$$

Hence:

$$\begin{aligned} \tilde{\epsilon}(\pi'(x')) &= \tilde{\zeta}(\tilde{\rho}g^\sharp(\tilde{x}')) \\ &= \tilde{\zeta}\left(\tilde{\rho}(\tilde{x}) - [p^n]\eta f^*(\pi'(\tilde{x}')) + [p^n]\epsilon(\pi'(x')) + [p^n]f^\sharp(\eta'\pi'(x'))\right) \\ &= -\eta(f^*(\pi'(x'))) + \epsilon(\pi'(x')) + f^\sharp(\eta'\pi'(x')) \end{aligned}$$

This proves (2), and statements (3) and (4) are immediate consequences. □

Corollary 1.7. *Let Y be a formal ϕ-scheme and let X_{n+1} be a closed subscheme of Y_{n+1}, flat over $\mathbf{Z}/p^{n+1}\mathbf{Z}$. Suppose that X_n is ϕ_n-aligned, and let ϵ be the element of $\mathrm{Hom}(I_1/I_1^2, F_{X*}(\mathcal{O}_{X_1}))$ defined as in (1) of Proposition 1.6. Then the obstruction to finding a ϕ_{n+1}-aligned lifting of X_n is the image of ϵ in the cokernel of the natural map*

$$\mathrm{Hom}(I_1/I_1^2, \mathcal{O}_X) \to \mathrm{Hom}(I_1/I_1^2, F_{X_1*}(\mathcal{O}_X)),$$

which, if X_1 is regularly embedded in Y_1, can be identified with the map

$$N_{X_1/Y_1} \to F_{X_1*}F_{X_1}^*(N_{X_1/Y_1}).$$

Proof. This follows from (3) of Proposition 1.6 and the fact that $\phi^*\colon I_1/I_1^2 \to I_1/I_1^2$ is the zero map, since $\phi^\sharp(I_{X_1'}) \subseteq I_{X_1}^p$. If X_1 is regularly embedded in Y_1, then I_1/I_1^2 is locally free of finite rank, so $N_{X_1/Y_1} := \mathcal{H}om(I_1, I_1^2, \mathcal{O}_X)$ shares this property, and

$$\mathcal{H}om(I_1/I_1^2, F_{X_1*}(\mathcal{O}_X)) \cong F_{X_1*}F_{X_1}^*(N_{X_1/Y_1}).$$

□

Example 1.8. Let $Y := \mathrm{Spf}\, W[x,y]\hat{}$, with $\phi(x) = x^p + py$ and $\phi(y) = y^p$, and let $X \subseteq Y$ be defined by (p, x). Then X is not ϕ_2-alignable, *i.e.*, there is no lift of X to Y_2 which is invariant under ϕ. Indeed, any such lift would be defined by $(p^2, x + pf)$ for some f such that $\phi(x + pf) \in (p^2, x + pf)$. Say $\phi(x + pf) = (x + pf)g \pmod{p^2}$, with $g \in W[x,y]\hat{}$. Then $x^p + py + p\phi(f) \equiv (x + pf)g \pmod{p^2}$, and so $g \equiv x^{p-1} \pmod p$. Write $g = x^{p-1} + ph$, and deduce that $y + \phi(f) = hx + x^{p-1}f \pmod p$. But this implies that $y + f^p \equiv 0 \pmod{(x,p)}$, which is impossible. In fact, the element ϵ described in Corollary 1.7 is the map ${I_1/I_1}^2 \to F^*(\mathcal{O}_X)$ taking the class of x to the class of y, and its image in $F_*F^*(N_{X'/Y_1'})/N_{X'/Y_1'}$ is nonzero.

Example 1.9. Let us also note that if Y/W is a flat formal scheme over W endowed with two Frobenius lifts ϕ and ϕ', then the diagonal embedding of Y_1 in $(Y \times Y, \phi \times \phi')$, may not be ϕ_2-alignable. For example, let $Y = \mathrm{Spf}\, W[x]\hat{}$

with $\phi(x) = x^p$ and $\phi'(x) = x^p + px$. Then the ideal of the diagonal is generated by $\xi := 1 \otimes x - x \otimes 1$, and (ϕ, ϕ') sends $x \otimes 1$ to $x^p \otimes 1$ and ξ to

$$1 \otimes x^p + 1 \otimes px - x^p \otimes 1 = \xi \left(\sum_{i=1}^{p-1} x^i \otimes x^{p-i} \right) + p \otimes x.$$

Thus ϵ is the map taking ξ to x, which lies in $F_*F^*(N_{Y_1/Y_1 \times Y_1})$ but not in $N_{Y_1/Y_1 \times Y_1}$.

In this next corollary we draw some conclusions about Frobenius lifts and ϕ-alignment. In the first part, we "decouple" the relationship between X_n and X'_n, in that we don't require that X'_n be obtained by base change from X_n.

Corollary 1.10. *Let Y/S be a morphism formal ϕ-schemes, where S_1 is regular, and let X_1 be a reduced closed subscheme of Y_1. Let $X'_1 \to Y'_1$ be the pullback of $X_1 \to Y_1$ along the Frobenius endomorphism of S and let $g\colon Y \to Y'$ be the map induced by ϕ.*

1. *For every n, there is at most one lift X'_n of X'_1 in Y'_n for which there exist some lift X_n of X such that g_n maps X_n to X'_n.*

2. *If $X_1 \subseteq Y_1$ is locally ϕ_n-alignable, then it is ϕ_n-alignable. If X_1 is locally ϕ-alignable, then it admits a lifting as a formal ϕ-subscheme of Y.*

Proof. Suppose that $X_n, \tilde{X}_n \subseteq Y_n$ and $X'_n, \tilde{X}'_n \subseteq Y'_n$ are flat liftings of X_1 and X'_1 respectively and that g takes X_n to X'_n and $\tilde{X}_n$ to $\tilde{X}'_n$. We prove that $X'_m = \tilde{X}'_m$ by induction on m. Assuming that this is true for m, let η (resp. η') measure the difference between X_{m+1} and $\tilde{X}_{m+1}$ (resp. between $X'_{m+1}$and $\tilde{X}'_{m+1}$). Then we have a diagram as in (4) of Proposition 1.6. But the top horizontal arrow in this diagram is the zero map, as we have seen in the proof of the previous corollary, and the bottom arrow is injective, since X_1 is reduced. It follows that the vertical map η' also vanishes, and hence that $\tilde{X}'_{m+1} = X'_{m+1}$.

Before proving (2), let us translate (1) into a uniqueness statement for ϕ_n-aligned liftings. Suppose that X_n and $\tilde{X}_n$ are ϕ_n-aligned liftings of X_1 in Y_n. Then (1) implies that their Frobenius pullbacks X'_n and $\tilde{X}'_n$ in Y'_n agree. Since S_1 is regular, the Frobenius lifting of S is p-completely flat, hence $Y'_n \to Y_n$ is faithfully flat, and this implies that X_n and $\tilde{X}_n$ also agree. If X_n is locally ϕ_n-alignable, it follows that local ϕ_n-aligned liftings agree on overlaps and hence patch together to a global aligned lifting. The statement for formal schemes follows immediately. $\square$

We note that in statement (1) of Corollary 1.10, the lifting of X_n is not unique. To the contrary, if g maps X_{n+1} to X'_{n+1}, then it also takes $\tilde{X}_{n+1}$ to X'_{n+1} for every lifting $\tilde{X}_{n+1}$ of X_n. Indeed, if g maps X'_{n+1} to X_{n+1}, then since f^* in the diagram in (4) of Proposition 1.6 vanishes, the diagram commutes with $\eta' = 0$ and any η. For an explicit example, let $Y := \operatorname{Spf} W[x]\hat{}$, and let $Y' := \operatorname{Spf} W[x']\hat{}$, with $g^\sharp(x') = x^p$, with X and X' defined by (x, p) and (x', p)

respectively. Suppose X_2' is defined by (x', p^2) and $\tilde{X}_2$ is defined by $(\tilde{x}, p^2)$, with $\tilde{x} := x + pf$ for some $f \in W[x]\hat{}$. Then

$$g^\sharp(x') \equiv -p^p f^p \pmod{\tilde{x}}$$

and hence belongs to $(\tilde{x}, p^2)$ for every f. Note, however, that further lifts may not be possible. For example, if X_2 is defined by $(x + p, p^2)$, there is no W_{p+1}-lift of $\tilde{X}_2$ in Y_{p+1} which maps to X'_{p+1}. Indeed, such a lift would be defined by $(p^{p+1}, \tilde{x})$ with $\tilde{x} = x + pf$ with $f \equiv 1 \pmod{p^2}$. Then $f^p \equiv 1 \pmod{p^2}$ and $g^\sharp(x') \equiv -p^p f^p$ and so does not belong to $(\tilde{x}, p^{p+1})$.

The following result shows that morphisms of ϕ-schemes are remarkably rigid.

Proposition 1.11. *Let T/S and Y/S be formal ϕ-schemes over a formal ϕ-scheme S. Assume that T_1 is reduced. Then a morphism of ϕ-schemes $f\colon T/S \to Y/S$ is uniquely determined by its restriction to T_1.*

Proof. It is enough to prove that for each $n \geq 1$, the restriction of f to T_{n+1} is uniquely determined by its restriction to T_n. Suppose then that $\tilde{f}\colon T_{n+1} \to Y$ is a morphism compatible with the ϕ-structures on T_{n+1} and Y and that $\tilde{f}_n = f_n\colon T_n \to Y_n$. Then $\tilde{f}_n^\sharp - f_n^\sharp$ is given by a homomorphism

$$\delta\colon f_n^* \Omega^1_{Y/S} \to p^n \mathcal{O}_{T_{n+1}} \cong \mathcal{O}_{T_1}.$$

Since $\tilde{f}$ and f are both compatible with the Frobenius liftings, we have:

$$\begin{aligned} \phi_T^\sharp \circ \tilde{f}^\sharp &= \tilde{f}^\sharp \circ \phi_Y^\sharp \\ \phi_T^\sharp \circ (f_n^\sharp + \delta) &= (f_n^\sharp + \delta) \circ \phi_Y^\sharp \\ \phi_T^\sharp \circ f_n^\sharp + \phi_T^\sharp \circ \delta &= f_n^\sharp \circ \phi_Y^\sharp + \delta \circ \phi_Y^* \\ \phi_T^\sharp \circ \delta &= \delta \circ \phi_Y^* \end{aligned}$$

Here ϕ_Y^* is the map $\Omega^1_{Y_1/S} \to \Omega^1_{Y_1/S}$ induced by the Frobenius endomorphism of Y_1, hence vanishes. On the other hand, since T_1 is reduced, $F_{T_1}^\sharp$ is injective, and it follows that $\delta = 0$. $\square$

The following result is inspired by [20, 3.4], which it generalizes. It will play a key role in the construction (6.6) of coverings of the final object in the prismatic topos and hence in the computation of prismatic cohomology.

Theorem 1.12. *Let $(Y, \phi_Y) \to (S, \phi_S)$ and $(T, \phi_T) \to (S, \phi_S)$ be morphisms of formal ϕ-schemes, where the underlying morphism Y/S is p-completely smooth.*

1. *Suppose that $n > 0$ and that we are given an S-morphism $f_n\colon T_n \to Y$ which is compatible with the Frobenius liftings. Then there exists a p-completely faithfully flat morphism of ϕ-schemes $u\colon (\widetilde{T}, \phi_{\widetilde{T}}) \to (T, \phi_T)$ and a morphism of S-ϕ-schemes $\tilde{f}\colon (\widetilde{T}, \phi_{\widetilde{T}}) \to (Y, \phi_Y)$ whose restriction to $\widetilde{T}_n$ agrees with the map $f_n \circ u_n$.*

2. *In the situation of statement (1), suppose in addition that $i\colon Z \to T$ is a closed immersion of ϕ-schemes and that we are given a morphism $g\colon Z \to Y$ which is compatible with the Frobenius lifts and such that $f_n \circ i_n = g_n$. Then there exists a commutative diagram of ϕ-schemes*

$$\begin{tikzcd}
\tilde{Z} \arrow[r, "\tilde{i}"] \arrow[d, "v"'] & \widetilde{T} \arrow[d, "u"] \arrow[ddr, "\tilde{f}"] & \\
Z \arrow[r, "i"] \arrow[drr, "g"'] & T & \\
 & & Y
\end{tikzcd}$$

such that $\tilde{f}_n = f_n \circ u_n$. and the morphisms u and v are p-completely faithfully flat. (Note: the square in this diagram may not be Cartesian.)

Proof. For statement (1), we first show that, after a suitable cover $\widetilde{T} \to T$, we can find a lift $\tilde{f}_{n+1}$ of f_n. Replacing T by an open affine cover, which is certainly p-completely flat (although not necessarily flat!), we may and shall assume that T is affine. Since the ideal $p^n\mathcal{O}_{T_{n+1}}$ of T_n in T_{n+1} is a square zero ideal and Y/S is formally smooth, there exists an S-morphism $f_{n+1}\colon T_{n+1} \to Y$ extending f_n, not necessarily compatible with the Frobenius liftings.

Consider the following diagram, where $\phi_{T/S}\colon T \to T' := T \times_{\phi_S} S$ and $\phi_{Y/S}\colon Y \to Y' := Y \times_{\phi_S} S$ are the relative lifted Frobenius maps.

$$\begin{tikzcd}
T_{n+1} \arrow[r, "\phi_{T/S}"] \arrow[d, "f_{n+1}"] & T'_{n+1} \arrow[r] \arrow[d, "f'_{n+1}"] & T_{n+1} \arrow[d, "f_{n+1}"] \\
Y \arrow[r, "\phi_{Y/S}"] & Y' \arrow[r] & Y
\end{tikzcd}$$

Here the right square is commutative, but the left square may not be. Since it does commute after restriction to T_n, the maps $f'_{n+1} \circ \phi_{T/S}$ and $\phi_{Y/S} \circ f_{n+1}$ from T_{n+1} to Y' differ by an element ϵ of

$$\begin{aligned}
\mathrm{Hom}(f_{n+1}^*\phi_{Y/S}^*(\Omega^1_{Y'/S}), p^n\mathcal{O}_{T_{n+1}}) &\cong \mathrm{Hom}(f_{n+1}^*\phi_{Y/S}^*(\Omega^1_{Y'/S}), \mathcal{O}_{T_1}) \\
&\cong \mathrm{Hom}(f_1^* F_{Y_1/S}^*(\Omega^1_{Y_1'/S}), \mathcal{O}_{T_1} \\
&\cong \mathrm{Hom}(F_{T_1}^* f_1^*(\Omega^1_{Y_1/S}), \mathcal{O}_{T_1}) \\
&\cong \mathrm{Hom}(f_1^*(\Omega^1_{Y_1/S}), F_{T_1*}(\mathcal{O}_{T_1}))
\end{aligned}$$

Since we are allowed to replace T by an affine cover, we may assume that $f_1^*(\Omega^1_{Y/S})$ is freely generated by elements $\omega_1, \ldots, \omega_N$. By [7, 2.12], which we restate and reprove as Proposition 2.25 below, we may find a p-completely faithfully flat map of ϕ-schemes $u\colon \widetilde{T} \to T$ and sections $\tilde{b}_i$ of $\mathcal{O}_{\widetilde{T}}$ such that $u^\sharp(\epsilon(\omega_i)) = \phi^\sharp_{\widetilde{T}}(b_i)$ for all i. These sections define a map $\delta\colon u^* f_1^* \Omega^1_{Y_1/S} \to \mathcal{O}_{\widetilde{T}_1}$ such that $F^*_{\widetilde{T}}(\delta) = u^*(\epsilon)$. Let $\tilde{f}_{n+1}\colon \widetilde{T}_{n+1} \to Y$ be the S-morphism corresponding to $f_{n+1} \circ u + \delta$. Then $\tilde{f}_{n+1} \circ u \circ \phi_{\widetilde{T}} = \tilde{f}_{n+1} \circ \phi_T \circ u$ corresponds to

$$f_{n+1} \circ \phi_T \circ u + F^*_{\widetilde{T}}(\delta) = f_{n+1} \circ \phi_T \circ u + u^*(\epsilon) = \phi_Y \circ f_{n+1} \circ u,$$

while $\phi_Y \circ \tilde{f}_{n+1} \circ u$ corresponds to

$$\phi_Y \circ f_{n+1} \circ u + F^*_Y(\delta) = \phi_Y \circ f_{n+1} \circ u,$$

since $F^*_Y\colon \Omega^1_{Y_1/S} \to \Omega^1_{Y_1/S}$ is the zero map. Thus $\phi_Y \circ \tilde{f}_{n+1} \circ u = \tilde{f}_{n+1} \circ u \circ \phi_T$ as desired.

Repeating this process, we find for $m \geq n$, affine and p-completely faithfully flat morphisms of ϕ-schemes $u^{(m)}\colon T^{(m+1)} \to T^{(m)}$, and Frobenius compatible maps $f^{(m)}\colon T^{(m)}_m \to Y$, such that $f^{(m+1)}_m = f^{(m)} \circ u^{(m)}_m$. Let $T^{(\infty)} := \varprojlim T^{(m)}$, endowed with the endomorphism ϕ inherited from that of each $T^{(m)}$. For each m, the map $T^{(\infty)}_m \to \varprojlim T^{(n)}_m$ is an isomorphism; this is just because direct limits of algebras commute with reduction modulo p^m. Our construction implies that, if $m'' \geq m' \geq m$, the diagram

$$\begin{tikzcd}
T^{(m'')}_m \arrow[d] \arrow[dr, "f^{(m'')}_m"] & \\
T^{(m')}_m \arrow[r, "f^{(m')}_m"] & Y
\end{tikzcd}$$

commutes. The vertical arrow is compatible with the Frobenius liftings, and the same is true of the horizontal arrow when $m' = m$. It follows that remaining arrow is also compatible with the Frobenius lifting. Thus we find a map

$$f^{(\infty)}_m := \varinjlim \{f^{(m')}_m : m' \geq m\}\colon T^{(\infty)}_m \to Y,$$

also compatible with Frobenius liftings. Now let $\widetilde{T}$ be the p-adic completion of $T^{(\infty)}$. By an argument of Temkin [9, 2.3.8] (which we review in Proposition 7.2) in fact $\widetilde{T}_m = T^{(\infty)}_m$ for all m, so $\widetilde{T} = \varinjlim T^{(\infty)}_m$, and the map

$$\tilde{f} := \varprojlim f^{(\infty)}_m \colon \widetilde{T} \to Y$$

is a map of formal ϕ-schemes. Since each $\widetilde{T}_m \to T_m$ is faithfully flat, the morphism $\widetilde{T} \to T$ is p-completely faithfully flat, by Proposition 7.13.

Let us sketch how to modify the proof of statement (1) to obtain statement (2). We again begin by assuming that T is affine. The morphisms $g_{n+1}\colon Z_{n+1} \to$

Y and $f_n\colon T_n \to Y$ agree on $Z_{n+1} \cap T_n = Z_n$, and hence glue to define an S-morphism $h_n\colon Z_{n+1} \cup T_n \to Y$, again compatible with the Frobenius lifts. The ideal I'' of $Z_{n+1} \cup T_n$ in T_{n+1} is contained in the ideal $p^n \mathcal{O}_{T_{n+1}}$ of T_n in T_{n+1} and, in particular, is a square zero ideal. Since Y/S is formally smooth, there exists an S-morphism $f_{n+1}\colon T_{n+1} \to Y$ extending h_n, not necessarily compatible with the Frobenius liftings. Then the element ϵ measuring this incompatibility considered in the proof of statement (1) in fact belongs to

$$\mathrm{Hom}(f_{n+1}^* \phi_{Y/S}^*(\Omega^1_{Y'/S}), I'').$$

Since Z is p-torsion free, in fact $I'' = p^n \mathcal{O}_{T_{n+1}} \cap I_{Z/T} \cong I_{Z_1/T_1}$, so the elements $b_i := \epsilon(\omega_i)$ belong to $F_{T_1 *}(I_{Z_1/T_1})$.

Let $u\colon \widetilde{T} \to T$ be the universal morphism of ϕ-schemes endowed with sections $\tilde{b}.$ such that $u^\sharp(b.) = \phi(\tilde{b}_i)$ as described in Proposition 2.25 below. This morphism is p-completely faithfully flat, and, since $i^\sharp(b.) = 0$, admits a section $\tilde{i}\colon Z \to \widetilde{T}$. Then we can take $v := \mathrm{id}_Z$ to form the diagram in statement (2). Now let $\tilde{f}\colon \widetilde{T} \to Y$ be the S-morphism corresponding to $f_{n+1} \circ u + \delta$. Just as before, we see that $\phi_Y \circ \tilde{f}_{n+1} \circ u = \tilde{f}_{n+1} \circ u \circ \phi_T$ as desired. Furthermore, since $\tilde{i}^\sharp(b.) = 0$, it follows that $\tilde{f}_{n+1} \circ \tilde{i} = g_{n+1} \circ v$, as claimed. The rest of the proof proceeds as before. □

Theorem 1.12 suggests that a morphism of ϕ-schemes whose underlying morphism of formal schemes is formally smooth should satisfy an infinitesimal lifting property, provided one works locally in the p-completely flat topology. Although we shall not need this result here, let us take the time to formulate it precisely. The proof is immediate from the theorem.

Corollary 1.13. *Let $i\colon Z \to T$ be a closed immersion of S-ϕ-schemes defined by a nilpotent ideal and let $g\colon Z \to Y$ and $Y \to S$ be morphisms of S-ϕ-schemes, where the underlying morphism $Y \to SS$ is p-completely smooth. Then there exists a commutative diagram as in statement (2) of Theorem 1.12.* □

2 Tubular neighborhoods

Our aim in this section is to describe and compare some of the notions of p-adic tubular neighborhoods that have appeared in the literature and to explain their relation to the new theory of prisms. In general, X will be a scheme embedded as a closed subscheme of a p-torsion free scheme or formal scheme Y, and $\mathbb{T}_X(Y)$ will denote the "tubular neighborhood" of X in Y. For example, $\mathbb{T}_X(Y)$ could be one of the following:

$\mathbb{F}_X(Y)$, the formal completion of Y along X,

$\mathbb{P}_X(Y)$, the p-torsion free divided power envelope of X in Y, which we call the "divided power enlargement" of X in Y.

$\mathbb{D}_X(Y)$, the p-adic dilatation of X in Y,

$\mathbb{A}_X(Y)$ the prismatic envelope of X in Y. (In this case, we require that Y be endowed with a suitable lift of Frobenius; see Theorem 2.19 for a precise statement.)

2.1 Divided power enlargements

We begin by discussing divided power enlargements, which underly the first successful attempt at integral p-adic cohomology. Since we are restricting our attention here to torsion-free algebras, the intricacies of divided powers are considerably simplified.

Let B be a torsion free ring. For each $x \in B$, let $x^{[n]} := x^n/n! \in \mathbf{Q} \otimes B$, and recall the following formulas.

$$
\begin{aligned}
(x+y)^{[n]} &= \sum_{i+j=n} x^{[i]} y^{[j]} \\
(bx)^{[n]} &= b^n x^{[n]} \\
x^{[m]} x^{[n]} &= \binom{m+n}{n} x^{[m+n]} \\
(x^{[n]})^{[m]} &= \prod_{i=1}^{m-1} \binom{in+n-1}{n-1} x^{[mn]}
\end{aligned}
$$

An ideal I of B is said to be a *PD-ideal* if $x^{[n]} \in I$ whenever $x \in I$ and $n > 0$. If Y is a p-torsion free scheme or formal scheme, a closed immersion $X \to Y$ is a *PD-immersion* if its defining ideal is a PD-ideal. It follows from the above formulas that the set of all $x \in B$ such that $x^{[n]}$ belongs to B for all $n \in \mathbf{N}$ forms an ideal $I_{\mathbb{P}}(B)$ of B and that $x^{[m]} \in I_{\mathbb{P}}(B)$ if $x \in I_{\mathbb{P}}(B)$. This ideal is the largest PD-ideal of B. It is the unit ideal if and only if B contains $\mathbf{Q}$, and if B is a $\mathbf{Z}_{(p)}$-algebra, the ideal $I_{\mathbb{P}}(B)$ contains p. Let us record these facts in the following definition.

Definition 2.1. *If B is a torsion free ring, let*

$$I_{\mathbb{P}}(B) := \{x \in B : x^{[n]} \in B \text{ for all } n \in \mathbf{N}\},$$

the largest PD-ideal of B. If T is a p-torsion free formal p-adic scheme over W, define $I_{\mathbb{P}} \subseteq \mathcal{O}_T$ in the same way, and note that $p \in I_{\mathbb{P}}$. If $I_{\mathbb{P}}$ is quasi-coherent, let $T_{\mathbb{P}}$ denote the closed subscheme it defines.

If T is a p-torsion free formal p-adic scheme, the ideal $I_{\mathbb{P}}$ of $\mathcal{O}_T$ is the inverse image of its image in $\mathcal{O}_{T_1}$, since it contains p, and quasi-coherence of $I_{\mathbb{P}}$ amounts to quasi-coherence of this image. This does not seem to be automatic: $I_{\mathbb{P}}$ is the intersection of a countable sequence of quasi-coherent sheaves, which need not be quasi-coherent. Note that, if $T_{1\mathrm{red}}$ is the reduced subscheme of T_1, then $I_{\mathbb{P}} \subseteq I_{T_{1\mathrm{red}}}$. Indeed, if $x \in I_{\mathbb{P}}$, then $x^{[p]} \in \mathcal{O}_T$, hence $x^p = p! x^{[p]} \in p\mathcal{O}_T$, hence x belongs to the ideal $I_{T_{1\mathrm{red}}}$.

Definition 2.2. *If X is a k-scheme, a PD-enlargement of X is a p-torsion free p-adic formal scheme T together with a PD-immersion $i_T \colon X_T \to T$ and a* *morphism $z_T: X_T \to X$. A PD-enlargement is small if z_T is flat. A morphism of PD-enlargements of X is a pair of morphisms $(f, f_X): (T, X_T) \to (T', X_{T'})$ such that $z_{T'} \circ f_X = z_T$ and $f \circ i_T = i_{T'} \circ f_X$. If $i: X \to Y$ is a closed immersion of X into a p-torsion free p-adic formal scheme, then a PD-enlargement of X in Y is PD-enlargement (T, z_T, i_T) of X together with a morphism $\pi_T: T \to Y$ such that $i \circ z_T = \pi_T \circ i_T$. The PD-dilatation of X in Y is the final object in the category of PD-enlargements of X in Y.*

Our "PD-enlargements" differ from the "PD-thickenings" usually considered in the crystalline setting in two respects: we require the ambient formal scheme T to be p-torsion free and we allow $z_T: X_T \to X$ to be any morphism, not necessarily an open immersion. The "smallness" condition will allow us to partially compensate for this extra generality.

The following result establishes the existence and basic properties of PD-dilatations; these may not be the same as PD-envelopes in general.

Proposition 2.3. *Let Y be a p-torsion free p-adic formal scheme and let X be a closed subscheme of Y_1. Then the PD-dilatation of X in Y is representable by a diagram:*

$$\begin{tikzcd} X_{\mathbb{P}} \arrow[r, "i_{\mathbb{P}}"] \arrow[d, "z_{\mathbb{P}}"'] & \mathbb{P}_X(Y) \arrow[d, "\pi_{\mathbb{P}}"] \\ X \arrow[r] & Y. \end{tikzcd}$$

The morphisms $\pi_{\mathbb{P}}: \mathbb{P}_X(Y) \to Y$ and $z_{\mathbb{P}}: X_{\mathbb{P}} \to X$ are affine, and $\mathbb{P}_X(Y)$ is a closed formal subscheme of the usual PD-envelope $D_X(Y)$ of X in Y. Furthermore, the following statements are verified.

1. *Formation of $\mathbb{P}_X(Y)$ is functorial: a morphism of pairs $g: (Y', X') \to (Y, X)$ induces a morphism $g: \mathbb{P}_{X'}(Y') \to \mathbb{P}_X(Y)$. If $g: Y' \to Y$ is p-completely flat and $X' = g^{-1}(X)$, then the corresponding map $\mathbb{P}_{X'}(Y') \to \mathbb{P}_X(Y) \times_Y Y'$ is an isomorphism.*

2. *Suppose that $X \to Y$ is an immersion of formal S-schemes and that Y/S is a formally smooth morphism of formal ϕ-schemes. If X is reduced, the map $z_{\mathbb{P}}: X_{\mathbb{P}} \to X$ is an isomorphism, and if $X \to Y_1$ is a regular immersion, then the maps $z_{\mathbb{P}}: X_{\mathbb{P}} \to X$ and $\mathbb{P}_X(Y) \to D_X(Y)$ are isomorphisms.*

3. *If $\phi: Y \to Y$ is a Frobenius lifting, then $\mathbb{P}_X(\phi): \mathbb{P}_X(Y) \to \mathbb{P}_X(Y)$ is also a Frobenius lifting.*

Proof. Let $D_X(Y)$ be the p-adic completion of the usual PD-envelope described in [6, 3.19]. (Strictly speaking, this is direct limit of the PD-envelopes of X in

Y_n, for $n > 0$.) This fits into a universal

$$\begin{tikzcd} X \arrow[r] \arrow[dr] & D_X(Y) \arrow[d] \\ & Y, \end{tikzcd}$$

where $X \to D_X(Y)$ is a PD-immersion. This diagram might not define a PD-enlargement, since $D_X(Y)$ might not be torsion free. To remedy this, let I_{tor} denote the sheaf of p-torsion sections of the structure sheaf $\mathcal{D}_X(Y)$ of $D_X(Y)$ and let $\overline{I}_{\mathrm{tor}}$ denote its closure in the p-adic topology. It is straightforward to check that these sheaves are ideals of $\mathcal{D}_X(Y)$ whose formation is compatible with localization (see Corollary 7.18). Then $\overline{I}_{\mathrm{tor}}$ defines a closed immersion $\mathbb{P}_X(Y) \to D_X(Y)$, and $\mathbb{P}_X(Y)$ is p-torsion free. Furthermore, the intersection of $\overline{I}_{\mathrm{tor}}$ with the ideal $\overline{I}_X$ of X in $\mathcal{D}_X(Y)$ is closed under divided powers, so that $\overline{I}_X/\overline{I}_X \cap \overline{I}_{\mathrm{tor}}$ is a divided power ideal of $\mathcal{O}_{\mathbb{P}_X(Y)}$ and hence defines a PD-immersion $X_{\mathbb{P}} \to \mathbb{P}_X(Y)$. It follows easily that this construction gives the universal PD-enlargement of X. The functoriality of the formation of $\mathbb{P}_X(Y)$ follows from its universal property. If $g\colon Y' \to Y$ is p-completely flat, then each $g_n\colon Y'_n \to Y_n$ is flat (see Proposition 7.13), and if in addition $X' = g^{-1}(X)$, it follows that each map $D_{X'}(Y'_n) \to D_X(Y_n) \times_{Y_n} Y'_n$ is an isomorphism [3, 2.7.1]. Then $D_{X'}(Y') \to D_X(Y) \times_Y Y'$ is also an isomorphism, and it follows that $D_{X'}(Y') \to D_X(Y)$ is p-completely flat, by Proposition 7.14. Then $\overline{I}_{tor}(X') \subset \mathcal{D}_{X'}(Y')$ is the pullback of $\overline{I}_{tor}(X) \subseteq \mathcal{D}_X(Y)$, by Corollary 7.18. It follows that $\mathbb{P}_{X'}(Y') \to \mathbb{P}_X(Y) \times_Y Y'$ is also an isomorphism.

Statement (2) can be verified locally, so we may assume that X and Y are affine. If X is reduced, recall from [15, §1.4] that the embedding of X in $WX := \operatorname{Spf} W(\mathcal{O}_X)$ is a PD-immersion into a p-torsion free p-adically complete and p-torsion free sheaf of rings. Since Y/S is formally smooth $X \to WX$ and X is affine, we can extend the formal embedding $X \to WX$ to a map $WX \to Y$. Then the universal property of $\mathbb{P}_X(Y)$ produces maps of pairs:

$$(X \subseteq WX) \longrightarrow (X_{\mathbb{P}} \subseteq \mathbb{P}_X(Y)) \longrightarrow (X \subseteq Y).$$

since the map $X_{\mathbb{P}} \to X$ is, by construction, a closed immersion, it must in fact be an isomorphism. If $X \to Y_1$ is a regular immersion, then $D_X(Y)$ is already p-torsion free. (To see this, one can use the compatibility of formation of PD-envelopes with flat base extension to reduce to the case in which X is the zero section of affine space.) Thus $\mathbb{P}_X(Y) \to D_X(Y)$ is an isomorphism in this case.

It suffices to check statement (3) in the affine case, and we use affine notation, with $Y = \operatorname{Spf} B$ and $X \subseteq Y$ defined by the ideal I. As we have seen, then $\mathbb{P}_X(Y)$ is the p-adic completion C of the spectrum of the B-algebra $D_I(B)/\overline{I}_{\mathrm{tor}}$ and hence is topologically generated by elements of the form $x^{[n]}$ for all $x \in I$. Our claim is that $\phi(c) \equiv c^p \pmod{pC}$ for all $c \in C$. Since this is true for elements of B, it will suffice to check that it holds on a set of topological generators for the B-algebra C, and hence for the elements $x^{[n]}$ with $x \in I$ and $n > 0$. Since each of these belongs to a divided power ideal, $(x^{[n]})^p \in pC$. On the other hand, $\phi(x^{[n]}) = \phi(x)^{[n]} \equiv (x^p)^{[n]} \pmod{pC}$ and $(x^p)^{[n]} = (p!x^{[p]})^{[n]} = (p!)^n x^{[p]})^{[n]} \in pC$. Thus $(x^{[n]})^p$ and $\phi(x^{[n]})$ are both congruent to zero mod pC. $\square$

The following result is a reformulation of [7, 2.35].

Proposition 2.4. *Let Y be a formal ϕ-scheme, and let $\phi(Y_1) \subset Y$ denote the scheme-theoretic image of the restriction of ϕ to Y_1, i.e., the closed subscheme defined by $I_\phi := \{c \in \mathcal{O}_Y : c^p \in p\mathcal{O}_Y\}$. Then $\phi(Y_1) \to Y$ is a PD-immersion. In fact, $\phi(Y_1) = Y_{\mathbb{P}}$, the smallest PD-subscheme of Y_1 (see Definition 2.1).*

Proof. We claim that the ideal I_ϕ defining the closed immersion $\phi(Y_1) \to Y$ is a PD-ideal. Assume without loss of generality that $Y = \operatorname{Spf} C$.

We check first that if $c \in I_\phi$ and $c^p = pc_1$, then also $c_1 \in I_\phi$. in fact:

$$\begin{array}{rcl} \phi(c)^p & = & p\phi(c_1) \\ (c^p + p\delta(c))^p & = & p(c_1^p + p\delta(c_1)) \\ c^{p^2} + p^2(\cdots) & = & pc_1^p + p^2\delta(c_1) \\ p^p c_1^p + p^2(\cdots) & = & pc_1^p + p^2\delta(c_1) \\ (p^{p-1} - 1)c_1^p & = & p(\delta(c_1) - \cdots). \end{array}$$

This last equation implies that $c_1^p \in pC$, as claimed.

Continuing by induction, we find a sequence $c_0, c_1, c_2 \ldots$ such that $c_0 = c$ and $c_i^p = pc_{i+1}$ for $i \geq 0$. It follows that $c^{p^i} = p^{1+p+\cdots+p^{i-1}} c_i$ for all i, and if $a_i \in \mathbf{N}$, that $c^{a_i p^i} \in p^{a_i(1+p+\cdots+p^{i-1})}C$. If n is a natural number, let $n = \sum a_i p^i$ be its p-adic expansion, and let $\sigma := \sum a_i$. Then $c^n \in p^m C$, where

$$m := \sum_i a_i(1 + p + \cdots + p^{i-1}) = \sum_i a_i \frac{p^i - 1}{p - 1} = \frac{n - \sigma}{p - 1}.$$

The expression on the right is the p-adic ordinal of $n!$ [6, 3.3], so we conclude that $c^{[n]} := c^n/n! \in C$. Moreover, since $c \in I_\phi$, we can write $\phi(c) = pb$, and then $\phi(c^{[n]}) = \phi(c)^{[n]} = (pb)^{[n]} = b^n p^{[n]} \in pC$, so $c^{[n]}$ again belongs to I_ϕ. This shows that I_ϕ is indeed a PD-ideal as claimed.

Finally, suppose that $X \subseteq Y_1$ is a closed subscheme whose ideal $I \subseteq C$ is invariant under divided powers. For each $c \in I$, we have $c^p = p!c^{[p]}$, so $c^p \in pC$, hence $c \in I_\phi$. $\square$

2.2 p-adic Dilatations

The following special PD-enlargements form the starting point for the theory of convergent and rigid cohomology, and were first explicitly discussed in [22] and further developed in [28].

Recall that if T is a p-adic formal scheme, we write either T_1 or $\overline{T}$ for the closed subscheme defined by (p).

Definition 2.5. *Let X be a p-adic formal scheme for example (and usually) an $\mathbf{F}_p$-scheme. A p-adic enlargement of X is a p-torsion free p-adic formal scheme T together with a morphism $z_T\colon \overline{T} \to X$. A p-adic enlargement is small if z_T is flat. A morphism of p-adic enlargements of X is a morphism $f\colon T \to T'$ such that $z_{T'} \circ f_1 = z_T$. If $i\colon X \to Y$ is a closed immersion of X into a formal scheme Y, then a p-adic enlargement of X in Y is a p-adic enlargement (T, z_T) of X together with a map $\pi_T\colon T \to Y$ such that $i \circ z_T = \pi_{T|_{\overline{T}}}$:*

$$\begin{tikzcd} \overline{T} \arrow[r] \arrow[d, "z_T"'] & T \arrow[d, "\pi_T"] \\ X \arrow[r, "i"] & Y. \end{tikzcd}$$

As an alternate phrasing, we may say *p-adic X-enlargement* instead of "p-adic enlargement of X," and we may say a *p-adic X-enlargement over Y* instead of a "p-adic enlargement of X in Y." Note that the categories of p-adic enlargements of X and of $\overline{X}$ are identical. A p-adic enlargement of X is also a PD-enlargement, since the ideal (p) always has divided powers.

Suppose that $i\colon X \to Y$ is an embedding of X in a p-adic formal scheme Y and that (T, z_T, π_T) is a p-adic enlargement of X in Y. The map $\overline{T} \to T$ necessarily factors through $\overline{X} \times_Y T$, and the map $\overline{X} \times_Y T \to T$ necessarily factors through $\overline{T}$. It follows that the these factorizations are isomorphisms, so the diagram in Definition 2.5 is Cartesian when $X = \overline{X}$. Thus the image of the map $\pi_T^\sharp\colon \pi_T^*(I_{\overline{X}/Y}) \to \mathcal{O}_T$ is the principal ideal $p\mathcal{O}_T$. Consequently there is a unique $\mathcal{O}_Y$-linear map:

$$\rho_T\colon I_{\overline{X}/Y} \to \pi_{T*}(\mathcal{O}_T) \tag{2.1}$$

such that $p\rho_T(a) = \pi_T^\sharp(a)$ for all $a \in I_{\overline{X}/Y}$.

Definition 2.6. *If $i\colon X \to Y$ is a closed immersion of a p-adic formal scheme X into a p-torsion free p-adic formal scheme Y, then the p-adic dilatation of X in Y, denoted by $\mathbb{D}_X(Y)$, is the final object of the category of p-adic enlargements of X in Y:*

$$\begin{tikzcd} \overline{\mathbb{D}}_X(Y) \arrow[r] \arrow[d, "z_{\mathbb{D}}"'] & \mathbb{D}_X(Y) \arrow[d, "\pi_{\mathbb{D}}"] \\ X \arrow[r, "i"] & Y \end{tikzcd}$$

Warning: $\mathbb{D}_X(Y)$ could well be empty. For example, let $V := W[x]/(x^m - p)$ with $m > 1$, let $Y := \operatorname{Spf} V$ and $X := \operatorname{Spec} k$. If $V \to B$ defines a p-adic enlargement of X in Y, then $\pi B = pB$, hence $B = \pi^{m-1}B$, and since B is p-adically complete, it follows that $B = 0$. This difficulty was addressed in [22] by considering more general enlargements of X: p-adic formal schemes T

endowed with a map from the reduced subscheme of $\overline{T}$ to X. The cohomology of the corresponding topos corresponds to the so-called "convergent cohomology" of X [23]. We should also mention another variant, introduced by Oyama [25] and further studied by Xu [28]: p-adic formal schemes T equipped with a map from the scheme-theoretic image of $F_{\overline{T}}$ to X. We shall not follow either of these approaches here. See, however, Remark 2.16 and section 6.5.

As is well known, $\mathbb{D}_X(Y)$ is representable. The following result summarizes what we shall need about the construction. The references [22] and [28] contain additional details.

Theorem 2.7. *Let Y be a p-torsion free scheme or formal p-adic scheme and let X be closed subscheme of Y. Then $\mathbb{D}_X(Y)$ is representable, and the morphisms $\pi_{\mathbb{D}} \colon \mathbb{D}_X(Y) \to Y$ and $z_{\mathbb{D}} \colon \overline{\mathbb{D}}_X(Y) \to X$ are affine. Furthermore, the following statements are verified.*

1. *$\mathbb{D}_X(Y)$ is the open subset $D^+(p)$ of the formal blowup of $\overline{X}$ in Y defined by the element p of the ideal of $\overline{X}$ in Y.*

2. *Formation of $\mathbb{D}_X(Y)$ is functorial: a morphism $g \colon Y' \to Y$ sending a closed subscheme X' of Y' to X induces a morphism $\mathbb{D}_X(g) \colon \mathbb{D}_{X'}(Y') \to \mathbb{D}_X(Y)$. If g is p-completely flat and $X' = g^{-1}(X)$, then the natural map*

$$\mathbb{D}_{X'}(Y') \to \mathbb{D}_X(Y) \times_Y Y'$$

   is an isomorphism.

3. *Assume that $X = \overline{X}$ and that $X \to \overline{Y}$ is a very regular immersion, i.e., that it is locally defined by a finite sequence $(x_1, \ldots, x_r)$ every permutation of which is regular (see Definition 7.23.) Then the following hold.*

   (a) *If $\pi_T \colon T \to Y$ is any morphism of formal schemes, the set of Y-morphisms $T \to \mathbb{D}_X(Y)$ identifies naturally with the set of homomorphisms $\rho \colon I_{X/Y} \to \pi_* \mathcal{O}_T$ such that $\rho(p) = 1$.*

   (b) *The map $z_Y \colon \overline{\mathbb{D}}_X(Y) \to X$ is faithfully flat and is naturally a torsor under the action of the conormal bundle of X in $\overline{Y}$.*

   (c) *Let $I_{X/Y}$ be the ideal of X in Y and let $I_{X/\overline{Y}} = I_{X/Y}/(p \cap I_{X/Y})$ be the ideal of X in $\overline{Y}$. Then there is an exact sequence:*

$$0 \to \mathcal{O}_X \xrightarrow{[p]} I_{X/Y}/I^2_{X/Y} \longrightarrow I_{X/\overline{Y}}/I^2_{X/\overline{Y}} \to 0,$$

   and $\overline{\mathbb{D}}_X(Y)$ identifies with the relative spectrum of the $\mathcal{O}_X$-algebra $\varinjlim S^n(I_{X/Y}/I^2_{X/Y})$, where the maps

$$S^n(I_{X/Y}/I^2_{X/Y}) \to S^{n+1}(I_{X/Y}/I^2_{X/Y})$$

   are given by multiplication by $[p]$.

Proof. We may and shall assume without loss of generality that $X = \overline{X}$, that $Y = \operatorname{Spf} B$ (or $\operatorname{Spec} B$), and that I is the defining ideal of X in Y. Let B_I be the Rees-algebra $B \oplus I \oplus I^2 \oplus \cdots$ and let $[p]$ be the element of degree one defined by p. Statement (1) asserts that $\mathbb{D}_X(Y)$ is the formal spectrum of the p-adic completion of the degree zero part of the localization of B_I by $[p]$. This is proved in [22] and [28]. Alternatively, one can take the completion of the quotient of B_I by the ideal generated by $[p] - 1$, or of the direct limit of the system $B \xrightarrow{[p]} I \xrightarrow{[p]} I^2 \xrightarrow{[p]} \cdots$, with the natural B-algebra structure.

For yet another construction, suppose that $(p, x_1, \ldots, x_r)$ is a set of generators for I and let $B' := B[y_1, \ldots, y_r]/(py_1 - x_1, \ldots, py_r - x_r)$, modulo the closure of its p-torsion. The p-adic completion $B'^{\hat{}}$ of B' is again p-torsion free (see Lemma 7.15), and $IB'^{\hat{}}$ is the principal ideal p. Thus $B \to B'^{\hat{}}$ defines a p-adic enlargement of X over Y, and it is clearly universal. This construction looks more down-to-earth, but computing the p-torsion of B' may be difficult.

In the situation of statement (2), let $f \colon X' \to X$ be the morphism induced by g. Then $(\mathbb{D}_{X'}(Y'), g \circ \pi_{Y'}, f \circ z_{Y'})$ is a p-adic enlargement of X over Y, so we find the desired morphism $\mathbb{D}_{X'}(Y') \to \mathbb{D}_X(Y)$ by the universal property of $\mathbb{D}_X(Y)$. If $X' = g^{-1}(X)$, then the projection map

$$(\mathbb{D}_X(Y) \times_Y Y')_1 = \mathbb{D}_X(Y)_1 \times_{Y_1} Y'_1 \to Y'_1$$

factors through a map $z_{Y'} \colon (\mathbb{D}_X(Y) \times_Y Y')_1 \to X'$, and if g is flat, then $\mathbb{D}_X(Y) \times_Y Y'$ is again p-torsion free, by statement (1) of Proposition 7.14. Then $(\mathbb{D}_X(Y) \times_Y Y', \pi_{Y'}, z_{Y'})$ defines a p-adic enlargement of X' over Y', and so there is a map $\mathbb{D}_X(Y) \times_Y Y' \to \mathbb{D}_{X'}(Y')$, inverse to the map map $\mathbb{D}_{X'}(Y') \to \mathbb{D}_X(Y) \times_Y Y'$ coming from functoriality.

Now suppose that $X \to Y_1$ is a very regular immersion. By Proposition 7.24, the map $X \to Y$ is also a very regular immersion, and then Proposition 7.26 implies that the natural map $S^{\cdot} I \to B_I$ from the symmetric algebra of I to the Rees algebra is an isomorphism. Thus if B' is any B-algebra, to give a B-algebra homomorphism $B_I \to B'$ sending $[p] \in I \subseteq \oplus_n I^n$ to 1 is equivalent to giving a B-algebra homomorphism $S^{\cdot} I \to B'$ sending $[p] \in I \subseteq \oplus_n S^n I$ to 1, which in turn is equivalent to giving a B-module homomorphism $I \to B'$ sending p to 1. This proves (2a). Furthermore, I/I^2 is locally free over B/I and the maps $B/I \otimes S^{\cdot} I \to S^{\cdot} I/I^2 \to \oplus_n I^n/I^{n+1}$ are isomorphisms. Statement (2a) implies that, if B' is a B/I-algebra, a B'-valued point of $\mathbb{D}_X(Y)$ amounts to a homomorphism $\rho \colon I/I^2 \to B'$ sending the class of p to 1. We have an exact sequence:

$$0 \to B/I \xrightarrow{[p]} I/I^2 \longrightarrow \overline{I}/\overline{I}^2 \to 0,$$

where $\overline{I} := I/p \cap I$ is the ideal of X in $\overline{Y}$. Thus the set of such ρ's identifies with the set of splittings of this sequence, which is naturally a torsor under $\operatorname{Hom}(\overline{I}/\overline{I}^2, B/I)$, which in turn identifies with the spectrum of $S^{\cdot} N_{X/\overline{Y}}$. It follows that $\overline{\mathbb{D}}_X(Y)$ is faithfully flat over X. Furthermore, we have seen that $\mathbb{D}_X(Y)$ is the completion of the formal spectrum of the direct limit of $B \xrightarrow{[p]} I \xrightarrow{[p]} I^2 \xrightarrow{[p]} \cdots$, and it follows that $\overline{\mathbb{D}}_X(Y)$ is the spectrum of the limit of $B/I \xrightarrow{[p]} I/I^2 \xrightarrow{[p]} I^2/I^3 \xrightarrow{[p]} \cdots$. In the case of a regular immersion each I^n/I^{n+1} is isomorphic to $S^n I/I^2$. This proves (3). $\square$

Example 2.8. Let A be a p-torsion free and p-adically complete ring, let B be the p-adic completion of the polynomial algebra $A[x_1, \ldots, x_r]$, and let I be the ideal of B generated by $(p, x_1, \ldots, x_r)$. Then the p-adic dilatation of $\operatorname{Spf} B/I$ in $\operatorname{Spf} B$ is given by the formal spectrum of the p-adic completion of

$$B' := B[t_1, \ldots, t_r]/(x_1 - pt_1, \ldots, x_r - pt_r)$$

Indeed, it is clear that this ring is already p-torsion free, that $IB' = pB'$, and that it is universal with this property. The following result generalizes this fact.

Proposition 2.9. *Suppose that B is a p-torsion free p-adically complete ring and that I is the ideal of B generated by a finite regular sequence $(p, x_1, \ldots, x_r)$. Then the p-adic dilatation of B/I in B is given by the homomorphism $B \to B'$ where B' is the p-adic completion of $B[t_1, \ldots, t_r]/(pt_1 - x_1, \ldots, pt_r - x_r)$.*

Proof. If B is noetherian, this follows from the Example 2.8, because the map from the completed polynomial algebra $W[X_1, \ldots, X_r]\hat{}$ to B sending X_i to x_i is flat, and statement (2) of Theorem 2.7 tells us that formation of p-adic dilatations commutes with flat base change. To avoid the noetherian hypothesis, we can argue as follows. Thanks to Lemma 2.11 below, we can use induction on r to reduce to the case in which I is generated by a regular sequence (p, x). Our claim is simply that $B[t]/(pt - x)\hat{}$ is p-torsion free. This follows from the following lemma.

Lemma 2.10. *Let A be a p-torsion free ring, let (p, a) be an A-regular sequence, and let $A[t]\hat{}$ be the completed polynomial algebra in t over A. Then the ring $A[t]\hat{}/(pt - a)$ is p-torsion free.*

Proof. It is clear that $A[t]\hat{}$ is p-torsion free (see Lemma 7.15). Furthermore, its reduction modulo p is the polynomial algebra $(A/pA)[t]$, which is flat over A/pA. Since multiplication by a is injective on A/pA, it is also injective on $(A/pA)[t]$. Thus the sequence (p, a) is $A[t]\hat{}$-regular. Now if f and g are elements of $A[t]\hat{}$ and $pg = f(pt - a)$, it follows that $fa = p(ft - g)$, and the $A[t]\hat{}$-regularity of (p, a) implies that $f = p\tilde{f}$ for some $\tilde{f} \in A[t]\hat{}$. Then $g = \tilde{f}(pt - a)$. $\square$

$\square$

Lemma 2.11. *Let Y be p-torsion free p-adic formal scheme, let $X \subseteq X' \subseteq Y_1$ be closed immersions, and $\tilde{X} := \pi_Y^{-1}(X) \subseteq \mathbb{D}_{X'}(Y)$. Then the natural map*

$$\mathbb{D}_{\tilde{X}}(\mathbb{D}_{X'}(Y)) \to \mathbb{D}_X(Y)$$

is an isomorphism.

Proof. The map of pairs $(\tilde{X} \subseteq \mathbb{D}_{X'}(Y)) \to (X \subseteq Y)$ induces the "natural map" in the statement. On the other hand, the map of pairs $(X \subseteq Y) \to (X' \subseteq Y)$ induces a morphism $\mathbb{D}_X(Y) \to \mathbb{D}_{X'}(Y)$. Since $\mathbb{D}_X(Y)_1$ is the inverse image of

X, this map sends $\mathbb{D}_X(Y)_1$ to $\tilde{X}$, hence defines an $\tilde{X}$-dilatation over $\mathbb{D}_{X'}(Y)$, hence factors uniquely through a map $\mathbb{D}_X(Z) \to \mathbb{D}_{\tilde{X}}(\mathbb{D}_Y(X))$. The two maps are inverses to each other because of the uniqueness of the factorizations. $\square$

Proposition 2.12. *Let $j\colon Y \to Z$ be a closed immersion of p-torsion free p-adic formal schemes.*

1. *The map j factors uniquely through the p-adic dilatation of Y in Z:*

$$\begin{tikzcd} & \mathbb{D}_Y(Z) \arrow[d, "\pi_{\mathbb{D}}"] \\ Y \arrow[ur, "\tilde{j}"] \arrow[r, "j"] & Z. \end{tikzcd}$$

2. *Let $J_{Y/Z}$ be the ideal of the closed immersion j and let $J_{Y/D}$ be the ideal of the closed immersion $\tilde{j}$. Assume that $J_{Y/Z}$ is locally finitely generated. Then the ideal $J_{Y/D}$ is generated by the set of sections c of $\mathcal{O}_{\mathbb{D}_Y(Z)}$ such that $pc \in J_{Y/Z}$.*

3. *Suppose in addition that $j_1\colon Y_1 \to Z_1$ is a regular immersion. Then j, $\tilde{j}$, and $\tilde{j}_1$ are also regular immersions, and the map ρ (see 2.1) induces an isomorphism:*
$$J_{Y/Z}/J^2_{Y/Z} \to J_{Y/D}/J^2_{Y/D}.$$

Proof. Since Y is p-torsion free, it is a p-adic enlargement of itself, so the existence and uniqueness of $\tilde{j}$ follows from the universal property of $\mathbb{D}_Y(Z)$. Let us write the rest of the proof in the affine case, with $Y = \operatorname{Spf} B$, $Z = \operatorname{Spf} C$, and $\pi_{\mathbb{D}}$ given by $\operatorname{Spf}(\theta\colon C \to C')$. If $x \in J_{Y/Z}$, then $p\rho(x) = \theta(x)$, hence

$$p\tilde{j}^\sharp(\rho(x)) = \tilde{j}^\sharp\theta(x) = j^\sharp(x) = 0.$$

Since B is p-torsion free, it follows that $\tilde{j}^\sharp(\rho(x)) = 0$, hence that $\rho(x) \in J_{Y/D}$. This shows that ρ factors through $J_{Y/D}$. Choose a set of generators $x_1, \ldots, x_n$ for $J_{Y/Z}$. As we saw in the proof of Theorem 2.7, the algebra C' is the p-adic completion of the p-torsion free quotient of $C[y_1, \ldots, y_n]/(py_1 - x_1, \ldots, py_n - x_n)$. In particular, $y_i = \rho(x_i)$, and these elements topologically generate C'. Thus every $c' \in C'$ can be written, in multi-index notation, as a p-adically convergent sum $c' = \sum \theta(c_I)\rho(x)^I$, with $c_I \in C$. If $c' \in J_{Y/D}$, then $\tilde{j}^*(c') = 0$, and hence $j^*(c_0) = \tilde{j}^*\theta(c_0) = 0$, hence $c_0 \in J_{Y/Z}$. Then $c' = p\rho(c_0) + \sum_{I>0} \theta(c_I)\rho(x)^I$ belongs to the image of ρ. This completes the proof of statement (2).

Now suppose that j_1 is a regular immersion. Since B is p-torsion free, the ideal defining j_1 is $J_{Y/Z}/pJ_{Y/Z}$. Choose a C_1-regular sequence generating this ideal, and lift it to a sequence $(x_1, \ldots, x_n)$ of elements of $J_{Y/Z}$. Since $J_{Y/Z}$ is closed in C, it is p-adically separated and complete, and it follows that this sequence also generates $J_{Y/Z}$. By construction, the sequence $(p, x_1, \ldots, x_n)$ is

C-regular. By Lemma 7.25, it follows that $(x_1, p, x_2, \dots, x_n)$ is also C-regular and that C/x_1C is again p-torsion free and p-adically separated and complete. Repeating the argument with the ideal of C/x_1C generated by $(x_2, \dots, x_n)$ and continuing by induction, we conclude that $(x_1, \dots, x_n)$ is C-regular. Thus j is a regular immersion. Furthermore, Proposition 2.9 shows that C' identifies with $C[y_1, \dots, y_n]\hat{}/(py_1 - x_1, \dots, py_n - x_n)$. Thus C'_1 identifies with the polynomial algebra $B_1[y_1, \dots, y_n]$, in which the sequence $(y_1, \dots, y_n)$ is very regular. Then by Proposition 7.24, we can conclude that $(p, y_1, \dots, y_n)$ is a very regular sequence in C', and in particular that $(y_1, \dots, y_n)$ is C'-regular and C'_1-regular. It follows that the image of this sequence forms a basis for $J_{Y/D}/J^2_{Y/D}$. Since $(x_1, \dots, x_n)$ gives a basis for $J_{Y/Z}/J^2_{Y/Z}$ and $\rho(x_i) = y_i$ for all i, statement (3) is proved. □

Statement (2) of Theorem 2.7 shows that formation of p-adic dilatations commutes with flat base change. The flatness hypothesis is used to control possible p-torsion in the fiber product. The following generalization, in the case of regular immersions, will be important in applications.

Proposition 2.13. *Let $g\colon Y \to Z$ be a morphism of p-torsion free formal schemes and $i\colon X \to Z_1$ a regular closed immersion.*

1. *Suppose that the ideal of X in Z_1 is locally defined by a regular sequence which remains regular in $\mathcal{O}_{Y_1}$, and let $X' := g^{-1}(X) \subseteq Y_1$. Then the natural map*
$$\mathbb{D}_{X'}(Y) \to \mathbb{D}_X(Z) \times_Z Y$$
is an isomorphism.

2. *Suppose that $i\colon X \to Z_1$ factors as the composite of regular immersions: $X \to Y_1$ and $g_1\colon Y_1 \to Z_1$. Let $\tilde{Y} \to \mathbb{D}_Y(Z)$ be the canonical section defined by g. Then the natural map*
$$\mathbb{D}_X(Y) \to \mathbb{D}_X(Z) \times_{\mathbb{D}_Y(Z)} \tilde{Y}$$
is an isomorphism,

Proof. We can check these assertions locally, and hence we may and shall assume that $j\colon Y \to Z = \operatorname{Spf}(C \to B)$ and that $X \subseteq Z_1$ is defined by a C_1-regular sequence $(x_1, \dots, x_r)$. Our hypothesis in statement (1) asserts that this sequence is also B_1-regular. Then as we saw in Proposition 2.9,

$$\begin{aligned} \mathbb{D}_X(Z) &= \operatorname{Spf} C[t_1, \dots, t_r]\hat{}/(pt_1 - x_1, \dots, pt_r - x_r) \text{ and} \\ \mathbb{D}_{X'}(Y) &= \operatorname{Spf} B[t_1, \dots, t_r]\hat{}/(pt_1 - x'_1, \dots, pt_r - x'_r), \end{aligned}$$

where x'_i is the image of x_i in B. This proves statement (1).

In the situation of statement (2), we first find a sequence $(x_1, \dots, x_r)$ of elements of C which generates the ideal J of Y_1 in Z_1 and is C_1-regular. Since $X \to Y_1$ is also a regular immersion, we may then find a B_1-regular sequence $(y_1, \dots, y_m)$ of elements of B which generates the ideal of X in Y_1. Then $\tilde{Z} :=$

$\mathbb{D}_Y(Z) = \operatorname{Spf} C'$, where $C' := C[t_1, \ldots, t_r]\hat{}/(pt_1 - x_1, \ldots pt_i - x_r)$, and $\tilde{Y}$ is defined by the ideal $(t_1, \ldots, t_r)$. Let $\tilde{X} := X \times_{Y_1} \mathbb{D}_Y(Z) \subseteq \mathbb{D}_Y(Z)_1$. The ideal of $\tilde{X}$ in $\mathbb{D}_Y(Z)$ is generated by the sequence $(p, x_1, \ldots, x_r, y_1, \ldots, y_m)$, or in fact just by $(p, y_1, \ldots, y_m)$ since $x_i \in pC'$. Since C'_1 is just a polynomial algebra over B_1, it follows that $(y_1, \ldots, y_r)$ is also C'_1-regular, so the map map $\tilde{X} \to \tilde{Z}_1$ is a regular immersion. The map $\tilde{X} \cap \tilde{Y}_1 \to \tilde{Y}_1$ identifies with the map $X \to Y_1$, and hence is also a regular immersion. Then statement (1), applied to the morphisms $\tilde{j}\colon \tilde{Y} \to \tilde{Z}$ and $\tilde{i}\colon \tilde{X} \to \tilde{Z}$, implies that the map

$$\mathbb{D}_{\tilde{X}\cap\tilde{Y}}(\tilde{Y}) \to \mathbb{D}_{\tilde{X}}(\tilde{Z}) \times_{\tilde{Z}} \tilde{Y}$$

is an isomorphism. Lemma 2.11 tells us that the map $\mathbb{D}_{\tilde{X}}(\tilde{Z}) \to \mathbb{D}_X(Z)$ is an isomorphism, and since the map of pairs $(\tilde{X} \cap \tilde{Y} \subseteq \tilde{Y}) \to (X \subseteq Y)$ is an isomorphism, so is the map $\mathbb{D}_{\tilde{X}\cap\tilde{Y}}(\tilde{Y}) \to \mathbb{D}_X(Y)$. This concludes the proof of statement (2). □

Corollary 2.14. *Suppose that X and X' are two regularly immersed subschemes of Y_1 which meet transversally. Then the natural map*

$$\mathbb{D}_{X\cap X'}(Y) \to \mathbb{D}_X(Y) \times_Y \mathbb{D}_{X'}(Y)$$

is an isomorphism.

Proof. We may assume that $Y = \operatorname{Spf} B$ and that X (resp. X') is defined by a B/pB-regular sequence $(x_1, \ldots, x_r)$ (resp. $(x'_1, \ldots, x'_{r'})$. Then $\mathbb{D}_{X'}(Y) = \operatorname{Spf}(B')$, where

$$B' \quad = \quad B[t'_1, \ldots, t'_{r'}](pt'_1 - x'_1, \ldots, pt'_{r'} - x'_{r'})$$

Since X and X' meet transversally, the sequence $(x_1, \ldots, x_r)$ remains regular in $B/(p, x'_1, \ldots, x'_{r'})$, and hence also in $B'/pB' \cong B/(p, x'_1, \ldots, x'_{r'})[t'_1, \ldots, t'_{r'}]$. Note that $\tilde{X} := X \times_Y \mathbb{D}_{X'}(Y) = \pi_Y^{-1}(X \cap X')$. Statement (1) of Proposition 2.13, applied to the map $\mathbb{D}_{X'}(Y) \to Y$, implies that

$$\mathbb{D}_{\tilde{X}}(\mathbb{D}_{X'}(Y)) \cong \mathbb{D}_X(Y) \times_Y \mathbb{D}_{\tilde{X}}(Y),$$

and Lemma 2.11 implies that $\mathbb{D}_{\tilde{X}}(\mathbb{D}_{X'}(Y)) \cong \mathbb{D}_{X\cap X'}(Y)$. □

Example 2.15. Let $i\colon X \to Y$ be the closed immersion corresponding to the map $W[x]\hat{} \to k$. Then $\mathbb{D}_X(Y) \to Y$ corresponds to the homomorphism $W[x]\hat{} \to W[t]\hat{}$ sending x to pt, and $\mathbb{D}_X(Y) \times_Y \mathbb{D}_X(Y) \cong \operatorname{Spf} W[t_1, t_2]\hat{}/(pt_1 - pt_2)$. Thus $t_2 - t_1$ is a p-torsion element, and in fact the diagonal morphism $\mathbb{D}_X(Y) \to (\mathbb{D}_X(Y) \times_Y \mathbb{D}_X(Y))_{\mathrm{tf}}$ is an isomorphism.

Remark 2.16. Let $i\colon X \to Y$ be a closed immersion of a scheme into a p-torsion free p-adic formal -scheme. Then we have seen that $\mathbb{D}_X(Y)$ can be empty. However, this cannot happen if Y is a formal ϕ-scheme. To see this, recall from [22, 2.6.1] that, for $m \gg 0$, $\mathbb{D}_{X_m}(Y)$ is not empty, where X_m is the closed subscheme of Y_1 defined by $I_X^m + p\mathcal{O}_Y$. The map ϕ^m maps X_m to X, and hence induces a morphism $\mathbb{D}_{X_m}(Y) \to \mathbb{D}_X(Y)$. Since the first of these is not empty, neither is the second. The main idea of [22] was to consider p-adic enlargements of all X_m as enlargements of X. It appears that if Y is a ϕ-scheme, this is unnecessary.

2.3 Prisms and prismatic envelopes

We now shift our attention to prisms and prismatic envelopes, as introduced in [7]. We restrict out attention to the special case of crystalline prisms, *i.e.*, the case in which the invertible ideal I in Definition 3.2 of [7] is generated by p. We fix a formal ϕ-scheme (1.1) S as base, and recall that, by definition, such a scheme is p-torsion free, so (p) is invertible.

Definition 2.17. *Let S be a formal ϕ-scheme and let X/S be a formal scheme over S (typically over $\overline{S}$). An X-prism is a formal ϕ-scheme endowed with a ϕ-morphism $T \to S$ and an S-morphism $z_T \colon \overline{T} \to X$. An X-prism is small if z_T is flat. A morphism $(T, z_T) \to (T', z_{T'})$ of X-prisms is a morphism of ϕ-schemes $f \colon T \to T'$ such that $z_{T'} \circ f_1 = z_T$. If $i \colon X \to Y$ is a closed immersion from X into a p-torsion free formal scheme endowed with a an endomorphism ψ, then an X-prism over Y is an X-prism (T, z_T) together with a morphism of formal schemes $\pi_T \colon T \to Y$ such that $i \circ z_T = \pi_{T|_{\overline{T}}}$ and $\psi \circ \pi_T = \pi_T \circ \phi$:*

$$\begin{tikzcd}
\overline{T} \arrow[r] \arrow[d, "z_T"'] & T \arrow[r, "\phi"] \arrow[d, "\pi_T"] & T \arrow[d, "\pi_T"] \\
X \arrow[r, "i"] & Y \arrow[r, "\psi"] & Y
\end{tikzcd}$$

If (T, z_T) is an X-prism and $g \colon T' \to T$ is a morphism of formal ϕ-schemes, then g induces a morphism $\overline{g} \colon \overline{T}' \to \overline{T}$, and $(T', \overline{g} \circ z_T)$ is also an X-prism. If $f \colon X' \to X$ is a morphism of S-schemes and (T, z_T) is an X'-prism, then $(T, f \circ z_T)$ is an X-prism. In particular, the map $\overline{X} \to X$ induces an isomorphism on the respective categories of prisms.

Definition 2.18. *Let $i \colon X \to Y$ be a closed immersion from an $\mathbf{F}_p$-scheme into a p-torsion free formal scheme Y endowed with an endomorphism ψ. The prismatic envelope of X in Y, denoted by $\mathbb{\Delta}_X(Y)$, is the final object of the category of X-prisms over Y.*

The existence of prismatic envelopes, when Y is a ϕ-scheme and $X \to \overline{Y}$ is a regular immersion, is proved in [7, 3.13], using the formalism of δ-rings. Our construction is more general, in that we do not require that ψ be a Frobenius lift, and more explicit, expressing the prismatic envelope as the limit of a sequence of dilatations. For an example of an application of this extra generality, see Proposition 2.25; the extra generality is also needed because of the inductive structure of the proof On the other hand, the construction in [7] works for more general prisms (not necessarily crystalline).

Theorem 2.19. *Suppose that Y is p-torsion free p-adic formal scheme with an endomorphism ψ and that $i\colon X \to Y$ is a closed immersion such that $\psi \circ i = i \circ F_X$. Then the prismatic envelope of X in Y is represented by an X-prism $(\mathbb{\Delta}_X(Y), z_{\mathbb{\Delta}})$ endowed with a map $\pi_{\mathbb{\Delta}}\colon \mathbb{\Delta}_X(Y) \to Y$. The morphisms π_Y and $z_{\mathbb{\Delta}}$ are affine, but not necessarily of finite type. The morphism $\pi_{\mathbb{\Delta}}$ can be viewed as a completion of a limit of p-adic dilatations. Specifically, there exists a sequence of morphisms of closed immersions:*

$$(X^{(n+1)} \subseteq Y^{(n+1)}) \longrightarrow (X^{(n)} \subseteq Y^{(n)}) \longrightarrow \cdots (X \subseteq Y)$$

such that $Y^{(n+1)} \to Y^{(n)}$ is the p-adic dilatation of $X^{(n)}$ in $Y^{(n)}$ and such that the p-adic completion of $\varprojlim Y^{(n)}$ is the prismatic envelope $\mathbb{\Delta}_X(Y)$ of X in Y. Furthermore, the following statements are verified.

1. *Formation of $\mathbb{\Delta}_X(Y)$ is functorial: a morphism $g\colon Y' \to Y$ sending $X' \subseteq Y'$ to $X \subseteq Y$ induces a morphism $\mathbb{\Delta}_{X'}(Y') \to \mathbb{\Delta}_X(Y)$. If g is p-completely flat and $X' = g^{-1}(X)$, then the resulting morphism $\mathbb{\Delta}_{X'}(Y') \to \mathbb{\Delta}_X(Y) \times_Y Y'$ is an isomorphism.*

2. *Suppose that $X \to \overline{Y}$ is a regular immersion. Then the same is true of each $X^{(n)} \to \overline{Y}^{(n)}$, and $\overline{\mathbb{\Delta}}_X(Y) \to X$ is faithfully flat. In particular, $\mathbb{\Delta}_X(Y)$ is small.*

Proof. The main computations are in the following lemma, which we formulate in the affine setting.

Lemma 2.20. *Let I be an ideal in a p-torsion free ring B. Assume that $p \in I$ and that B is equipped with an endomorphism ψ such that $\psi(b) \equiv b^p \pmod{I}$, for all $b \in B$; note that ψ necessarily maps I to I. Let $\theta\colon B \to B'$ be the homomorphism corresponding to the p-adic dilatation of I in B, let ψ' be the endomorphism of B' induced by ψ, and let $\rho\colon I \to B'$ be the homomorphism defined in (2.1). For $b \in B$, set*

$$\begin{array}{rcl} \delta'(b) & := & \psi(b) - b^p \in I \\ \epsilon(b) & := & \rho(\delta'(b) = p^{-1}(\theta(\psi(b)) - \theta(b)^p) \in B', \end{array}$$

and for $x \in I$, set

$$\epsilon'(x) \quad := \quad \psi'(\rho(x)) - \rho(x)^p \in B'.$$

Then the following statements are verified.

1. *If $x, y \in I$ and $b \in B$, then*

$$\begin{array}{rcl} \epsilon'(x+y) & \equiv & \epsilon'(x) + \epsilon'(y) \pmod{pB'} \\ \epsilon'(bx) & \equiv & \theta(b^p)\epsilon'(x) \pmod{pB'} \end{array}$$

2. *If $x \in I$, then*

$$\begin{array}{rcl} \rho(\psi(x)) & = & \psi'(\rho(x)) \\ \rho(\psi(x)) & \equiv & \epsilon(x) \pmod{pB'} \\ \epsilon'(x) & \equiv & \epsilon(x) - \rho(x)^p \pmod{pB'}. \end{array}$$

3. *Let I' be the ideal of B' generated by $\epsilon'(I)$ and pB'.Then $\psi'(b') \equiv b'^p \pmod{I'}$ for all $b' \in B'$.*

4. *if I/pB is generated by a B/pB-regular sequence, then I'/pB' is generated by a B'/pB'-regular sequence and B'/I' is faithfully flat over B/I. Moreover, e' induces an isomorphism*

$$F^*(\overline{I}/\overline{I}^2) \to \overline{I}'/\overline{I}'^2,$$

where $\overline{I}$ (resp. $\overline{I}'$) is the image of I in B/pB (resp., of I' in B'/pB').

Proof. First note that $\epsilon(p) = 1 - p^{p-1}$ and that $\epsilon'(p) = 0$. The first equation in (1) follows from the definition and the additivity of ρ. If $x \in I$, we have:

$$\begin{aligned}
\epsilon'(bx) &= \psi'(\rho(bx)) - (\rho(bx))^p \\
&= \psi'(\theta(b)\rho(x)) - (\theta(b)\rho(x))^p \\
&= \psi'(\theta(b))\psi'(\rho(x)) - (\theta(b))^p(\rho(x))^p \\
&= \psi'(\theta(b))\psi'(\rho(x)) - \psi'(\theta((b))(\rho(x))^p + \psi'(\theta(b)(\rho(x))^p - (\theta(b))^p(\rho(x))^p \\
&= \theta(\psi(b))\epsilon'(x) + \theta(\delta'(b))(\rho(x))^p \\
&= \theta(b^p + \delta'(b))\epsilon'(x) + \theta(\delta'(b))(\rho(x))^p \\
&= \theta(b^p)\epsilon'(x) + \theta(\delta'(b))\epsilon'(x) + \theta(\delta'(b))(\rho(x))^p
\end{aligned}$$

Since $\delta'(b) \in I$ and $\theta(I) \subseteq pB'$, this proves the second equation in (1).

If $x \in I$, we have

$$p\rho(\psi(x)) = \theta(\psi(x)) = \psi'(\theta(x)) = \psi'(p\rho(x)) = p\psi'(\rho(x)),$$

and so $\rho(\psi(x)) = \psi'(\rho(x))$. Moreover, since x and $\psi(x)$ belong to I, we have:

$$\begin{aligned}
\theta(\psi(x)) &= \theta(x)^p + p\epsilon(x) \\
p\rho(\psi(x)) &= (p\rho(x))^p + p\epsilon(x) \\
\rho(\psi(x)) &= p^{p-1}\rho(x))^p + \epsilon(x) \\
\rho(\psi(x)) &\equiv \epsilon(x) \pmod{pB'}
\end{aligned}$$

Since $\epsilon'(x) = \psi'(\rho(x)) - \rho(x)^p$, this completes the proof of statement (2).

To prove (3), first suppose that $b' = \theta(b)$ with $b \in B$. Since $\psi(b) - b^p \in I$, it follows that

$$\psi'(b') - b'^p = \theta(\psi(b) - b^p) \in \theta(I) \subseteq pB' \subseteq I'.$$

On the other hand, if $x \in I$, then $\psi'(\rho(x)) - \rho(x)^p = \epsilon'(x) \in I'$, and hence $\psi'(\rho(x)) \equiv \rho(x)^p \pmod{I'}$. Since the B-algebra B' is topologically generated by such elements $\rho(x)$, the claim holds for all $b' \in B'$.

Statement (1) implies that the ideal I' is generated by p and the set of elements $\epsilon'(x)$ as x ranges over any set of generators for I. Suppose that I is generated by a B-regular sequence $(p, x_1, \ldots, x_r)$. Then I' is generated by the sequence $(p, \epsilon'(x_1), \ldots, \epsilon'(x_r))$. We claim that this sequence is B'-regular. Indeed, as we saw in Proposition 2.9, B'/pB' is isomorphic to the polynomial algebra

$B/I[y_1, \ldots, y_r]$, where y_i is the reduction modulo pB' of $\rho(x_i)$. Then the image of I' in B'/pB' is generated by the image of the sequence $(\epsilon'(x_1), \ldots \epsilon'(x_r))$. Each $\delta'(x_i)$ belongs to I, and so there is a sequence $b_{i,1}, \ldots, b_{i,r}$ in B such that $\delta'(x_i) = \sum_j b_{i,j} x_j$. Then

$$\begin{aligned} \epsilon(x_i) &= \rho(\delta'(x_i)) \\ &= \sum_j \rho(b_{i,j} x_j) \\ &= \sum_j \theta(b_{i,j})\rho(x_j) \end{aligned}$$

Then, working mod pB', we have

$$\begin{aligned} \epsilon'(x_i) &= \epsilon(x_i) - \rho(x_i)^p \\ &= \sum \theta(b_{i,j}) y_j - y_i^p \end{aligned}$$

This is a monic polynomial whose leading term is $-y_i^p$. One sees easily by induction that the sequence $(\epsilon'(x_1), \ldots, \epsilon'(x_r))$ is B'/pB' regular and that the quotient is faithfully flat over B/IB. Statement (1) shows that ϵ induces a linear map $F^*\overline{I} \to \overline{I}'$, which is surjective by construction. If $(p, x_1, \ldots, x_r)$ is a B-regular sequence then $(x_1, \ldots, x_r)$ induces a basis for $\overline{I}/\overline{I}^2$, and we have just seen that $(\epsilon'(x_1), \ldots, \epsilon'(x_r))$ is a B'/pB'-regular sequence which generates $\overline{I}'$. Hence it induces a basis for $\overline{I}'/\overline{I}'^2$, concluding the proof of statement (4) $\square$

Let us prove the existence of prismatic envelopes using affine notation, assuming that $Y = \operatorname{Spf} B$ and $X = \operatorname{Spec} B/I$. Using the lemma, we inductively find a sequence:

$$(B, I, \epsilon^{(1)}) \to (B', I', \epsilon^{(2)}) \to (B'', I'', \epsilon^{(3)}) \to \cdots \to (B^{(n)}, I^{(n)}, \epsilon^{(n+1)}) \to \cdots,$$

where $B^{(n)} \to B^{(n+1)}$ is the p-adic dilatation of $I^{(n)}$, where $\epsilon^{(n+1)} \colon I^{(n+1)} \to B^{(n+1)}$ is a function, and where $I^{(n+1)}$ is the ideal of $B^{(n+1)}$ generated by p and $\epsilon^{(n+1)}(I^{(n)})$. Each $B^{(n)}$ inherits an endomorphism $\psi^{(n)}$, and these endomorphisms are compatible with the transition maps $B^{(n)} \to B^{(n+1)}$. Furthermore, if $b \in B^{(n)}$, we have $\psi^{(n)}(b) \equiv b^p \pmod{I^{(n)}}$ and hence the image $\psi^{(n)}(b) - b^p$ in $B^{(n+1)}$ is divisible by p. Let $(\mathcal{B}, \psi) := \varinjlim (B^{(n)}, \psi^{(n)})$. Then $\mathcal{B}/p\mathcal{B} = \varinjlim B^{(n)}/pB^{(n)}$, which inherits a B/pB-algebra structure. Since $\mathcal{B}$ is p-torsion free, its p-adic completion $\mathcal{B}\hat{}$ is also (see Lemma 7.15), and so defines a p-adic enlargement of X in Y. If $b \in \mathcal{B}$ comes from an element b_n of $B^{(n)}$, then $\psi^{(n)}(b_n) \equiv b_n^p \pmod{pB^{(n+1)}}$ and hence $\psi(b) \equiv b^p \pmod{p\mathcal{B}}$. It follows that this congruence also holds on all of $\mathcal{B}\hat{}$. Thus $\operatorname{Spf}(\mathcal{B}\hat{})$ defines an X-prism Y' over Y. If (T, π_T, z_T) is another, we claim that π_T factors uniquely through Y'. We may check this locally on T, and so may assume that T is affine, say $T = \operatorname{Spf} C$. Since T is a p-adic enlargement of X, the map $B \to C$ factors through $B' = B^{(1)}$; necessarily the endomorphisms ϕ of C and $\psi^{(1)}$ of $B^{(1)}$ are compatible. If $c \in C$ is any element of C, necessarily $\phi(c) - c^p \in pC$. This applies if c is the image of any element b' of B' and in particular if $b' = \rho(x)$ for some $x \in I$, we see that $\epsilon'(x)$ becomes divisible by p in C. Thus the homomorphism $B^{(1)} \to C$ factors through $B^{(2)}$. Continuing in this way, we find a homomorphism $\mathcal{B} \to C$ and then $\mathcal{B}\hat{} \to C$. Each step is unique, and hence so is this homomorphism. We conclude that Y' is indeed a final X-prism over Y.

The proofs of functoriality of prismatic envelopes and their compatibility with base change is the same as that for p-adic dilatations; of course they also can be found in [7].

If $X \to \overline{Y}$ is a regular immersion, we have seen that the same is true of each of the closed immersions defining each of the successive dilatations, and it follows from Theorem 2.7 that each map $B^{(n)}/I^{(n)} \to B^{(n+1)}/I^{(n+1)}$ is faithfully flat. Then the same is true for the map

$$B/I \to \varinjlim B^{(n)}/I^{(n)} = \mathcal{B}/p\mathcal{B} = \mathcal{B}\hat{}/p\mathcal{B}\hat{}.$$

□

The following result gives a more explicit description of prismatic envelopes in the case of a regular immersion into a formal ϕ-scheme.

Proposition 2.21. *Let $Y = \operatorname{Spf} B$ be an affine ϕ-scheme, with $\phi(b) = b^p + p\delta(b)$ for $b \in B$, and let X be the the closed subscheme of $\overline{Y}$ defined by a B/pB-regular sequence $(x_1, \ldots, x_r)$ of elements of B. Then the prismatic envelope of X in Y is the formal spectrum of the p-adic completion of the ring B^∞, described as follows. Let $\{t_{i,j} : 1 \leq i \leq r, j \in \mathbf{N}\}$ be free variables for $j > 0$, let $t_{i,0} := 0$, and set*

$$B^\infty := B[t_{i,j}]/(pt_{i,j+1} - \delta^j(x_i) + t_{i,j}^p) : i = 1, \ldots, r, j \in \mathbf{N}.$$

Proof. Let I be the ideal of B generated by $(p, x_1, \ldots, x_r)$. We define a sequence of rings $B = B^{(0)} \subseteq B^{(1)} \subseteq B^{(2)}$, where $B^{(j)}$ is the subring of B^∞ generated by $\{t_{i,j} : j \leq r\}$. Then

$$B^{(j+1)} := B^{(j)}[t_{1,j+1}, \ldots, t_{r,j+1}]/(pt_{i,j+1} - \delta^j(x_i) + t_{i,j}^p : i = 1, \ldots, r),$$

and $B^\infty = \varinjlim\{B^{(j)} : j \in \mathbf{N}\}$. Let $I^{(j)}$ be the ideal of $B^{(j)}$ generated by p and $\{\delta^j(x_i) - t_{i,j}^p : i = 1, \ldots, r\}$. This sequence of rings and ideals corresponds to the sequence of (uncompleted) p-adic dilatations used to construct the prismatic envelope as in the proof of Theorem 2.19. However, we write the proof here so that it can be read independently. We shall prove the following statements by induction on j.

1. For each $j > 0$, $B^{(j)}$ corresponds to the (uncompleted) p-adic dilatation of the ideal $I^{(j-1)}$ in $B^{(j-1)}$. In particular, each of these rings is p-torsion free.

2. For each $j > 0$, the ring $B^{(j)}$ admits a unique endomorphism $\psi^{(j)}$ compatible with the endomorphism ϕ of B, and

$$\psi^{(j)}(t_{i,j}) \equiv \delta^j(x_i) \pmod{pB^{(j)}}.$$

3. $\psi^{(j)}(b) \equiv b^p \pmod{I^{(j)}}$ for all $b \in B^{(j)}$.

It follows from the definition that $B^{(0)} = B$. Since $(x_1, \ldots, x_r)$ is B/pB-regular, Proposition 2.9 shows that

$$B^{(1)} := B[t_{1,1} \ldots, t_{r,1}]/(pt_{1,1} - x_1, \ldots, pt_{r,1} - x_r),$$

is the (uncompleted) p-adic dilatation of I, so statement (1) holds when $j = 1$. Moreover, $B^{(1)}/pB^{(1)} \cong B/I[t_{1,1}, \ldots, t_{1,r}]$, and so the sequence $(p, t_{1,1}^p - \delta(x_1), \ldots t_{r,1}^p - \delta(x_r))$ is $B^{(1)}$-regular. This implies that $B^{(2)}$ is the uncompleted p-adic dilation of $I^{(1)}$ in $B^{(1)}$, and in particular is p-torsion free. This same argument works for all j, proving (1).

To prove (2) when $j = 1$, we use the facts that $B^{(1)}$ is p-torsion free and that $t_{i,1} = p^{-1}x_i$. Calculating in $B \otimes \mathbf{Q}$, we find

$$\begin{array}{rcl} \phi(t_{i,1}) & = & p^{-1}\phi(x_i) \\ & = & p^{-1}(x_i^p + p\delta(x_i)) \\ & = & p^{p-1}t_{i,1}^p + \delta(x_i) \end{array}$$

This shows that ϕ induces an endomorphism $\psi^{(1)}$ of $B^{(1)}$ and that $\psi(t_{i,1}) \equiv \delta(x_i) \pmod{pB^{(1)}}$. Since $\delta(x_i) \equiv t_{i,1}^p \pmod{I^{(1)}}$, this equation implies that (3) holds for each of $t_{1,1}, \ldots, t_{r,1}$, and since it also holds for elements of $B^{(0)}$, it holds for all elements of $B^{(1)}$, completing the proof when $j = 1$.

To proceed by induction, note first that the congruence in (2) for j implies that each $\psi^{(j+1)}(t_{i,j+1})$ is well-defined, hence that $\psi^{(j+1)}$ is well-defined, and also that

$$(\psi^{(j)}(t_{i,j}))^p \equiv (\delta^j(x_i))^p \pmod{p^2B^{(j)}}.$$

Now we compute:

$$\begin{array}{rcl} p\psi^{(j+1}(t_{i,j+1}) & = & \phi(\delta^j(x_i)) - \psi^{(j)}(t_{i,j}^p) \\ & \equiv & \phi(\delta^j(x_i)) - (\delta^j(x_i))^p \pmod{p^2B^{(j+1)}} \\ & \equiv & (\delta^j(x_i))^p + p\delta^{j+1}(x_i) - (\delta^j(x_i))^p \pmod{p^2B^{(j+1)}} \\ & \equiv & p\delta^{j+1}(x_i) \pmod{p^2B^{(j+1)}} \end{array}$$

This implies that $\phi^{(j+1)}(t_{i,j+1}) \equiv \delta^{j+1}(x_i) \pmod{pB^{(j+1)}}$, as claimed in (2). Statement (3) follows as before.

One concludes easily that the endomorphism ψ of B^∞ is a Frobenius lift and that the p-adic completion of B^∞ satisfies the universal property of the prismatic envelope. $\square$

The description of prismatic envelopes in explained in Proposition 2.21 depends on certain choices of a generating set for the ideals $I^{(n)}$. The formulas there look quite explicit, and perhaps appealing, but in some cases the computation of the iterates δ^n can be daunting, and other choices may be more convenient. Here is an example, which arose from an attempt to compare prismatic envelopes with differing Frobenius liftings.

Example 2.22. Write $Y := W[x]^\wedge$ with $\phi(x) = x^p$ and $Y' := W[y]^\wedge$ with $\phi(y) = y^p + p\delta(y)$ for some $\delta(y)$. We have maps of formal ϕ-schemes

$$Y \leftarrow Y \times Y' \rightarrow Y',$$

and if we identify the underlying schemes of Y_1 and Y_1' as X, we then get maps

$$Y \leftarrow \mathbb{A}_X(Y \times Y') \rightarrow Y.'$$

Here $Y \times Y' = \operatorname{Spf} W[x, y]^\wedge \cong \operatorname{Spf} W[x, \xi]^\wedge$, where $y = x + \xi$, and X is defined by (p, ξ).

Claim 2.23. *With the notation above, the prismatic envelope of X in $Y \times Y'$ is the p-adic completion of the ring*

$$W[x, \eta_1, \eta_2, \eta_3, \cdots]/(p\eta_1 - y, p\eta_2 - \delta(y) + \eta_1^p, p\eta_3 - \delta^2(y) + \eta_2^p, \cdots)$$

To show this, note first that $\mathbb{D}_X(Y \times Y')$ is obtained by adjoining η_1, with $p\eta_1 = \xi$. Now we calculate:

$$\begin{aligned}
\phi(\xi) &= \phi(y - x) \\
&= y^p + p\delta(y) - x^p \\
&= (x + \xi)^p - x^p + p\delta(y) \\
&= \xi \sum_{i=1}^{p-1} \binom{p}{i} x^i \xi^{p-i-1} + \xi^p + p\delta(y),
\end{aligned}$$

Thus,

$$\delta(\xi) = \xi \sum_{i=1}^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i \xi^{p-i-1} + \delta(y),$$

a rather unwieldy expression, and computation of $\delta^n(\xi)$ seems hopeless. However, the fact that the summed term is divisible by ξ means it can be neglected. Indeed:

$$\begin{aligned}
p\phi(\eta_1) &= \phi(\xi) \\
&= \xi p \sum_1^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i \xi^{p-i-1}) + \xi^p + p\delta(y) \\
&= p^2 \eta_1 \sum_1^{p-1} \frac{(p-1)!}{i!(p-i)!} x^i p^{p-i-1} \eta^{p-i-1} + p^p \eta^p + p\delta(y)
\end{aligned}$$

Hence

$$\phi(\eta_1) \equiv \delta(y) \pmod{p}$$

Thus the ideal of the next dilatation is generated by $(p, \eta_1^p - \delta(y))$. Let $p\eta_2 := \delta(y) - \eta_1^p$ and compute:

$$\begin{aligned}
\phi(p\eta_2) &= \phi(\delta(y)) - \phi(\eta_1)^p \\
&\equiv \phi(\delta(y)) - \delta(y)^p \pmod{p^2}
\end{aligned}$$

$$
\begin{aligned}
&\equiv \quad p\delta^2(y) \pmod{p^2}, \\
\phi(\eta_2) &\equiv \quad \delta^2(y) \pmod{p}
\end{aligned}
$$

The remaining calculations proceed in the same way.

Corollary 2.24. *Suppose that $Y \to S$ is a morphism of formal ϕ-schemes, that $X \to \overline{S}$ is flat, and that $X \to \overline{Y}$ is a regular immersion. Then $\mathbb{\Delta}_X(Y) \to S$ is p-completely flat.*

Proof. By construction, $\mathbb{\Delta}_X(Y)$ is p-torsion free, and statement (2) of Theorem 2.19 tells us that $z_{\mathbb{\Delta}} \colon \overline{\mathbb{\Delta}}_X(Y) \to X$ is flat. Then $\overline{\mathbb{\Delta}}_X(Y) \to \overline{S}$ is also flat, and Proposition 7.13 tells us that $\mathbb{\Delta}_X(Y) \to S$ is p-completely flat. □

The construction in the Theorem 1.12 used the fact that ϕ is locally surjective in the p-completely flat topology. This is proved by a universal construction in [7, 2.12]. It is also an easy consequence of Theorem 2.19, as we explain in the following proposition.

Proposition 2.25. *Let T be ϕ-scheme and let $b. := (b_1, \ldots, b_n)$ be a sequence of sections of $\mathcal{O}_T$. Then there exist a p-completely flat morphism of ϕ-schemes $u \colon \widetilde{T} \to T$ and a sequence $\tilde{b}.$ of sections of $\mathcal{O}_{\widetilde{T}}$ such that $u^\sharp(b.) = \phi(\tilde{b}.)$, and which enjoys the following universal property. If $u' \colon T' \to T$ is a morphism of ϕ-schemes and $b'.$ a sequence of sections of $\mathcal{O}_{T'}$ such that $u'^\sharp(b.) = \phi(b'.)$, then there is a unique morphism of ϕ-schemes $v \colon T' \to \widetilde{T}$ such that $u' = u \circ v$ and $v^\sharp(\tilde{b}.) = b'..$ Formation of $\widetilde{T}$ is functorial in $(T, b.)$ and compatible with base change.*

Proof. Suppose for simplicity of notation that $T = \operatorname{Spf} B$. Let C be the p-adic completion of the polynomial algebra $B[x_1, \ldots, x_n]$, extend ϕ to a ϕ_B-linear endomorphism ψ of C by letting $\psi(x_i) := b_i$, and let I be the ideal of C generated by p and $\{x_i^p - b_i : i = 1, \ldots, n\}$. Since $\psi(x_i) = b_i \equiv x_i^p \pmod{I}$, this endomorphism induces the Frobenius endomorphism of C/I. Let $\operatorname{Spec}(C \to \tilde{B})$ be the prismatic envelope of I described in Theorem 2.19. Since $x.^p - b.$ is a C/pC-regular sequence, statement (2) of the theorem tells us that $C/I \to \tilde{B}/p\tilde{B}$ is faithfully flat. Since $B/p \to C/I$ is also faithfully flat, so is the map $B/pB \to \tilde{B}/p\tilde{B}$. Then $u^\sharp \colon B \to \tilde{B}$ is p-completely faithfully flat by Proposition 7.13. Let $\tilde{b}_i$ be the image of x_i in $\tilde{B}$; then $\phi(\tilde{b}_i)$ is the image of $\psi(x.)$, which is indeed $u^\sharp(b_i)$.

To check that the universality of this construction, suppose that $u' \colon T' \to T$ is a morphism of ϕ-schemes and $b'.$ is a sequence of sections of $\mathcal{O}_{T'}$. Then there is a unique T-morphism $T' \to \operatorname{Spf} C$ sending x_i to b'_i. If $\phi(b'_i) = u'^\sharp(b_i)$, then this homomorphism sends $x_i^p - b_i$ to $b_i'^p - u'^\sharp(b_i) = b_i'^p - \phi(b) \in p\mathcal{O}_{T'}$, so T' becomes a $\operatorname{Spec} C/I$-prism over C and the map u' factors uniquely through $\tilde{T}$.

The universality of $\widetilde{T}$ guarantees its functoriality. To see that its formation commutes with base change, suppose that $f \colon T' \to T$ is a morphism of ϕ-schemes and $b'. = f^\sharp(b.)$. Since $\tilde{T} \to T$ is p-completely flat and T' is p-torsion free, it follows from Proposition 7.14 that $\tilde{T} \times_T T'$ is again p-torsion free and hence a ϕ-scheme. We claim that the map $\widetilde{T}' \to T' \times_T \widetilde{T}$ induced by $\tilde{f}$ is an isomorphism. But

$$\phi_{\widetilde{T}'}(\pi^{\sharp}_{\widetilde{T}}(\tilde{b}.)) = \pi^{\sharp}_{\widetilde{T}}(\phi_{\widetilde{T}}(\tilde{b}.)) = \pi^{\sharp}_{\widetilde{T}}(u^{\sharp}(b.)) = f^{\sharp}(b.)) = b'_.,$$

and so the universal property of $\widetilde{T}'$ guarantees the existence of the desired inverse morphism. □

Remark 2.26. The construction in Proposition 2.25 is universal but not especially efficient. For example, if $T = \operatorname{Spf} B$ and $b. = 0.$, then $\tilde{T}$ is the prismatic envelope of $(p, x.^p)$ in $B[x.]^\wedge$, which, as we shall see in Theorem 2.27, is the p-adically completed divided power envelope $B\langle x.\rangle^\wedge$; with $\phi(x.) = 0$. Note, however, that in this case the sequence $0.$ in B satisfies $\phi(0.) = 0.$, so no p-completely flat cover is necessary. The universal property of $\tilde{T}$ gives us a section $T \to \tilde{T}$ of $\tilde{T} \to T$ sending $\tilde{0}.$ to $0.$.

In some cases, prismatic envelopes can be expressed as PD-dilatations. The second of the following two results appears in [7, 2.38] and is the key to its main cohomology comparison theorems.

Theorem 2.27. *Let Y be a formal ϕ-scheme and let $i\colon X \to Y_1$ be a closed immersion.*

1. *Suppose that we are given a p-torsion free lifting $j\colon \tilde{X} \to Y$ of i. Let $\tilde{j}\colon \tilde{X} \to \mathbb{D}_X(Y)$ be the resulting section and let $s_{\tilde{X}}\colon X \to \mathbb{D}_X(Y)$ be its restriction to X. If $\tilde{X}$ is ϕ_2-aligned (see 1.4) in Y, then $\mathbb{\Delta}_X(Y)$ identifies with the PD-dilatation of this section. Moreover, if i is a regular immersion, the same is true of j, $\tilde{j}$, and $s_{\tilde{X}}$.* [3]
2. *Let X^ϕ be the inverse image of X under ϕ. The canonical map $\mathbb{P}_X(Y) \to Y$ induces a map $z\colon \overline{\mathbb{P}}_X(Y) \to X^\phi$, identifying $(\mathbb{P}_X(Y), z, \pi)$ with the prismatic envelope of X^ϕ in Y.*

Proof. The universal property of $\mathbb{D}_X(Y)$ guarantees the existence of $\tilde{j}$, and the statements about regular immersions are explained in Proposition 2.12.

Let $\mathbb{P}_{\tilde{X}}(\mathbb{D}_X(Y))$ be the PD-dilatation of $s_{\tilde{X}}$ defined in Proposition 2.3, which is p-torsion free by construction and which agrees with the usual PD-envelope if $s_{\tilde{X}}$ is a regular immersion. The morphism $\overline{\mathbb{P}}_{\tilde{X}}(\mathbb{D}_X(Y)) \to Y$ factors through X, and so $\mathbb{P}_{\tilde{X}}(\mathbb{D}_X Y))$ can also be viewed as a p-adic enlargement of X over Y.

The endomorphism ϕ of Y induces endomorphisms of $\mathbb{D}_X(Y)$ and $\mathbb{P}_{\tilde{X}}(\mathbb{D}_X(Y))$, both of which we denote by ϕ. Although ϕ may not be a Frobenius lift on $\mathbb{D}_X(Y)$, we claim that it is so on $\mathbb{P}_{\tilde{X}}(\mathbb{D}_X(Y))$ if $\tilde{X}$ is ϕ_2-aligned. Let us check this locally, assuming that $\mathbb{D}_X(Y) \to Y$ is given by $\theta\colon C \to C'$, that $C' \to C''$ defines the PD-dilatation of $s_{\tilde{X}}$, and that J is the ideal of $\tilde{X}$ in Y. If $x \in J$, then $\phi(x) = x^p + p\delta(x) \in J + p^2C$, since $\tilde{X}$ is ϕ_2-aligned. It follows that $p\delta(x) \in J + p^2C$, and since $\tilde{X}$ is p-torsion free, that $\delta(x) \in I := J + pC$. The ideal $\tilde{J}$ of $\tilde{j}$ is generated by elements $\rho(x)$ with $x \in J$, and for such x we have:

$$p\phi(\rho(x)) \quad = \quad \phi(\theta(x))$$

[3] Recall from Corollary 1.10 that if X is reduced, a ϕ-aligned lifting $\tilde{X}$ is in fact unique.

$$
\begin{aligned}
&= \theta(\phi(x)) \\
&= \theta(x^p + p\delta(x)) \\
&= p^p\rho(x)^p + p\theta(\delta(x)) \\
&= p^p\rho(x)^p + p^2\rho(\delta(x))
\end{aligned}
$$

It follows that $\phi(\rho(x)) \in pC'$. The C-algebra C' is topologically generated by $\rho(J)$, and the divided power envelope C'' of the ideal generated by $\rho(J)$ is topologically generated by the divided powers of the elements of $\rho(J)$. Since we already know that $\phi(c) \equiv c^p \pmod p$ for $c \in C$, it will therefore suffice to check that the same holds for all such divided powers. Write $\phi(\rho(x)) = p\tilde{x}$ with $\tilde{x} \in C'$, and note that in fact $\tilde{x}$ belongs to $\tilde{J}$. Then $(\phi(\rho(x))^{[n]} = p^{[n]}\tilde{x}^n \in pC''$, On the other hand, $\rho(x)^{[n]}$ belongs to a PD-ideal of C'', so $(\rho(x)^{[n]})^p$ also belongs to pC''. Thus $\phi(c'') \equiv (c'')^p \pmod{pC''}$ for all $c'' \in C''$, as required.

To complete the proof of statement (1), it remains only to show that the X-prism $\mathbb{P}_X(\mathbb{D}_X(Y))$ is universal. Suppose that T is another X-prism over Y. In particular it is a p-adic enlargement of X over Y, and hence maps to $\mathbb{D}_X(Y)$. We saw above that ϕ maps the ideal $\tilde{J}$ of $s_{\tilde{X}}$ to $p\mathcal{O}_{\mathbb{D}}$, and hence ϕ maps the ideal $\tilde{J}\mathcal{O}_T$ to $p\mathcal{O}_T$. In other words, $\tilde{J}$ is contained in the ideal of $\phi(\overline{T}) \subseteq T$. As we saw in in Proposition 2.4, this is a PD-ideal. It follows that T factors uniquely through $\mathbb{P}_X(\mathbb{D}_X(Y))$.

To prove (2), note first that it follows from (3) of Proposition 2.3 that the endomorphism $\mathbb{P}_X(\phi_Y)$ endows $\mathbb{P}_X(Y)$ with the structure of a formal ϕ-scheme. Furthermore, the ideal of X^ϕ is generated by p and the pth powers of elements of I, and since I maps into a PD-ideal in $\mathbb{P}_X(Y)$, all these elements become divisible by p in $\mathcal{O}_{\mathbb{P}_X(Y)}$. Thus $\mathbb{P}_X(Y)$ defines an X^ϕ-prism over Y. To see that it is final, let (T, z_T, π_T) be another such. We claim that there is a commutative diagram:

$$
\begin{tikzcd}
T_1 \arrow[r] \arrow[d, "z_T"'] & \phi(T_1) \arrow[r] \arrow[d, dashed] & T_1 \arrow[d, "z_T"] \\
X^\phi \arrow[r] & X \arrow[r] & X^\phi
\end{tikzcd}
$$

and the composite horizontal arrows are the Frobenius endomorphisms. Indeed, if x is a local section of the ideal of X in Y, then $\phi_T(\pi_T^\sharp(x)) = \pi_T^\sharp(\phi_Y(x)) \in p\mathcal{O}_T$, since $\phi_Y(x)$ belongs to the ideal of X^ϕ. But Proposition 2.4 tells us that $\phi(T_1) = T_{\mathbb{P}}$, and, since $T_{\mathbb{P}} \to T$ is a PD-immersion, the arrows $T_{\mathbb{P}} \to X$ and $T \to Y$ then give T the structure of a PD-enlargement of X over Y which must factor through $\mathbb{P}_X(Y)$. $\square$

Remark 2.28. The two statements of Theorem 2.27 are related. In fact, statement (2) can be deduced from statement (1), using the fact that X^ϕ is always ϕ_2-aligned in Y. To see this suppose that $(p, x_1, \ldots, x_r)$ is a sequence generating the ideal of X in Y. Then $(p, x_1^p, \ldots, x_r^p)$ generates the ideal of X^ϕ, and

$$\phi(x_i^p) = \phi(x_i)^p = (x_i^p + p\delta(x_i))^p \equiv (x_i^p)^p \pmod{p^2}.$$

Thus we find a section $\tilde{X} \subseteq \mathbb{D}_X(Y)$ defined by $(\rho(x_1^p), \dots \rho(x_r^p))$, and statement (1) tells us that $\mathbb{A}_X(Y)$ is the PD-dilatation of $\tilde{X}$ in $\mathbb{D}_{\tilde{X}}(Y)$. Since $\rho(x_i^p) = x_i^p/p$, this coincides with the PD-dilatation of X in Y.

Example 2.29. Let us remark that, as observed in [7, 2.40], statement (2) of Theorem 2.27 implies that $\mathbb{A}_{X^\phi}(Y)$ does not depend on the Frobenius lifting ϕ of Y, but this is not true more generally. By way of example, it is shown that if $Y = \operatorname{Spf} W[x]\hat{}$ and X is defined by the ideal (x, p), then the choices $\phi(x) = x^p$ and $\phi(x) = x^p + p$ give different prismatic envelopes. Statement (1) of Theorem 2.27 allows us to be a little more explicit about these examples, since X is ϕ_2-alignable in both cases. In the first case, since $\phi(x) = x^p$, one finds that $\mathbb{A}_X(Y)$ is the formal spectrum of $W\langle\langle x/p\rangle\rangle$. In the second case, note that $\phi(x-p) = x^p \equiv (x-p)^p \pmod{p^2}$, so $\mathbb{A}_X(Y)$ is the formal spectrum of $W\langle\langle x/p - 1\rangle\rangle$.

For another example, consider the formal ϕ-scheme Y given by $\operatorname{Spf} W[x,y]\hat{}$, with $\phi(x) = x^p + py$ and $\phi(y) = y^p$, and let X be the closed subscheme defined by the ideal (p, x). As we saw in Example 1.8, this X is not ϕ_2-alignable in Y. We claim that the prismatic envelope $\mathbb{A}_X(Y) \to Y$ is given by the homomorphism:

$$W[x,y]\hat{} \mapsto W[s]\langle t\rangle\hat{} : x \mapsto ps, y \mapsto pt + s^p.$$

Let us check this using the constructions and notations of the proof of Theorem 2.19. Thus we let $B := W[x,y]\hat{}$ and let $B \to B'$ be the p-adic dilatation of the ideal $I := (p, x)$, *i.e.*, B' is the (p-adic completion of) $B[s]/(x - ps)$. Then $\epsilon'(x) = p^{p-1}x^p + y - s^p$, so I' is the ideal of B' generated by p and $y - s^p$, and $B'' = B[t]/(pt' - (y - s^p))\hat{}$. Let us compute:

$$\begin{array}{rcl} \psi'(y - s^p) & = & \psi'(y) - p^{-p}\psi'(x)^p \\ & = & y^p - p^{-p}(x^p + py)^p \\ & = & y^p - p^{-p}(p^p s^p + py)^p \\ & = & y^p - (p^{p-1}s^p + y)^p \\ & \in & p^2 B' \end{array}$$

Thus $\psi''(t) \in pB''$, and it follows from Proposition 2.4 that the prismatic envelope contains all the divided powers of t, and hence all of $W[s]\langle t\rangle$. In this ring,

$$\psi(s) = p^{p-1}s^p + y = p^{p-1}s^p + pt + s^p \equiv s^p \pmod{(p)}$$

and

$$\psi(t) = p^{-1}\psi(y - s^p) \equiv 0 \pmod{(p)} \equiv t^p,$$

this ring really is the prismatic envelope.

Example 2.30 (semi-final prisms)**.** Suppose that $S = \operatorname{Spf} R$ is affine and that $X/\overline{S}$ is an affine scheme, say $X = \operatorname{Spec} A$. Choose a set of generators for A/R, indexed by a natural number I, or more generally, any set. Then we find a closed immersion of X into the affine space $\mathbf{A}^I$ over S. Recall from Example 1.2 that there is a formal ϕ-scheme $\mathbf{A}^I_\phi$ equipped with a universal morphism $r\colon \mathbf{A}^I_\phi \to \mathbf{A}^I$. Let $X_\phi := r^{-1}(X) \subseteq \mathbf{A}^I_\phi$, let $\mathbb{A}_{X_\phi}(\mathbf{A}^I_\phi)$ denote its prismatic envelope, and let

$z_{\mathbb{\Delta}}$ be the composite map $\overline{\mathbb{\Delta}}_{X_\phi}(\mathbf{A}^I_\phi) \to X_\phi \to X$. Then $(\mathbb{\Delta}_{X_\phi}(\mathbf{A}^I_\phi), z_{\mathbb{\Delta}})$ is an X-prism, and it is semi-final in the sense that it receives a morphism (not unique) from every affine X-prism (T, z_T). Indeed, if (T, z_T) is such a prism, the map $\overline{T} \to T$ is a closed immersion and consequently the map $\overline{T} \to X \to \mathbf{A}^I$ lifts to a map $T \to \mathbf{A}^I$ which in turn lifts uniquely to a map of formal ϕ-schemes $T \to \mathbf{A}^I_\phi$. The map $\overline{T} \to \mathbf{A}^I_\phi$ factors through X_ϕ, giving T the structure of an X_ϕ-prism over $\mathbf{A}^I_\phi$, and this map $T \to \mathbf{A}^I_\phi$ factors through $\mathbb{\Delta}_{X_\phi}(\mathbf{A}^I_\phi)$. This construction, which I believe is due to Koshikawa, was communicated to be by Shiho, and now appears in [7, 4.16]. We discuss generalizations and variations of this construction in §6.2.

As a subexample, take $X = \overline{S}$ and $I = 1$. Then $\mathbf{A}^1 = \operatorname{Spf} R[x]\hat{}$, and $\mathbf{A}^1_\phi$ is the universal δ-ring (B, δ) on the variable x. Explicitly, B is the completed polynomial algebra $R[x_0, x_1, \ldots]\hat{}$, with $x = x_0$, and $\phi(x_i) = x_i^p + px_{i+1}$ for all i. The ideal J generated by x as a δ-ideal is generated by $x_0, x_1, \ldots$ as an ideal, and is ϕ-invariant. Thus Theorem 2.27 applies, and we find that the prismatic envelope $\mathbb{\Delta}_{r^{-1}(X)}(\mathbf{A}^1_\phi)$ can be identified with the PD-envelope of the ideal J' of the section of the p-dilatation B' of (p, J) defined by J. In fact, $B' \cong W[y_0, y_1, \ldots,]$ with $x_i = py_i$, and so this envelope is the spectrum of the completed PD-polynomial algebra $W\langle y_0, y_1, \ldots\rangle\hat{}$.

The next two results are formal but useful.

Lemma 2.31. *Let Y be a formal ϕ-scheme, let $X \subseteq X' \subseteq Y_1$ be closed immersions, and $\tilde{X} := \pi_Y^{-1}(X) \subseteq \mathbb{\Delta}_{X'}(Y)$. Then the natural map*

$$\mathbb{\Delta}_{\tilde{X}}(\mathbb{\Delta}_{X'}(Y)) \to \mathbb{\Delta}_X(Y)$$

is an isomorphism.

Proof. This is proved in the same way as Lemma 2.11. □

Lemma 2.32. *Suppose that $g\colon Y \to Z$ is a closed immersion of formal ϕ-schemes admitting a p-completely flat retraction $r\colon Z \to Y$. If X is a closed subscheme of $\overline{Y}$, let $Y' := \mathbb{\Delta}_X(Y)$ and $Z' := Y' \times_Y Z$ and let $g'\colon Y' \to Z'$ be the closed immersion $\mathrm{id}_{Y'} \times_Y g$. Then the natural map*

$$\mathbb{\Delta}_{Y'}(Z') \to \mathbb{\Delta}_X(Z)$$

is an isomorphism.

Proof. We should first observe that, since r is p-completely flat and Y' is p-torsion free, necessarily Z' is also p-torsion free, so Z' is again a formal ϕ-scheme. Since $\overline{Y}' \to \overline{Y}$ factors through X, the projection $Z' \to Z$ induces a map of pairs: $(\overline{Y}' \subseteq Z') \to (X \subseteq Z)$ and hence a map of prismatic envelopes $\mathbb{\Delta}_{Y'}(Z') \to \mathbb{\Delta}_X(Z)$, uniquely determined by its projection to Z. On the other hand, the map $\mathbb{\Delta}_X(Z) \to Z \to Y$ is an X-prism over Y, and hence it factors uniquely through $Y' = \mathbb{\Delta}_X(Y)$. Thus we find a morphism $\mathbb{\Delta}_X(Z) \to Y' \times_Y Z = Z'$, which we claim factors (uniquely) through $\mathbb{\Delta}_{Y'}(Z')$. In fact, the map $\overline{\mathbb{\Delta}}_X(Z) \to \overline{Y}' \times_Y \overline{Z}$ factors through $\overline{Y}' \times_Y X = \overline{Y}$, so $\mathbb{\Delta}_X(Z) \to Z'$ is indeed a Y'-prism over Z'.

The uniqueness of these maps allows one to check that they are inverse to each other, proving the lemma. □

Corollary 2.33. *Let $j \colon Y \to Z$ be a closed immersion of formal ϕ-schemes and assume that $\overline{Y} \to Z$ is a regular immersion. Then j lifts uniquely to a closed immersion $\tilde{j} \colon Y \to \mathbb{\Delta}_{\overline{Y}}(Z)$, and the ideal $J_{Y/\mathbb{\Delta}}$ of $\tilde{j}$ is a PD-ideal. In fact, $J_{Y/\mathbb{\Delta}}$ is the PD-ideal generated by the sections t of $\mathcal{O}_{\mathbb{\Delta}_Y(Z)}$ such that pt belongs to the ideal $J_{Y/Z}$ of Y in Z.*

Proof. Since $\overline{Y} \to Z$ is regularly immersed and ϕ_2-alignable, statement (1) of Theorem 2.27 implies that $\mathbb{\Delta}_Y(Z)$ identifies with the PD-envelope of the section $\tilde{j} \colon Y \to \mathbb{D}_Y(Z)$. As we saw in Proposition 2.12, the ideal of this section is generated by the sections t of $\mathcal{O}_{\mathbb{D}_Y(Z)}$ such that $pt \in J_{Y/Z}$. The corollary follows. □

Corollary 2.34. *Let Y/S be a morphism of formal ϕ-schemes, with relative Frobenius morphism $\phi_{Y/S} \colon Y \to Y'$ (1.1). If X is a closed subscheme of $\overline{Y}$, there is a commutative diagram:*

$$\begin{tikzcd}
\mathbb{\Delta}_X(Y) \arrow[r, "\Psi"] \arrow[dr, "\mathbb{\Delta}_X(\phi_{Y/S})"'] & \mathbb{P}_X(Y) \arrow[d, "\Phi_{Y/S}"'] \arrow[dr, "\mathbb{P}_X(\phi_{Y/S})"] & \\
& \mathbb{\Delta}_{X'}(Y') \arrow[r, "\Psi'"] & \mathbb{P}_{X'}(Y')
\end{tikzcd}$$

If Y/S is p-completely smooth, then $\Phi_{Y/S}$ is p-completely faithfully flat.

Proof. Let $X^\phi \subseteq Y$ be the inverse image of X by the endomorphism ϕ of Y; equivalently, the inverse image of X' by the relative Frobenius morphism $\phi_{Y/S}$. We then have morphisms of pairs:

$$\begin{tikzcd}
(Y, X) \arrow[r, "(\mathrm{id}, \mathrm{inc})"] \arrow[dr, "(\phi_{Y/S}, \phi_{X/S})"'] & (Y, X^\phi) \arrow[d] \arrow[dr, "(\phi_{Y/S}, \phi_{X^\phi_{X/S}})"] & \\
& (Y', X') \arrow[r, "(\mathrm{id}, \mathrm{inc})"] & (Y', X^{\phi\prime}),
\end{tikzcd}$$

where the vertical map is induced by $\phi_{Y/S}$, which maps X^ϕ to X'. Taking the corresponding diagram of prismatic envelopes and using statement (2) of Theorem 2.27 to identify $\mathbb{\Delta}_{X^\phi}(Y)$ with $\mathbb{P}_X(Y)$ and $\mathbb{\Delta}_{X^{\phi\prime}}(Y')$ with $\mathbb{P}_{X'}(Y')$, we obtain the diagram in the statement of the corollary. Let us note for future reference that the morphism Ψ in the diagram factors naturally:

$$\Psi = \mathbb{\Delta}_X(Y) \to \mathbb{D}_X(Y) \to \mathbb{P}_X(Y). \tag{2.2}$$

If Y/S is p-completely smooth, then $\phi_{Y/S}$ is p-completely faithfully flat, and since X^ϕ is the inverse image of X', the square in the diagram

$$\begin{tikzcd}
\Phi\colon \mathbb{P}_X(Y) \arrow[r, "\cong"] \arrow[dr] & \mathbb{\Delta}_{X^\phi}(Y) \arrow[r] \arrow[d] & \mathbb{\Delta}_X(Y) \arrow[d] \\
& Y \arrow[r, "\phi_{Y/S}"] & Y'
\end{tikzcd} \tag{2.3}$$

is Cartesian, by statement (1) of Theorem 2.19. Then $\Phi_{Y/S}$ is also p-completely faithfully flat. $\square$

Example 2.35. Suppose that $X \to Y$ admits a ϕ-invariant lift $\tilde{X} \to Y$, and let $\tilde{X} \to \mathbb{D}_X(Y)$ be the induced section (see Proposition 2.12). Then by Theorem 2.27, we can identify $\mathbb{\Delta}_X(Y)$ with $\mathbb{P}_{\tilde{X}}(\mathbb{D}_X(Y))$. Furthermore, the map of pairs $(\mathbb{D}_X(Y), \tilde{X}) \to (Y, X)$ induces a map of divided power envelopes:

$$\mathbb{\Delta}_X(Y) \cong \mathbb{P}_{\tilde{X}}(\mathbb{D}_X(Y)) \to \mathbb{P}_X(Y)$$

which is none other than the morphism Ψ.

The next proposition discusses finite inverse limits of prisms; as we shall see, these are better behaved if we restrict our attention to small prisms.

Proposition 2.36. *Let S be a formal ϕ-scheme and $X/\overline{S}$ a morphism of finite type.*

1. *In the category of X-prisms over S, inverse limits over finite index sets are representable.*

2. *If $X \to Y$ is a closed S-immersion from X into a formal ϕ-scheme Y, then in the category of X-prisms over Y, inverse limits over finite index sets are representable.*

3. *Suppose that X/S_1 is smooth and that T' and T'' are small X-prisms. Then their product $T' \times_S T''$ (in the category of X-prisms over S) is also small, and its projection mapping to T' and T'' are p-completely flat.*

Proof. To show that a category admits finite nonempty inverse limits, it is enough to show that the product of every pair of objects and the fibered product of every pair of morphisms are representable. With this strategy, statement (1) is straightforward. If T' and T'' are X-prisms, let $\widetilde{T} := (T' \times_S T'')_{\mathrm{tf}}$ be their product in the category of formal ϕ-schemes over S as described in Remark 1.3, and let be $\tilde{X}$ be the inverse image of the diagonal under the natural map $\widetilde{T}_1 \to T'_1 \times_S T''_1 \to X \times_S X$. Then it is straightforward to verify that $\mathbb{\Delta}_{\tilde{X}}(\widetilde{T})$ represents the product of T' and T'' in the category of X-prisms over S. The analogous construction works for fibered products. Since the category of X-prisms over Y is equivalent to the category of morphisms of X-prisms $T \to \mathbb{\Delta}_X(Y)$, statement (1) implies statement (2).

Now suppose that T' and T'' are small. Since $X \to S_1$, $T'_1 \to X$, and $T''_1 \to X$ are flat, the maps $T'_1 \to S_1$ and $T''_1 \to S_1$ are flat. Thus $T' \to S$ and $T'' \to S$ are p-completely flat, by Proposition 7.13, and it follows that $\widetilde{T} := T' \times_S T''$ is p-torsion free (Proposition 7.14). Furthermore, the map $\widetilde{T}'_1 = T'_1 \times_S T''_1 \to X \times_S X$ is flat, and since X/S_1 is smooth, the diagonal $X \to X \times_S X$ is a regular immersion. Then $\tilde{X} \to \widetilde{T}'_1$ is also a regular immersion, and it follows from Theorem 2.19 that $(\mathbb{\Delta}_{\tilde{X}}(\widetilde{T}))_1 \to \tilde{X}$ is flat. Since $\tilde{X} \to X$ is also flat, we conclude that $(\mathbb{\Delta}_{\tilde{X}}(\widetilde{T}))_1 \to X$ is flat, so $\mathbb{\Delta}_{\tilde{X}}(T' \times_S T'')$ is small. We have seen that $(\mathbb{\Delta}_{\tilde{X}}(\widetilde{T}))_1 \to \tilde{X}$ is flat, and $\tilde{X} \to T'_1$ is flat because it is obtained by base change from the flat map $T''_1 \to X$. Thus the the map $(\mathbb{\Delta}_{\tilde{X}}(\widetilde{T}))_1 \to T'_1$ is flat, and since $\mathbb{\Delta}_{\tilde{X}}(\widetilde{T})$ is p-torsion free, it follows that $\mathbb{\Delta}_{\tilde{X}}(\widetilde{T}) \to T'$ is p-completely flat. The case of the projection to T'' follows by symmetry. $\square$

Remark 2.37. If $T' \to T$ and $T'' \to T$ are morphisms of small X-prisms, and if $T' \times_T T''$ is p-torsion free, then it is also small. For example, this holds if $T' \to T$ or $T'' \to T$ is p-completely flat. However I do not know if this is true in general.

Proposition 2.38. *Let $g\colon Y \to Z$ be a morphism of formal ϕ-schemes and $i\colon X \to Z_1$ a regular closed immersion.*

1. *Suppose that the ideal of X in Z_1 is locally defined by a regular sequence which remains regular in $\mathcal{O}_{Y_1}$, and let $X' := g^{-1}(X) \subseteq Y_1$. Then the natural map*
$$\mathbb{\Delta}_{X'}(Y) \to \mathbb{\Delta}_X(Z) \times_Z Y$$
is an isomorphism.

2. *Suppose that i factors as a composite of regular immersions $X \to Y_1$ and $g_1\colon Y_1 \to Z$. Then the natural map*
$$\mathbb{\Delta}_X(Y) \to \mathbb{\Delta}_X(Z) \times_{\mathbb{\Delta}_Y(Z)} \tilde{Y}$$
is an isomorphism.

Proof. We shall deduce statement (1) from its analog in Proposition 2.13 and the explicit construction of prismatic envelopes in Theorem 2.19. Recall that $\mathbb{\Delta}_X(Z)$ is a (completed) limit of flat W-schemes $Z^{(n)}$, where each $Z^{(n+1)} \to Z^{(n)}$ is obtained as the dilatation of a subscheme $X^{(n)}$. The envelope $\mathbb{\Delta}_{X'}(Y)$ is constructed in the same way, and in fact there is a commutative diagram

$$\begin{tikzcd} Y^{(n+1)} \arrow[r] \arrow[d] & Z^{(n+1)} \arrow[d] \\ Y^{(n)} \arrow[r] & Z^{(n)}. \end{tikzcd}$$

When $n = 0$, the vertical map on the right (resp. left) is the dilatation of X in Z (resp. of X' in Y), and the diagram is Cartesian by Proposition 2.13. It follows from Statement (2) of Theorem 2.19 that each $X^{(n)} \to Z^{(n)}$ and each ${X'}^{(n)} \to Y^{(n)}$ is again a regular immersion, and from the construction of the ideal defining $X^{(n)}$ described in (3) of Lemma 2.20, that ${X'}^{(n)} = X^{(n)} \times_{Z^{(n)}} Y^{(n)}$. Then Proposition 2.13 implies that that the diagram is Cartesian for every n. We deduce that $Y^{(n)} \cong Y \times_Z Z^{(n)}$ for every n, and statement (1) follows.

In the situation of statement (2), we follow the method of proof of statement (2) of Proposition 2.13, working locally in an affine setting with the same notation. We first find a sequence $(x_1, \ldots, x_r)$ in C which generates the ideal J of Y in Z and is C/pC-regular. Since $X \to Y_1$ is also a regular immersion, we may then find a B/pB-regular sequence $(y_1, \ldots, y_m)$ in C whose image generates the ideal of X in Y_1. Then by Theorem 2.27, the prismatic envelope $\tilde{Z} := \mathbb{A}_Y(Z)$ of Y in Z corresponds to the PD-envelope $\mathbb{P}_{\tilde{Y}}(\mathbb{D}_Y(Z))$ of the section $\tilde{Y}$ of $\mathbb{D}_Y(Z)$ defined by $Y \to Z$. The ideal of this section is generated by the regular sequence $(t_1, \ldots, t_r)$, where $t_i := \rho(x_i)$, by Proposition 2.12, and hence this PD-envelope is given by $\operatorname{Spf} \tilde{C}$, where $\tilde{C}$ is the completion of the PD-polynomial algebra $B\langle t_1, \ldots, t_r\rangle$. The inverse image $\tilde{X}$ of X in $\mathbb{D}_Y(Z)$ is defined by the ideal generated by the sequence $(p, x_1, \ldots, x_r, y_1, \ldots, y_m)$, or in fact just by $(p, y_1, \ldots, y_m)$ since $x_i \in p\tilde{C}$. Since $\tilde{C}/p\tilde{C}$ is a PD-polynomial algebra over B/pB, it follows that $(y_1, \ldots, y_r)$ is also $\tilde{C}/p\tilde{C}$-regular. Thus the maps $\tilde{X} := \pi_Z^{-1}(X) \to \tilde{Z}_1$ and $\tilde{X} \cap \tilde{Y}_1 \to \tilde{Y}_1$ are regular immersions. Then statement (1), applied to the morphisms $\tilde{j} \colon \tilde{Y} \to \tilde{Z}$ and $\tilde{i} \colon \tilde{X} \to \tilde{Z}$, implies that the map

$$\mathbb{A}_X(Y) = \mathbb{A}_{\tilde{X} \cap \tilde{Y}}(\tilde{Y}) \to \mathbb{A}_{\tilde{X}}(\tilde{Z}) \times_{\tilde{Z}} \tilde{Y}$$

is an isomorphism. Lemma 2.31 tells us that $\mathbb{A}_{\tilde{X}}(\tilde{Z}) \cong \mathbb{A}_X(Z)$, proving statement (2). $\square$

Corollary 2.39. *If Y is a formal ϕ-scheme X and X' are regularly immersed in Y_1 and meet transversally, then the natural map*

$$\mathbb{A}_{X \cap X'}(Y) \to \mathbb{A}_X(Y) \times_Y \mathbb{A}_{X'}(Y)$$

is an isomorphism.

Proof. This is proved in the same way as Corollary 2.14 $\square$

The various envelopes we have constructed can be thought of as having differing sizes and shapes. If Y is a formal ϕ-scheme and X is a closed subscheme of $\overline{Y}$, the following diagram illustrates some of their relations:

$$\begin{tikzcd}
 & \mathbb{A}_X(Y) \arrow[r] \arrow[d] & \mathbb{A}_{X^\phi}(Y) \arrow[r] \arrow[d, "\cong"] & \mathbb{D}_{X^\phi}(Y) \arrow[r] \arrow[d] & \mathbb{D}_{X^{(\phi^2)}}(Y) \cdots \arrow[dl] \\
\mathbb{A}_X(Y) \arrow[r] \arrow[ur, "="] & \mathbb{D}_X(Y) \arrow[r] & \mathbb{P}_X(Y) \arrow[r] & Y &
\end{tikzcd} \tag{2.4}$$

A crude measure of the size of these envelopes is the radius of the analytic tubes they define. It may be interesting to compute these. For simplicity, let $i\colon X \to Y$ be the embedding of the point $\operatorname{Spec} k$ in the formal affine line $\operatorname{Spf} W[X]\hat{}$. If $f = \sum a_i X^i \in W[X]\hat{}$ and $x \in \overline{K}\hat{}$, then $\sum a_i x^i$ is guaranteed to converge if $|x| \leq 1$, so the rigid space Y_K associated to $W[X]\hat{}$ corresponds to the closed unit disc $\{x : |x| \leq 1\}$. Now $\mathbb{D}_X(Y) = \operatorname{Spf} W[T]\hat{}$, with $X = pT$, and the image of $\mathbb{D}_X(Y)_K$ in Y_K is the closed disc of radius $|p|^{-1}$. On the other hand, an element of the completed divided power algebra $W\langle X\rangle\hat{}$ converges at x if $\operatorname{ord}_p(x^{p^n}/p^n!) \geq 0$ for all n. But

$$\begin{aligned}
\operatorname{ord}_p(x^{p^n}/p^n!) &= p^n \operatorname{ord}_p x - \operatorname{ord}_p(p^n!) \\
&= p^n \operatorname{ord}_p x - (1 + p + p^2 + \cdots + p^{n-1}) \\
&= p^n(\operatorname{ord}_p x - p^{-n} + \cdots + p^{-1}).
\end{aligned}$$

Thus $\operatorname{ord}_p(x^{p^n}/p^n!) \geq 0$ for all n if and only if $\operatorname{ord}_p x \geq 1/(p-1)$, so $\mathbb{P}_X(Y)_K$ corresponds to the closed disc of radius $|p|^{\frac{-1}{p-1}}$. Since $\mathbb{\Delta}_X(Y) = \operatorname{Spf} W\langle X/p\rangle\hat{}$, its image in Y_K is the closed disc of radius $|p|^{\frac{-p}{p-1}}$. Finally, $\mathbb{D}_X^{(p^n)}(Y) = \operatorname{Spf} W[x^{p^n}/p]\hat{}$, so the associated rigid space is the closed disc of radius $p^{-p^{-n}}$. The union of these is the open disc of radius one, which is the neighborhood corresponding to convergent cohomology.

We have seen that the construction of prismatic envelopes is, under some circumstances, independent of the choice of Frobenius lifting. The following result illustrates that prismatic envelopes are functorially independent of Frobenius lifts under certain circumstances. This result will not be used in an essential way in the remainder of the current manuscript; its was motivated by an unsuccessful attempt to prove a prismatic Poincaré lemma without recourse to p-completely flat localization.

Proposition 2.40. *Let $j\colon Y \to Z$ be a closed immersion of formal ϕ-schemes and let $X \to Y_1$ be a regular closed immersion. Suppose that j admits a retraction $r\colon Z \to Y$, not necessarily compatible with the Frobenius lifts. Then there is a unique retraction $\tilde{r}$ of $\mathbb{\Delta}_X(j)$ making the following diagram commute:*

$$\begin{tikzcd}
& \mathbb{\Delta}_X(Y) \arrow[r] \arrow[dl, "\mathbb{\Delta}_X(j)"'] \arrow[ddl] & Y \arrow[dl, "j"'] \arrow[ddl, "\mathrm{id}"] \\
\mathbb{\Delta}_X(Z) \arrow[r, "\mathrm{id}"'] \arrow[d, "\tilde{r}"'] & Z \arrow[d, "r"] & \\
\mathbb{\Delta}_X(Y) \arrow[r] & Y &
\end{tikzcd}$$

Proof. We work locally, assuming that j is given by a surjection $\pi\colon (C, \phi_C) \to (B, \phi_B)$, with kernel J. Then r is given by an injective homomorphism $\rho\colon B \to C$. Here π is compatible with the endomorphisms ϕ_C and ϕ_B of C and B, but ρ may not be. We shall follow the step-by-step procedure for the construction of prismatic envelopes described in Theorem 2.19. Thus $\mathbb{\Delta}_X(Z)$ (resp. $\mathbb{\Delta}_X(Y)$) is the p-adic completion of $\mathcal{C} := \varinjlim C^{(n)}$ (resp., of $\mathcal{B} := \varinjlim B^{(n)}$), where $C^{(n)}$ and $B^{(n)}$ are obtained by certain (uncompleted) dilatations as described in Lemma 2.20. The splitting ρ induces a splitting $\mathrm{id} \otimes \rho \colon \mathbf{Q} \otimes B \to \mathbf{Q} \otimes C$, and, since $\mathcal{C} \subseteq \mathbf{Q} \otimes C$ and $\mathcal{B} \subseteq \mathbf{Q} \otimes B$, it will suffice to show that $\mathrm{id} \otimes \rho$ maps $\mathcal{B}$ to $\mathcal{C}$. The following lemma, which we prove by induction on n, establishes this fact. To simplify the notation, we view the injective map ρ as an inclusion, so $C = B \oplus J$.

Lemma 2.41. *For each n, the following statements hold.*

1. *$\psi_C(B^{(n)}) \subseteq B + pC^{(n)}$.*
2. *$\psi_B(b) \equiv \psi_C(b) \pmod{pC^{(n)}}$ for all $b \in B^{(n)}$.*
3. *The map $\mathbf{Q} \otimes B \to \mathbf{Q} \otimes C$ sends $B^{(n)}$ to $C^{(n)}$.*

Proof. Since J is the kernel of a homomorphism of ϕ-algebras, it is invariant under ϕ_C. If $c \in C$, write $\phi_C(c) = c^p + p\delta(c)$; since $C/J = B$ is p-torsion free, $\delta(c) \in J$ if $c \in J$. Let $K \subseteq B$ be the ideal of $X \subseteq Y$; then the ideal $I \subseteq C = B \oplus J$ of X in Z is $K \oplus J$. If $c \in C$, write $\delta(c) = \beta(c) + \gamma(c)$, with $\beta(c) \in B$ and $\gamma(c) \in J$. Since $\pi \colon C \to B$ is compatible with the Frobenius lifts, it necessarily sends each $C^{(n)}$ to $B^{(n)}$; furthermore, $\psi_B(b) = b^p + p\beta(b)$ for $b \in B$.

Note that, in general, statements (1) and (3) imply statement (2). Indeed, if $b \in B^{(n)}$, (1) implies that $\psi_C(b) = b_0 + pc$ for some $b_0 \in B$ and $c \in C^{(n)}$. Then $\pi(c) \in B^{(n)}$, statement (3) implies that $\pi(c) \in C^{(n)}$, and so

$$\psi_B(b) = \pi\psi_C(b) = \pi(b_0 + pc) = b_0 + p\pi(c) = \psi_C(b) - pc + p\pi(c).$$

If $n = 0$, the inclusion $B \to C$ is given, and if $b \in B$,

$$\psi_C(b) = b^p + p\delta(b) \in B + pC,$$

and $\psi_B(b) = b^p + p\beta(b)$. This implies (1) and (2) when $n = 0$.

Now suppose $n = 1$. By construction, $B \to B^{(1)}$ is the dilatation of K and $C \to C^{(1)}$ is the dilatation of $I = K \oplus J$. Then $B^{(1)} \subseteq C^{(1)}$ because $K \subseteq I$, so (3) is automatic. To prove (1), it will suffice to check that $\psi_C(b) \in B + pC^{(1)}$ as b ranges over a set of generators of $B^{(1)}$ as a B-algebra, e.g., for elements of the form x/p with $x \in K$. We compute:

$$\begin{aligned} \psi_C(x/p) &= p^{-1}\phi_C(x) \\ &= p^{-1}\left(x^p + p\beta(x) + p\gamma(x)\right) \\ &= p^{p-1}(x/p)^p + \beta(x) + \gamma(x) \end{aligned}$$

Here $\gamma(x) \in J \subseteq I$, and since $IC^{(1)} = pC^{(1)}$, it follows that $\psi_C(x/p) \in B + pC^{(1)}$.

For the induction step, we assume that $n \geq 1$ and that the lemma is proved for n. We first verify statement (3), *i.e.*, that $B^{(n+1)} \subseteq C^{(n+1)}$. Because of the induction hypothesis, it suffices to check this for a set of generators for $B^{(n+1)}$ as a $B^{(n)}$-algebra. Recall that $C^{(n+1)}$ (resp. $B^{(n+1)}$) is the dilatation of the ideal $I^{(n)}$ of $C^{(n)}$ (resp. of the ideal $K^{(n)}$ of $B^{(n)}$), where $I^{(n)}$ (resp. $K^{(n)}$) is generated by elements of the form $\psi_C(c) - c^p$ with $c \in C^{(n)}$ (resp. $\psi_B(b) - b^p$ with $b \in B^{(n)}$). So it will suffice to check that, for each $b \in B^{(n)}$,

$$\tilde{b} := p^{-1}(\psi_B(b) - b^p)$$

belongs to $C^{(n+1)}$. By statement (2), there is some $c' \in C^{(n)}$ such that $\psi_B(b) = \psi_C(b) + pc'$. Then

$$\begin{aligned} \tilde{b} &:= p^{-1}(\psi_B(b) - b^p) \\ &= p^{-1}(\psi_C(b) + pc' - b^p) \\ &= p^{-1}(\psi_C(b) - b^p) + c' \end{aligned}$$

which belongs to $C^{(n+1)}$.

If statement (1) holds for n, it will also hold for $n+1$ if we verify it for a set of generators of $B^{(n+1)}$ as a $B^{(n)}$-algebra, *e.g.*, for elements of the form $\tilde{b} := p^{-1}(\psi_B(b) - b^p)$, where $b \in B^{(n)}$. By statement (1) for n, we may write $\psi_C(b) = b_0 + pc'$, with $b_0 \in B$ and $c' \in C^{(n)}$. Then $\psi_B(b) = b_0 + pb'$, where $b' := \pi(c') \in B^{(n)}$. Now write $\psi_C(b') = b'_0 + pc''$ with $b'_0 \in B$ and $c'' \in C^{(n)}$. Writing $(b_0 + pc')^p = b_0^p + p^2 c'''$, we calculate:

$$\begin{aligned} \psi_C(\tilde{b}) &= p^{-1}\psi_C\left(\psi_B(b) - b^p\right) \\ &= p^{-1}\psi_C\left(b_0 + pb' - b^p\right) \\ &= p^{-1}\psi_C\left(b_0 - b^p\right) + \psi_C(b') \\ &= p^{-1}\left(\phi_C(b_0) - (\psi_C(b))^p\right) + b'_0 + pc'' \\ &= p^{-1}\left(\phi_C(b_0) - (b_0 + pc')^p\right) + b'_0 + pc'' \\ &= p^{-1}\left(b_0^p + p\beta(b_0) + p\gamma(b_0) - b_0^p - p^2 c'''\right) + b'_0 + pc'' \\ &= \beta(b_0) + \gamma(b_0) - pc''' + b'_0 + pc'' \end{aligned}$$

Since $\gamma(b_0) \in JC \subseteq pC^{(1)}$, the lemma is proved. □

□

3 Connections, p-connections, and their cohomology

In this section we review the notion of p-connection and p-de Rham cohomology and prove a Poincaré lemma for these notions. We use this result to show that they can be used as the basis for a cohomology theory, which we later will relate to the cohomology of the prismatic topos.

3.1 Basic definitions

Let Y/S be a morphism of p-adic formal schemes. We write $\Omega^1_{Y/S}$ for the completed sheaf of Kahler differentials on Y/S and recall that the universal derivation $d\colon \mathcal{O}_Y \to \Omega^1_{Y/S}$ fits in the *de Rham complex*

$$(\Omega^{\cdot}_{Y/S}, d) := \mathcal{O}_Y \xrightarrow{d} \Omega^1_{Y/S} \xrightarrow{d} \Omega^2_{Y/S} \xrightarrow{d} \cdots .$$

Multiplying the differential by p, we find the *p-de Rham complex*

$$(\Omega^{\cdot}_{Y/S}, d') := \mathcal{O}_Y \xrightarrow{pd} \Omega^1_{Y/S} \xrightarrow{pd} \Omega^2_{Y/S} \xrightarrow{pd} \cdots$$

Note that if f and g are sections of $\mathcal{O}_Y$, then

$$d'(fg) := pd(fg) = pfdg + pgdf = fd'g + gd'f,$$

i.e., d' satisfies the usual Leibniz rule.

Remark 3.1. Bhatt has suggested the utility of a generalization of the notion of the p-de Rham complex. If Λ is an effective divisor on Y, then the canonical section $\mathcal{O}_Y \to \mathcal{O}_Y(\Lambda)$ defines a map $\Omega^1 \to \Omega^1(\Lambda)$, and we define $d_\Lambda\colon \mathcal{O}_Y \to \Omega^1_{Y/S}(\Lambda)$ to be the composite of d and this inclusion. Noting that $\Lambda^i\left(\Omega^1_{Y/S}(\Lambda)\right) \cong \Omega^i_{Y/S}(\Lambda^i)$, one checks that d_Λ extends naturally to maps $d_\Lambda\colon \Omega^i_{Y/S}(\Lambda^i) \to \Omega^{i+1}_{Y/S}(\Lambda^{i+1})$. The successive composition of any two of these is zero, and we call the resulting complex the "Λ-de Rham complex of Y/S," which we denote by $\Omega^{\cdot}_{Y/S}(\Lambda^{\cdot})$,. If the ideal of the divisor Λ is principally generated by $\lambda \in \mathcal{O}_Y$ and $d\lambda = 0$, then there is an isomorphism of complexes:

$$\begin{tikzcd}
\mathcal{O}_Y \arrow[r, "d_\Lambda"] \arrow[d, "\mathrm{id}"'] & \Omega^1_{Y/S}(\Lambda) \arrow[r, "d_\Lambda"] \arrow[d, "\lambda"] & \Omega^2_{Y/S}(\Lambda^2) \arrow[r] \arrow[d, "\lambda^2"'] & \cdots \\
\mathcal{O}_Y \arrow[r, "\lambda d"] & \Omega^1_{Y/S} \arrow[r, "\lambda d"] & \Omega^2_{Y/S} \arrow[r] & \cdots
\end{tikzcd} \tag{3.1}$$

Thus in this case the complex $\Omega^{\cdot}_{Y/S}(\Lambda^{\cdot})$ is isomorphic to the λ-de Rham complex.

Recall that a *connection* on a sheaf of $\mathcal{O}_Y$-modules E is an additive homomorphism $\nabla\colon E \to \Omega^1_{Y/S} \otimes E$ such that $\nabla(fe) = f\nabla e + df \otimes e$ for $f \in \mathcal{O}_Y$ and $e \in E$, that such a ∇ induces maps $\Omega^i_{Y/S} \otimes E \to \Omega^{i+1}_{Y/S} \otimes E$ satisfying $\nabla(\omega \otimes e) = d\omega \otimes e + (-1)^i \omega \wedge \nabla e$, and that ∇ is *integrable* if $\nabla^2 = 0$. In this case one can form the *de Rham complex of* (E, ∇):

$$(\Omega^{\cdot}_{Y/S} \otimes E, d) := E \xrightarrow{\nabla} \Omega^1_{Y/S} \otimes E \xrightarrow{\nabla} \Omega^2_{Y/S} \otimes E \xrightarrow{\nabla} \cdots .$$

Let us also recall the notion of a *p-connection.*

Definition 3.2. *If Y/S is a morphism of p-adic formal schemes and E is a sheaf of $\mathcal{O}_Y$-modules, a p-connection on E is an $\mathcal{O}_S$-linear map*

$$\nabla' \colon E \to \Omega^1_{Y/S} \otimes E$$

such that $\nabla'(ae) = d'a \otimes e + a\nabla'(e)$ for all $a \in \mathcal{O}_Y$ and $e \in E$, where $d'a := pda$. A p-connection extends uniquely to a family of maps

$$\nabla' \colon \Omega^i_{Y/S} \otimes E \to \Omega^{i+1}_{Y/S} \otimes E$$

such that $\nabla'(\omega \otimes e) = d'\omega \otimes e + \omega \wedge \nabla'(e)$ for all $\omega \in \Omega^i_{Y/S}$ and $e \in E$. A p-connection is integrable if $\nabla'^2 = 0$. In this case one can form the p-de Rham complex $(E' \otimes \Omega^{\cdot}_{Y/S}, d')$ of (E, ∇').

Example 3.3. If ∇ is a connection on an $\mathcal{O}_Y$-module E, then $p\nabla$ is a p-connection. We call $(E, p\nabla)$ the *p-transform* of (E, ∇). There is a natural morphism of complexes:

$$b \colon (\Omega^{\cdot}_{Y/S} \otimes E, d) \to (\Omega^{\cdot}_{Y/S} \otimes E, pd) \tag{3.2}$$

given by

$$\begin{tikzcd}
E \arrow[r, "d"] \arrow[d, "\mathrm{id}"'] & \Omega^1_{Y/S} \otimes E \arrow[r, "d"] \arrow[d, "p"] & \Omega^2_{Y/S} \otimes E \arrow[r, "d"] \arrow[d, "p^2"] & \cdots \\
 & & & \cdots \\
E \arrow[r, "pd"] & \Omega^1_{Y/S} \otimes E \arrow[r, "pd"] & \Omega^2_{Y/S} \otimes E) \arrow[r, "pd"] & \cdots
\end{tikzcd}$$

We note that this morphism is an isogeny: if Y/S has dimension m, there is a morphism of complexes:

$$\tilde{b} \colon (\Omega^{\cdot}_{Y/S} \otimes E, pd) \to (\Omega^{\cdot}_{Y/S} \otimes E, d), \tag{3.3}$$

given by:

$$\begin{tikzcd}
E \arrow[r, "pd"] \arrow[d, "p^m"'] & \Omega^1_{Y/S} \otimes E \arrow[r, "pd"] \arrow[d, "p^{m-1}"] & \Omega^2_{Y/S} \otimes E \arrow[r, "pd"] \arrow[d, "p^{m-2}"] & \cdots \Omega^m_{Y/S} \otimes E \arrow[d, "\mathrm{id}"] \\
E \arrow[r, "d"] & \Omega^1_{Y/S} \otimes E \arrow[r, "d"] & \Omega^2_{Y/S} \otimes E \arrow[r, "d"] & \cdots \Omega^m_{Y/S} \otimes E
\end{tikzcd}$$

Evidently $b \circ \tilde{b}$ and $\tilde{b} \circ b$ are multiplication by p^m.

Let us also note that, if E is p-torsion free, then b factors through an isomorphism

$$(E \otimes \Omega^{\cdot}_{Y/S}, d) \cong L\eta(E \otimes \Omega^{\cdot}_{Y/S}, pd).$$

Indeed, since the differentials of the p-de Rham complex of (E, pd) are divisible by p, $L\eta(E \otimes \Omega^{\cdot}_{Y/S}, pd)$ in degree i is just $p^i E \otimes \Omega^i_{Y/S}$, the image of b in that degree.

We shall give a geometric interpretation of this trivial-looking construction in Theorem 6.29.

We omit the proof of the following analog of Katz's construction of Gauss-Manin connection.

Proposition 3.4. *Let $g\colon Y \to Z$ be a p-completely smooth morphism of p-completely smooth p-adic formal schemes over S and (E, ∇') a sheaf of $\mathcal{O}_Y$-modules with integrable p-connection relative to S. Then the composition*

$$\nabla'\colon E \to \Omega^1_{Y/S} \otimes E \to \Omega^1_{Y/Z} \otimes E$$

defines a p-connection relative to Z. Furthermore, the sheaves

$$E^j := R^j g_*(\Omega^{\cdot}_{Y/Z} \otimes E)$$

carry a natural p-connection relative to Z, and there is a spectral sequence with

$$E_2^{i,j} = H^i(Z, \Omega^{\cdot}_{Z/S} \otimes E^j) \Rightarrow H^{i+j}(Y, \Omega^{\cdot}_{Y/S} \otimes E). \quad \square$$

3.2 Dilatations and p-connections

The twisted differential d' has a geometric interpretation, analogous to the well-known interpretation of the usual differential d. Let $Y(1) := Y \times_S Y$, and let J be the ideal of the diagonal embedding, $\Delta_{Y/S}\colon Y \to Y(1)$. Recall that $\Omega^1_{Y/S} \cong J/J^2$, and that df is the class of $p_2^* f - p_1^* f$ under this identification.

To treat d', suppose that Y and S are p-torsion free, let $\mathbb{D}_Y(1) := \mathbb{D}_{Y_1}(Y(1))$, and consider the commutative diagram:

$$\begin{tikzcd}
 & \mathbb{D}_Y(1) \arrow[d, "\pi"] \\
Y \arrow[ur, "\tilde{\Delta}_{Y/S}"] \arrow[r, "\Delta_{Y/S}"] & Y(1),
\end{tikzcd}$$

where $\tilde{\Delta}_{Y/S}$ is the map induced from the universal property of the dilatation as discussed in Proposition 2.12. Let $\tilde{J}$ be the ideal of the locally closed immersion $\tilde{\Delta}_{Y/S}$. Proposition 2.12 implies that ρ induces an isomorphism $\overline{\rho}$ fitting in a diagram:

$$\begin{tikzcd}
J/J^2 \arrow[r, "\overline{\rho}"] & \tilde{J}/\tilde{J} \arrow[d, "p"] \arrow[dr, "\cong"] & \\
\mathcal{O}_Y \arrow[u, "d"] \arrow[ur, "d'"] \arrow[r] & J/J^2 \arrow[r, "\cong"] & \Omega^1_{Y/S}
\end{tikzcd} \tag{3.4}$$

in the notation of Remark 3.1. We find an isomorphism of complexes, in which the vertical arrows are induced by the exterior powers of $\overline{\rho}$:

$$\begin{tikzcd}
\mathcal{O}_Y \arrow[r, "d'"] & \tilde{J}/\tilde{J}^2 \arrow[r, "d'"] & \Lambda^2 \tilde{J}/\tilde{J}^2 & \cdots \\
\mathcal{O}_Y \arrow[r, "pd"] \arrow[u, "\mathrm{id}"] & \Omega^1_{Y/S} \arrow[r, "pd"] \arrow[u, "\cong"'] & \Omega^2_{Y/S} \arrow[u, "\cong"'] & \cdots
\end{tikzcd} \tag{3.5}$$

These constructions can be used to amplify Mazur's interpretation of the Cartier isomorphism.

Proposition 3.5. *Let $Y \to S$ be a p-completely smooth morphism of formal ϕ-schemes, and let $P^1_{Y/S}$ (resp. $D^1_{Y/S}$) be the first infinitesimal neighborhood of $\Delta_{Y/S}$ in $Y(1)$ (resp., of $\tilde{\Delta}_{Y/S}$ in $\mathbb{D}_{Y/S}(1)$). Then the composition*

$$\tilde{\phi}\colon P^1_{Y/S} \subseteq Y(1) \xrightarrow{\phi(1)} Y(1)$$

defines a p-adic enlargement of Y over $Y(1)$ and factors uniquely through a map $\tilde{F}\colon P^1_{Y/S} \to D^1_{Y/S}$, and there is a commutative diagram

$$\begin{tikzcd}
 & \tilde{J}/\tilde{J}^2 \arrow[r, "\tilde{F}^*"] & J/J^2 \\
J/J^2 \arrow[ur, "\overline{\rho}"] \arrow[r, "\cong"] & \Omega^1_{Y/S} \arrow[u] \arrow[r, "\zeta"] & \Omega^1_{Y/S}, \arrow[u, "\cong"']
\end{tikzcd} \tag{3.6}$$

where the top horizontal arrow is induced by $\tilde{F}$, and $p\zeta = d\phi$. □

Proof. We claim that the morphism $\phi(1)\colon Y(1) \to Y(1)$ induced by ϕ fits into a diagram:

$$\begin{tikzcd}
\overline{P}^1_{Y/S} \arrow[r] \arrow[d, dashed, "z"'] & P^1_{Y/S} \arrow[r] \arrow[d, "\tilde{\phi}"'] & Y(1) \arrow[dl, "\phi(1)"] \\
\overline{Y} \arrow[r, "\overline{\Delta}"] & Y(1). &
\end{tikzcd}$$

To see the existence of the dashed arrow, we note that the ideal of $\overline{\Delta}$ is generated by p and the set of sections of $\mathcal{O}_{Y(1)}$ of the form $1 \otimes f - f \otimes 1$ for $f \in \mathcal{O}_Y$. Then

$$\phi^*(1 \otimes f - f \otimes 1) \equiv 1 \otimes f^p - f^p \otimes 1 \equiv (1 \otimes f - f \otimes 1)^p \pmod{p},$$

which belongs to J^2 and hence belongs to the ideal of $\overline{P}^1_{Y/S}$ in $Y(1)$. The arrow z gives $P^1_{Y_1/S}$ the structure of a p-adic enlargement of Y in $Y(1)$ and hence the map $\tilde{\phi}$ factors through $\mathbb{D}_Y(1)$ as claimed. This map takes the diagonal of $P^1_{Y/S}$ to the diagonal section of $\mathbb{D}_Y(1)$) and hence induces a map $\tilde{F}\colon P^1_{Y/S} \to D^1_{Y/S}$ as claimed. It is then immediate to check the commutativity of the diagram 3.6. $\square$

Let (E, ∇) be a module with integrable connection on Y/S. Its reduction modulo p is a module with integrable connection on $\overline{Y}/S$, whose *p-curvature* can be viewed as a linear map

$$\psi\colon \overline{E} \to F^*\Omega^1_{\overline{Y}'/S} \otimes \overline{E}$$

where $F\colon \overline{Y} \to \overline{Y}'$ is the relative Frobenius morphism [17]. The map ψ induces maps $F^*\Omega^i_{\overline{Y}'/S} \otimes E \to F^*\Omega^{i+1}_{\overline{Y}'/S} \otimes \overline{E}$ which satisfy the integrability condition $\psi^2 = 0$, which is equivalent to the fact that the dual map $F^*T_{\overline{Y}'/S} \to \operatorname{End}\overline{E}$ extends to an algebra homomorphism $S^{\cdot}F^*T_{\overline{Y}'/S} \to \operatorname{End}\overline{E}$.

Let (E, ∇') be a module with integrable p-connection on Y/S. Then the reduction modulo p of ∇' is a linear map

$$\theta'\colon \overline{E} \to \Omega^1_{Y/S} \otimes \overline{E};$$

equivalently a linear map $T_{Y/S} \to \operatorname{End}(\overline{E})$. The integrability guarantees that the image of $T_{\overline{Y}/S}$ in $\operatorname{End}(\overline{E})$ is contained in a commutative subalgebra, *i.e.*that it extends to an algebra homomorphism $S^{\cdot}T_{\overline{Y}/S} \to \operatorname{End}(\overline{E})$. Such a map is often called a *Higgs field* on $\overline{E}$. A Higgs field is said to be *quasi-nilpotent* if for every local section e of $\overline{E}$, there is an n such that $S^nT_{\overline{Y}/S}$ annihilates e. We say that a p-connection is *quasi-nilpotent* if its reduction modulo p has this property, and that a connection is quasi-nilpotent if its p-curvature does.

3.3 Connections and p-connections on envelopes

If X is a closed subscheme of a smooth scheme Y over $\mathbf{C}$, a fundamental ingredient in the theory of de Rham cohomology is that fact that the differential of the de Rham complex of $Y/\mathbf{C}$ extends naturally to the formal completion $\hat{Y}$ along X, or, equivalently, that its structure sheaf $\mathcal{O}_{\hat{Y}}$, viewed as a sheaf of $\mathcal{O}_Y$-algebras, admits an integrable connection compatible with its structure as an $\mathcal{O}_Y$-algebra. The crystalline incarnation of this fact says that if Y/W is a smooth and $X \subseteq Y$ is a closed subscheme, then the structure sheaf of the divided power envelope $\mathbb{P}_X(Y)$, viewed as a sheaf of $\mathcal{O}_Y$-algebras, admits an integrable and quasi-nilpotent connection. In this section we explain that analogs hold true for p-adic dilatations and prismatic envelopes, but one must replace connections by p-connections. We will give a more conceptual geometric proof of the these results later, in section §5.1.

Proposition 3.6. *Let Y/S be a p-completely smooth morphism of p-torsion free p-adic formal schemes and let $i\colon X \to \overline{Y}$ be a closed immersion.*

1. *The p-connection $d' := pd$ of $\mathcal{O}_Y$ extends uniquely to a multiplicative integrable and quasi-nilpotent p-connection d' on the $\mathcal{O}_Y$-algebra $\mathcal{O}_{\mathbb{D}_X(Y)}$.*

2. *If Y is a formal ϕ-scheme and $X \to \overline{Y}$ is a regular immersion, then d' extends uniquely to a multiplicative integrable and quasi-nilpotent p-connection on the $\mathcal{O}_Y$-algebra $\mathcal{O}_{\mathbb{A}_X(Y)}$. This p-connection is compatible with ϕ, in the sense that the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{A}_X(Y)} \arrow[r, "d'"] \arrow[d, "\phi^\sharp"'] & \mathcal{O}_{\mathbb{A}_X(Y)} \otimes \Omega^1_{Y/S} \arrow[d, "\phi^\sharp \otimes \phi^*"] \\
\phi_*(\mathcal{O}_{\mathbb{A}_X(Y)}) \arrow[dr, "\phi_*(d')"'] & \phi_*(\mathcal{O}_{\mathbb{A}_X(Y)}) \otimes_{\mathcal{O}_Y} \phi_*(\Omega^1_{Y/S}) \arrow[d] \\
 & \phi_*\left((\mathcal{O}_{\mathbb{A}_X(Y)}) \otimes_{\mathcal{O}_Y} \phi_*(\Omega^1_{Y/S})\right)
\end{tikzcd}$$

Proof. For (1), we suppose without loss of generality that Y is affine, with $Y/S = \mathrm{Spf}(B/R)$. Let I be the ideal defining X and and let B' be the uncompleted version of the p-adic dilatation of I. Then $B' \subseteq B_{\mathbf{Q}} := \mathbf{Q} \otimes B$ and is generated as a B-algebra by elements of the form $\rho(x)$ with $x \in I$. The p-connection $d'_B := pd_B$ extends uniquely to an integrable p-connection $d'_{\mathbf{Q}} \colon B_{\mathbf{Q}} \otimes \Omega^1_{B/R}$. We claim that $d'_{\mathbf{Q}}$ maps B' to $B' \otimes \Omega^1_{B/R}$. In fact, for $x \in I$:

$$d'_{\mathbf{Q}}\rho(x) = pd_{\mathbf{Q}}\rho(x) = d_{\mathbf{Q}}(p\rho(x)) = dx \in \Omega^1_{B/R}. \tag{3.7}$$

Since d' satisfies the Leibnitz rule and B' is generated over B by $\rho(I)$, it follows that $d'_{\mathbf{Q}}b' \in \Omega^1_B \otimes B'$ for all $b' \in B'$. We write $d' \colon B' \to \Omega^1_{B/R} \otimes B'$ for the induced map, which is evidently an integrable p-connection on B' and is the unique such extending the canonical p-connection on B and which is multiplicative. This p-connection extends uniquely to the p-adic completion of B' by continuity. To see the quasi-nilpotence, choose an element τ of $T_{B/R}$, and observe from equation (3.7) that $d'_\tau(\rho(x)) \in B$ for all $x \in I$. Then $(d'_\tau)^2(\rho(x)) \in pB$, and, since B' is generated as a B-algebra by $\rho(I)$, the multiplicativity of d'_τ implies that it is quasi-nilpotent modulo p.

Now suppose that ϕ endows Y with the structure of a formal ϕ-scheme. We are assuming also that X is regularly immersed in $\overline{Y}$ for convenience, since it will allow us to apply the explicit description of $\mathbb{A}_X(Y)$ from Proposition 2.21. With the notation there, we calculate:

$$\begin{aligned}
pd't_{i,j+1} &= d'\delta^j(x_i) - d'(t^p_{i,j}) \\
&= pd\delta^j(x_i) - pt^{p-1}_{i,j}d't_{i,j} \\
d't_{i,j+1} &= d\delta^j(x_i) - t^{p-1}_{i,j}d't_{i,j}
\end{aligned} \tag{3.8}$$

This shows that the subring B^∞ of $\mathbf{Q} \otimes B$ is stable under the p-connection $d'_{\mathbf{Q}}$, and also that the resulting p-connection on B^∞ is quasi-nilpotent, since $d'\delta^j(x_i)$

is divisible by p. Furthermore, the diagram in statement (2) commutes with $B_{\mathbf{Q}}$ in place of B^∞ by functoriality, and hence also with its subring B^∞. These results extend to the p-adic completion by continuity. $\square$

Remark 3.7. We should point out that the more general prismatic envelopes constructed in Theorem 2.19 may not inherit a p-connection. For example, let $Y := \operatorname{Spf} \mathbf{Z}_p[x]\hat{}$, with $\psi(x) := x^p + x$, and let X be the closed subscheme of Y defined by (p, x). Then ψ restricts to the Frobenius endomorphism of X, and it is not difficult to compute that $\mathbb{\Delta}_X(Y)$ is given by the p-adic completion of the ring $\mathbf{Z}_p[x, s_1, s_2, \ldots]/(ps_1 = x, ps_2 = s_1^p - s_1, ps_3 = s_2^p + s_2, \cdots)$. But then $d's_1 = dx$ and $pd's_2 = (ps_1^{p-1} - 1)dx$, so $d's_2$ cannot belong to $B^\infty \otimes \Omega^1_B$. For a more geometric explanation of the difficulty, see Remark 5.4.

Example 3.8. Suppose that $(x_1, \ldots, x_n)$ is a local system of coordinates for Y/S, that $X \subseteq Y$ is defined by $(x_1, \ldots, x_r, p)$, and that $\phi(x_i) \in (x_1, \ldots, x_r, p^2)$ for $1 \leq i \leq r$. Then by (1) of Theorem 2.27, $\mathbb{\Delta}_X(Y)$ is the p-adic completion of $\mathcal{O}_Y\langle t_1, \ldots, t_r\rangle$, where $pt_i = x_i$. Then $d't_i = pdt_i = dx_i$, and, more generally, $d't_i^{[n]} = t_i^{[n-1]}dx_i$.

Remark 3.9. Our next goal is to define a suitable notion of module with prismatic connection. Suppose again that Y/S is a p-completely smooth morphism of formal ϕ-schemes and that $X \to \overline{Y}$ is a regular immersion. Let $(\mathbb{\Delta}_X(Y), z_Y, \pi_Y)$ denote the prismatic envelope of X in Y. We shall sometimes find it convenient to work with sheaves on Y, sometimes with sheaves on $\mathbb{\Delta}_X(Y)$, and sometimes with sheaves on X. Let us explain how we can shift among these points of view. As we have seen, the projection $\pi_Y \colon \mathbb{\Delta}_X(Y) \to Y$ is an affine morphism of formal ϕ-schemes. Since formation of prismatic envelopes commutes with p-completely flat base change $Y' \to Y$, the sheaf $\pi_*(\mathcal{O}_{\mathbb{\Delta}_X(Y)})$ is p-completely quasi-coherent, in the sense of Definition 7.20. Moreover, thanks to a (relative version of) Proposition 7.21, there is a natural equivalence from the category of sheaves of p-completely quasi-coherent $\mathcal{O}_Y$-modules on Y endowed with a (compatible) $\mathcal{O}_{\mathbb{\Delta}_X(Y)}$-module structure, and the category of p-completely quasi-coherent $\mathcal{O}_{\mathbb{\Delta}_X(Y)}$-modules on $\mathbb{\Delta}_X(Y)$. We shall allow ourselves to identify these categories when we feel that no confusion should result. Note also that, if E is a p-completely quasi-coherent sheaf of $\mathcal{O}_{\mathbb{\Delta}_X(Y)}$-modules, Proposition 7.21 tells us that $R^q\pi_{Y*}(E)$ vanishes for $q > 0$, so that cohomology computed with these two points of view coincide.

Next, observe that the underlying topological space of $\mathbb{\Delta}_X(Y)$ is $\overline{\mathbb{\Delta}}_X(Y)$, and that $z_Y \colon \overline{\mathbb{\Delta}}_X(Y) \to X$ is an affine morphism of schemes. Thus, if E is a p-completely quasi-coherent sheaf of $\mathcal{O}_{\mathbb{\Delta}_X(Y)}$-modules, by Proposition 7.21 also implies that $R^q z_{Y*}(E)$ vanishes for $q > 0$. Furthermore the support of any sheaf of $\pi_{Y*}(\mathcal{O}_{\mathbb{\Delta}_X(Y)}$-modules is contained in the subset X; and if F is any sheaf on Y with this property, then the natural map $F \to i_*i^{-1}F$ is an isomorphism, so we may indifferently view such a sheaf as living on on Y or on X.

Definition 3.10. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes, let $X \to \overline{Y}$ be a regular closed immersion, and let E be a p-completely* *quasi-coherent sheaf of $\mathcal{O}_{\mathbb{A}_X(Y)}$-modules (see the previous paragraph).*

1. *An integrable p-connection $\nabla' \colon E \to \Omega^1_{Y/S} \otimes E$ on E is compatible if*

$$\nabla'(ae) = d'a \otimes e + a\nabla'(e).$$

   for every pair of local sections $a \in \mathcal{O}_{\mathbb{A}_X(Y)}, e \in E$.

2. *A module with prismatic p-connection on $(X/Y/S)$ is a p-completely quasi-coherent sheaf of $\mathcal{O}_{\mathbb{A}_X(Y)}$-modules endowed with an integrable, quasi-nilpotent, and compatible p-connection.*

We denote the category of modules with prismatic p-connection on $X/Y/S$ by $\mathrm{MICP}(X/Y/S)$.

The analogous category $\mathrm{MIC}(X/Y/S)$ of p-completely quasi-coherent $\mathcal{O}_{\mathbb{P}_X(Y)}$-modules with compatible quasi-nilpotent connection is discussed, for example, in [6, §6]. In the next section, we shall show that the category $\mathrm{MICP}(X/Y/S)$ is, in a suitable sense, independent of Y.

3.4 The prismatic Poincaré Lemma

Let S be a formal ϕ-scheme and $X/\overline{S}$ a smooth morphism. Although we have not yet discussed the prismatic site, we can use "bare hands" methods to prove the existence of a functorial cohomology theory $H^*_{\mathbb{A}}(X/S)$, (including a theory of coefficients) that will agree with the cohomology of the prismatic topos $(X/S)_{\mathbb{A}}$. The key is to show that, if X is embedded in p-completely smooth Y/S, the category of modules with prismatic p-connection on $X/Y/S$ is independent of the choice of Y, and that the same independence holds for the cohomology of any such module. We accomplish this using our detailed study of the nature of prismatic envelopes in §2.3. (When the base S is perfect, another approach is possible, by reduction to the usual crystalline theory using the F-transform, but we shall not explain this here.) In the following theorem, we show that the category $\mathrm{MICP}(X/Y/S)$, and its cohomology, is independent of the choice of Y. Here we view an object (E, ∇') of this category as living on the topological space X, as explained in Remark 3.9.

Theorem 3.11 (The prismatic Poincaré Lemma)**.** *Let S be a formal ϕ-scheme, let $X/\overline{S}$ be a smooth morphism, and let $g \colon Y \to Z$ be a morphism of p-completely smooth formal ϕ-schemes over S such that $j := g \circ i$ is again a regular closed immersion.*

1. *The morphism g induces an equivalence of categories:*

$$MICP(X/Z/S) \to MICP(X/Y/S).$$

2. *If (E, ∇') is an object of* $\mathrm{MICP}(X/Z/S)$*, then the natural map*

$$(\Omega^{\cdot}_{Z/S} \otimes E, d') \to (\Omega^{\cdot}_{Y/S} \otimes g^*(E), d')$$

   is a strict quasi-isomorphism.

Proof. The proof will proceed by reducing to calculations in the case of prismatic neighborhoods of a point. Thus the key case occurs when $Y = S$, when g is a closed immersion, which we now denote by $s\colon S \to Z$, and when $X = \overline{S}$. Under these conditions, the category $\mathrm{MICP}(X/Y/S) = \mathrm{MICP}(\overline{S}/S/S)$ is just the category of p-completely quasi-coherent $\mathcal{O}_S$-modules, and the the statement itself reduces to the following lemma.

Lemma 3.12. *Let $h\colon Z \to S$ be a p-completely smooth morphism of formal ϕ-schemes. Assume that h admits a section $s\colon S \to Z$, compatible with ϕ.*

1. *The functor*

$$s^*\colon \mathrm{MICP}(\overline{S}/Z/S) \to \mathrm{MICP}(\overline{S}/S/S) : (E, \nabla') \mapsto E_S$$

is an equivalence, with quasi-inverse

$$h^*\colon \mathrm{MICP}(\overline{S}/S/S) \to \mathrm{MICP}(\overline{S}/Z/S) : E' \mapsto (\mathcal{O}_{\Delta_S(Z)} \otimes E', d' \otimes \mathrm{id}).$$

In particular, the natural maps

$$(E^{\nabla'} \hat{\otimes}_{\mathcal{O}_S} \mathcal{O}_{\Delta_S(Z)}, \mathrm{id} \otimes d') \to (E, \nabla') \quad \textit{and} \quad E^{\nabla'} \to E_S$$

are isomorphisms.

2. *If (E, ∇') is an object of $\mathrm{MICP}(\overline{S}/Z/S)$, the natural map*

$$E^{\nabla'} \to (E \otimes \Omega^{\cdot}_{Z/S}, d')$$

is a quasi-isomorphism.

Proof. Before embarking on the proof, let us note that, since $(S, \mathrm{id}_{\overline{S}}, \mathrm{id}_S)$ is an $\overline{S}$ prism over S, the map $\Delta_S(Z) \to S$ admits a section $\tilde{s}'\colon S \to Z$, and the functor s^* in statement (1) takes an object (E, ∇') of $\mathrm{MICP}(\overline{S}/Z/S)$ to its pullback via $\tilde{s}'$. As we saw in statement (1) of Theorem 2.27, the prismatic envelope $\Delta_S(Z)$ of S in Z can be identified with the divided power envelope $\mathbb{P}_{\tilde{S}}(\mathbb{D}_S(Z))$ of the section $\tilde{S}$ of the p-adic dilatation of $\overline{S}$ in Z defined by s. Working locally, we may assume that $Z \to S$ is $\mathrm{Spf}(B \to C)$ and that there is a set of local coordinates $(x_1, \ldots, x_r)$ for Z/S which generates the ideal of S in Z. Then the formal completion of S along Z is $\mathrm{Spf}\, B[[x_1, \ldots, x_r]]$, and $\mathbb{D}_S(Z)$ is $\mathrm{Spf}\, B[t_1, \ldots, t_r]\hat{}$, with x_i mapping to pt_i. Thus the map $\mathbb{D}_S(Z) \to S$ is also formally smooth.

Continuing to work locally with the aid of coordinates, we can write $\mathbb{P}_{\tilde{S}}(\mathbb{D}_S(Z))$ as the formal spectrum of the p-adic completion of the PD-algebra $B\langle t_1, \ldots, t_r\rangle$. Since $pt_i = x_i$ and $d'x_i = pdx_i$, it follows that $d't_i = dx_i$, and since $\mathcal{O}_{\Delta_S(Z)}$ is p-torsion free, that $d't_i^{[n]} = t_i^{[n-1]}dx_i$ for all n. (See Example 3.8.) Because the p-connection ∇' on E' is compatible with the p-connection on $\mathcal{O}_{\Delta_S(Z)}$, it behaves like a *connection* on E', viewed now as a $B\langle t_1, \ldots, t_r\rangle\hat{}$-module; furthermore, it is quasi-nilpotentI as a connection. In this light the lemma becomes a standard fact from the usual crystalline theory. Let us nevertheless write some details.

Let $\partial_i := \nabla'_{\partial/\partial x_i}$, and let

$$r := \sum_I (-t)^{[I]}\partial^I \in \mathrm{End}_{\mathcal{O}_S}(E). \tag{3.9}$$

This formula converges p-adically because of the quasi-nilpotence of ∇': if $e \in E$ and $n \in \mathbf{N}$, then $\partial^I e \in p^n E$ for $|I| \gg 0$. Then a standard formal verification shows that

$$\begin{array}{rcl} \nabla \circ r & = & 0 \\ \sum_I t^{[I]} r \circ \partial^I & = & \mathrm{id}. \end{array} \tag{3.10}$$

It follows that $E^{\nabla} \to E_{|S}$ is surjective, since for every $e \in E$, $r(e) \in E^{\nabla}$ and $r(e) \equiv e$ modulo the ideal of S in $\mathbb{A}_S(Z)$.

The equations (3.10) implies that every element e of E can be written as a p-adically convergent sum

$$e = \sum t^{[I]} e_I$$

with each $e_I \in E^{\nabla'}$. We claim that any such expression is unique, *i.e.*, that necessarily $e_I = r(\partial^I e)$. To see this, we calculate

$$\begin{array}{rcl} r\left(\sum t^{[I]}e_I\right) & = & \sum_J (-t)^{[J]}\partial^J\left(\sum_I t^{[I]}e_I\right) \\ & = & \sum_{I,J}(-t)^{[J]}t^{[I-J]}e_I \\ & = & \sum_{I,J}\binom{I}{J}(-1)^J t^{[I]}e_I \\ & = & \sum_I \sum_{J=0}^{I}\binom{I}{J}(-1)^{|J|}t^{[I]}e_I \\ & = & \sum_I (1-1)^{|I|}t^{[I]}e_I \\ & = & e_0 \end{array}$$

Applying this same calculation to $\partial^I e$, we verify that $r(\partial^I e) = e_I$ for all I. This proves that the map $s^*(E^{\nabla'}) \to E$ is an isomorphism. It follows that $E^{\nabla'} \to E_S$ is also an isomorphism, because $s^*h^* = \mathrm{id}$. This completes the proof of statement (1). [4]

We continue in our localized situation to address statement (2). Statement (1) implies that we have a natural isomorphism:

$$(E \otimes \Omega^{\cdot}_{Z/S}, d') \cong (E^{\nabla'} \otimes_B B\langle t_1, \ldots, t_r\rangle\hat{}, d)$$

[4] should I write $\tilde{s}$?

where d is the usual differential of the de Rham complex of the divided power polynomial algebra. Thus the result follows from the standard divided power Poincaré lemma [6, 6.12]. □

The remainder of the proof of the theorem will be series of reductions to the case handled in Lemma 3.12.

Lemma 3.13. *Theorem 3.11 holds if $Z \to S$ is the identity map and g admits a section s such that $i = s \circ j$.*

Proof. Since Y/S was by assumption p-completely smooth, and in this case g is the structural morphism of Y/S, it follows that g is p-completely smooth. Let $S' := \Delta_X(S)$ and let $Y' := S' \times_S Y$; recalling that since g is p-completely smooth, this fibered product is again p-torsion free, so Y'/S' is again a p-completely smooth morphism of formal ϕ-schemes. The section s induces a section s' of $g' \colon Y' \to S'$, and there is a (2-commutative) diagram:

$$\begin{array}{ccc} g^*\colon \mathrm{MICP}(X/S/S) & \longrightarrow & \mathrm{MICP}(X/Y/S) \\ \downarrow & & \downarrow \\ g'^*\colon \mathrm{MICP}(\overline{S}'/S'/S') & \longrightarrow & \mathrm{MICP}(\overline{S}'/Y'/S'). \end{array}$$

Lemma 3.12, tells us that g' is an equivalence, compatible with cohomology. It follows from the definitions that the category $\mathrm{MICP}(\overline{S}'/S'/S')$ is just the category of p-completely quasi-coherent S'-modules; similarly, $\mathrm{MICP}(X/S/S)$ is the category of p-completely quasi-coherent $\Delta_X(S)$-modules. Since $S' = \Delta_X(S)$, the left vertical arrow is an equivalence—in fact, just the identity functor. On the other hand, an object of $\mathrm{MICP}(X/Y/S)$ is a p-completely quasi-coherent sheaf E of $\mathcal{O}_{\Delta_X(Y)}$-modules equipped with with a compatible quasi-nilpotent p-connection

$$\nabla' \colon E \to \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} E,$$

and an object of $\mathrm{MICP}(\overline{S}'/Y'/S')$ is a p-completely quasi-coherent sheaf of $\mathcal{O}_{\Delta_{S'}(Y')}$-modules E together with a quasi-nilpotent p-connection:

$$\nabla' \colon E \to \Omega^1_{Y'/S'} \otimes_{\mathcal{O}_{Y'}} E.$$

But $S' = \Delta_X(S)$, and Lemma 2.32 tells us that $\Delta_{S'}(Y') \cong \Delta_X(Y)$. Furthermore, if E is an $\mathcal{O}_{\Delta_X(Y)}$-module, we have

$$\Omega^1_{Y'/S'} \otimes_{\mathcal{O}_{Y'}} E \cong \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} \mathcal{O}_{Y'} \otimes_{\mathcal{O}_{Y'}} E \cong \Omega^1_{Y/S} \otimes_{\mathcal{O}_Y} E,$$

we see that the right vertical arrow is also the identity functor. Furthermore, for any object E of this category, the respective p-de Rham complexes $E \otimes \Omega^{\cdot}_{Y/S}$ and $E \otimes \Omega^{\cdot}_{Y'/S}$ are the same. This concludes the proof of the lemma. □

The next step requires localization in the p-completely flat topology.

Lemma 3.14. *Theorem 3.11 holds if $Z \to S$ is the identity map.*

Proof. Thanks to the descent results in section §7, this can be checked p-completely flat locally on S, so we may and shall assume that S is affine. After replacing Y by its formal completion along X, we may find a section of g which is compatible with the embeddings of X in Y and S but not necessarily with the ϕ-structures. However, by Proposition 1.12, there exist a p-completely flat covering $\tilde{S} \to S$ of formal ϕ-schemes and a section $\tilde{s} \colon \tilde{S} \to Y \times_S \tilde{S}$ which is l compatible with the embeddings as well at the ϕ-structure. Then Lemma 3.13 applies. $\square$

Lemma 3.15. *Theorem 3.11 holds if g p-completely smooth.*

Proof. In this case, we can form the relative cohomology sheaves

$$E^q := R^q g_*(E \otimes \Omega^{\cdot}_{Y/Z}, d'),$$

which here are just the cohomology sheaves of the complex $(E \otimes \Omega^{\cdot}_{Y/Z}, d')$. (Here we work modulo each power of p, not in the exact category of p-adic sheaves described in §7.) Lemma 3.14 applies to the morphism g, with S replaced by Z, and tells us that in fact these sheaves vanish for $q > 0$ and that the natural map $g^*E^0 \to E$ is an isomorphism. As we saw in Proposition 3.4, these sheaves admit a Gauss-Manin p-connection, and there is a spectral sequence with

$$E_2^{p,q} \cong R^p(E^q \otimes \Omega^{\cdot}_{Z/S}, d') \Rightarrow R^{p+q}(E \otimes \Omega^{\cdot}_{Y/S}, d').$$

Since $E^q = 0$ for $q > 0$, in fact we have isomorphisms

$$R^n(E \otimes \Omega^{\cdot}_{Y/S}, d') \cong R^n(E^0 \otimes \Omega^{\cdot}_{Z/S}).$$

Since also $g^*E^0 \to E$ is an isomorphism, we see that the map

$$(E^0 \otimes \Omega^{\cdot}_{Z/S}, d') \to (E \otimes \Omega^{\cdot}_{Y/S}, d')$$

is a quasi-isomorphism, as claimed in the theorem. The fact that $g^*E \to E^0$ is an isomorphism also implies that g induces the claimed equivalence of categories. $\square$

We can now prove the general case of Theorem 3.11. First note that if f and g are composable morphisms and the result holds for any two of f, g, and $g \circ f$, then it also holds for all three, as is easily seen. Now in the situation of the theorem, consider the graph $\Gamma_g \colon Y \to Y \times_S Z$ of g. The projections $\pi_Y \colon Y \times_S Z \to Y$ and $\pi_Z \colon Y \times_S Z \to Y$ are p-completely smooth, and Lemma 3.15 implies that the theorem is true for each of these. Since $\pi_Y \circ \Gamma_g = \mathrm{id}_Y$, the result for π_Y and for id_Y imply the result for Γ_g. Since $g = \pi_Z \circ \Gamma_g$, the result for π_Z and for Γ_g imply the result for g. $\square$

Corollary 3.16. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes and let $X \to \overline{Y}$ be a closed immersion, with $X/\overline{S}$ smooth. Then if (E, ∇) is an object of* $\mathrm{MICP}(X/Y/S)$ *and $n > 0$, the cohomology sheaves $\mathcal{H}^q(\Omega^{\cdot}_{Y/S} \otimes E_n, d')$ vanish if $q > \dim(X/S)$.*

Proof. We can check this statement locally on X, and so we may and shall assume that there is a p-completely smooth formal ϕ-scheme $\tilde{X}/S$ lifting $X/\overline{S}$. The corollary is certainly true with $\tilde{X}$ in place of Y/S. Now we have morphisms of embeddings:

$$(\tilde{X}, X) \leftarrow (\tilde{X} \times Y, X) \to (Y, X)$$

and so Theorem 3.11 allows us to carry the result for $(\tilde{X}, X)$ over to (Y, X). □

4 The F-transform

The purpose of this section is to explore the relationship between the prismatic theory and some of its predecessors, especially attempts to develop p-adic non-abelian Hodge theory. The material in this section is not strictly needed for the proof of our main comparison theorems, but it will have some applications to the prismatic theory, for example in section §6.6.

4.1 Shiho's equivalence of categories

We recall the following construction, which has been used by many authors, for example, in [24], [26], [10], and [28]. The setup is slightly more general than what is provided by ϕ-schemes.

Proposition 4.1. *Let S be a p-torsion free p-adic formal scheme and Y/S a p-completely smooth morphism of p-adic formal schemes. Suppose that we are given a morphism $F \colon Y \to Y'$ lifting the relative Frobenius morphism $F_{Y/S} \colon Y_1 \to Y'_1$, and let*

$$\zeta := p^{-1}F^* \colon \Omega^1_{Y'/S} \to F_*\Omega^1_{Y/S}.$$

If (E', ∇') is an integrable p-connection on Y', there is a unique integrable connection ∇ on F^*E with the property that

$$\nabla(1 \otimes e') = (\zeta \otimes \mathrm{id})(\nabla'(e'))$$

for every local section e' of E'. We call (F^*E', ∇) the F-transform of (E', ∇'). □

Connections, p-connections, and the F-transform can also be interpreted geometrically. As is well-known, to give a connection on an $\mathcal{O}_Y$-module E is the same as to give an isomorphism of $\mathcal{O}_{P^1_{Y/S}}$-modules $\epsilon \colon p_2^*E \to p_1^*E$ whose restriction to the diagonal is the identity. The isomorphism ϵ and the connection ∇ are related by the formula

$$\epsilon(p_2^*e) = p_1^*e + \nabla e.$$

Similarly, giving a p-connection on an $\mathcal{O}_Y$-module E is equivalent to giving an isomorphism of $\mathcal{O}_{D^1_{Y/S}}$-modules $p_2^*E \to p_1^*E$ whose restriction to the diagonal is the identity. Using these interpretations of connections and p-connections, we find the following geometric interpretation of the F-transform. The proof is immediate from the definitions and diagram (3.6) of Proposition 3.5.

Proposition 4.2. *Let $F\colon Y \to Y'$ be a lift of the relative Frobenius morphism of a smooth p-adic formal scheme Y/S, and let $\epsilon'\colon {p'}_2^*E' \to {p'}_1^*E'$ be the isomorphism of $\mathcal{O}_{D^1_{Y'/S}}$-modules corresponding to a p-connection ∇' on E'. Let $\tilde{F}\colon P^1_{Y/S} \to D^1_{Y'/S}$ be the morphism* [5] *defined in Proposition 3.5 Then the F-transform of (E', ∇') corresponds to the isomorphism of $\mathcal{O}_{P^1_{Y/S}}$-modules:*

$$\tilde{F}^*(\epsilon')\colon \tilde{F}^*{p'_2}^*E' \to \tilde{F}^*{p'_1}^*E',$$

after the identifications $\tilde{F}^*{p'_i}^* \cong p_i^*F^*$. □

Remark 4.3. Note that the F-transform of E' depends on the choice of Frobenius lifting. Two Frobenius liftings F_1 and F_2 agree modulo p, and since the ideal (p) has a unique PD-structure, an integrable connection on E' would define an isomorphism $F_2^*E' \cong F_1^*E'$, but a p-connection is not enough to do this on its own. This issue was studied in [24], [25], and [28], and we will review it in Proposition 5.8.

The following important theorem is due to A. Shiho [26, 3.1]. As he explains, it is a "descent to level minus one" version of Berthelot's Frobenius descent ideas [4]. We shall review his proof, using the point of view developed in our current context, in subsection 5.2.

Theorem 4.4 (Shiho)**.** *Let $F\colon Y \to Y'$ be as in Proposition 4.1, and let (E, ∇) be the F-transform of an integrable p-connection (E', ∇') on Y'/S. Then ∇ is nilpotent if ∇' is. The F-transform $(E', \nabla') \mapsto (E, \nabla)$ defines an equivalence from the tensor category of nilpotent integrable p-connections on Y' to the tensor category of nilpotent integrable connections on Y.* □

Remark 4.5. Shiho shows in [26] that the F-transform of a quasi-nilpotent p-connection is a quasi-nilpotent connection. We shall give a more precise version of his argument by computing the p-curvature of the F-transform explicitly; see Theorem 4.14 below. We should also point out that the converse result is not true: the F-transform of a non-quasi-nilpotent p-connection can be nilpotent, even zero. In particular, the F-transform functor is not fully faithful on the category of all p-connections.

The following result is strengthening of a special case of a result of Shiho [26, 4.4] relating the cohomology of a module with quasi-nilpotent p-connection to the cohomology of its F-transform. See §7 for the terminology used in the following statement.

[5] The treatment there used the absolute instead of relative Frobenius, but the construction here is essentially the same.

Theorem 4.6. *With the notation of Theorem 4.4, let (E', ∇') be a p-completely quasi-coherent sheaf of $\mathcal{O}_{Y'}$-modules with quasi-nilpotent integrable p-connection and let (E, ∇) be its F-transform. Then the natural morphism of complexes:*

$$\zeta_E^{\cdot} \colon \Omega^{\cdot}_{Y'/S} \otimes E' \to F_*(\Omega^{\cdot}_{Y/S} \otimes E)$$

is a strict quasi-isomorphism.

Proof. First suppose that $(E', \nabla') = (\mathcal{O}_{\overline{Y}}, 0)$. Then $(E, \nabla) = (\mathcal{O}_{\overline{Y}}, d)$, and our assertion is that

$$\zeta_E^{\cdot} \colon (\Omega^{\cdot}_{\overline{Y}'/S}, 0) \to F_*(\Omega^{\cdot}_{\overline{Y}/S}, d)$$

is a quasi-isomorphism. This follows from the fact that in each degree i, ζ_E reduces to the inverse Cartier isomorphism:

$$C^{-1}_{\overline{Y}/S} \colon \Omega^i_{\overline{Y}'/S} \to F_* \mathcal{H}^i(\Omega^i_{\overline{Y}/S}).$$

Now let E' be any sheaf of $\mathcal{O}_{\overline{Y}'}$-modules. Since all the terms of the complex $F_*(\Omega^{\cdot}_{\overline{Y}/S})$, as well as its cohomology sheaves, are locally free over $\mathcal{O}_{\overline{Y}'}$, it follows that the map

$$\zeta_E^{\cdot} \colon (\Omega^{\cdot}_{\overline{Y}'/S} \otimes E', 0) \to F_*(\Omega^{\cdot}_{\overline{Y}/S} \otimes_{\mathcal{O}_{Y'}} E', d \otimes \mathrm{id})$$

is still a quasi-isomorphism. The right side is the de Rham complex of the F-transform of $(E', 0)$. Thus we have proved the theorem whenever E' is annihilated by p and by ∇'.

Note next that, since F is flat, the F-transform preserves exact sequences. Hence if $0 \to (E'_1, \nabla') \to (E'_2, \nabla') \to (E'_3, \nabla') \to 0$ is an exact sequence of modules with p-connection and the theorem is true for any two of the terms, then it is true for the third. Consequently the result also holds if E' has a finite filtration whose associated graded satisfies the claimed result.

Let us now check that the theorem is true whenever E' is annihilated by p. We may work locally, and in particular we may assume that Y is affine. Let $E'_1 := \mathrm{Ker}(\nabla')$, let E'_2 be the inverse image in E' of the kernel of ∇' acting on E'/E'_1, etc. We obtain an increasing filtration $E'_1 \subseteq E'_2 \subseteq \cdots$ on whose associated graded ∇' is trivial. It follows that the result holds for each E'_n. If E' is noetherian, this sequence terminates, and $E'_n = E'$ for $n \gg 0$ because ∇' was assumed to be locally nilpotent. In any case, $E' = \varinjlim E'_n$, and since formation of F-transforms and of cohomology commute with direct limits, the result also holds for E'.

It now follows by induction that the theorem is true for any E' annihilated by a finite power of p. To check the induction step, assume that $p^{n+1}E' = 0$, so that there is an exact sequence:

$$0 \to p^n E' \to E' \to E'/p^n E' \to 0.$$

The first term in this sequence is annihilated by p and is a quotient of E'/pE', hence its p-connection is quasi-nilpotent, and hence the theorem holds for this term. Since the p-connection of the last term is also quasi-nilpotent, the theorem also holds for it, and hence also for E'.

To complete the proof of the theorem, recall that, by definition, a morphism of complexes p-completely quasi-coherent sheaves is a strict quasi-isomorphism if and only if its reduction modulo each power p is a quasi-isomorphism. Since the F-transform commutes with reduction modulo each power of p, and and we have just checked that the result is true for every such reduction, the theorem is proved. $\square$

The following result relates the Frobenius pull-back of a module with p-connection (E', ∇') on Y'/S to the p-transform (explained in Example 3.3) of the F-transform of (E', ∇'). It will allow us to see that the prismatic Frobenius is an isogeny on cohomology.

Theorem 4.7. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes, with relative Frobenius morphism $\phi_{Y/S} \colon Y \to Y'$, and let (E', ∇') be an object of* $\mathrm{MICP}(Y'/S)$.

1. *The $\phi_{Y/S}$-pullback (E'', ∇'') of (E', ∇') as a module with p-connection is the p-transform (see Example 3.3) of the F-transform (E, ∇) of (E', ∇').*

2. *There exists a commutative diagram: of complexes:*

$$\begin{tikzcd}
(\Omega^{\cdot}_{Y'/S} \otimes E', d') \arrow[r, "\zeta^{\cdot}_E"] \arrow[dr, "c"'] & \phi_{Y/S*}(\Omega^{\cdot}_{Y/S} \otimes E, d) \arrow[d, "\phi_{Y/S*}(b)"] \\
& \phi_{Y/S*}(\Omega^{\cdot}_{Y/S} \otimes E'', d'),
\end{tikzcd}$$

where $\zeta^{\cdot}_E$ is the quasi-isomorphism of Theorem 4.6, where b is the morphism defined in Example 3.3, and where c is the canonical map induced by the morphism $\phi_{Y/S}$.

Proof. By construction $E := \phi^*_{Y/S}(E')$; let $\alpha \colon E' \to \phi_{Y/S*}(E)$ be the adjunction map. Then we have a commutative diagram:

$$\begin{tikzcd}
E' \arrow[r, "\nabla'"] \arrow[d, "\alpha"'] & \Omega^1_{Y'/S} \otimes E' \arrow[d, "\zeta \otimes \alpha"] \\
\phi_{Y/S*}(E) \arrow[r, "\nabla"] \arrow[d, "\mathrm{id}"'] & \phi_{Y/S*}(\Omega^1_{Y/S} \otimes E) \arrow[d, "p \otimes \mathrm{id}"] \\
\phi_{Y/S*}(E'') \arrow[r, "\nabla''"] & \phi_{Y/S*}(\Omega^1_{Y/S} \otimes E'').
\end{tikzcd}$$

The composition of the vertical arrows on the left is α, and on the right is $\phi^*_{Y/S} \otimes \alpha$. Then the rectangle commutes, by the definition of the pullback p-connection ∇''. The top square commutes by the definition of the F-transform connection ∇, and it follows that the bottom square commutes as well. This implies that $\nabla'' = p\nabla$, so ∇'' is the p-transform of ∇.

The diagram above extends to a diagram of complexes:

$$\begin{tikzcd}
E' \arrow[r, "d'"] \arrow[d, "\alpha"'] & \Omega^1_{Y'/S} \otimes E' \arrow[r, "d'"] \arrow[d, "p^{-1}\phi^*_{Y/S} \otimes \alpha"] & \Omega^2_{Y'/S} \otimes E \arrow[r, "d'"] \arrow[d, "p^{-2}\phi^*_{Y/S} \otimes \alpha"] & \cdots \\
\phi_{Y/S*}(E) \arrow[r, "d"] \arrow[d, "\mathrm{id}"'] & \phi_{Y/S*}(\Omega^1_{Y/S} \otimes E) \arrow[r, "d"] \arrow[d, "p"] & \phi_{Y/S*}(\Omega^2_{Y/S} \otimes E) \arrow[r, "d"] \arrow[d, "p^2"] & \cdots \\
\phi_{Y/S*}(E) \arrow[r, "pd"] & \phi_{Y/S*}(\Omega^1_{Y/S} \otimes E) \arrow[r, "pd"] & \phi_{Y/S*}(\Omega^2_{Y/S} \otimes E) \arrow[r, "pd"] & \cdots ,
\end{tikzcd}$$

which is the content of statement (2). ☐

Continuing to let Y/S be a p-completely smooth morphism of formal ϕ-schemes, we now suppose that X is closed subscheme of $\overline{Y}$ and is smooth over $\overline{S}$. We have seen in Proposition 3.6 that the (structure sheaf of) the prismatic neighborhood $\mathbb{\Delta}_{X'}(Y')$ carries a p-connection, while (that of) the divided power neighborhood $\mathbb{P}_X(Y)$ carries a connection. We shall see that the F-transform takes the former to the latter.

Recall that if $X^\phi := \phi^{-1}_{Y/S}(X')$, then $\mathbb{\Delta}_{X^\phi}(Y) \cong \mathbb{P}_X(Y)$, and, from Corollary 2.34 and diagram 2.3 in its proof, the commutative diagram:

$$\begin{tikzcd}
\mathbb{\Delta}_{X^\phi}(Y) \cong \mathbb{P}_X(Y) \arrow[r, "\Phi_{Y/S}"] \arrow[d] & \mathbb{\Delta}_{X'}(Y') \arrow[r] \arrow[d] & \mathbb{\Delta}_X(Y) \arrow[d] \\
Y \arrow[r, "\phi_{Y/S}"] & Y' \arrow[r, "\pi"] & Y.
\end{tikzcd} \tag{4.1}$$

whose left square is Cartesian. (If S is perfect or regular, the same is true of the right square.)

Theorem 4.8. *With the notation and hypotheses described in the previous paragraphs, the canonical connection*

$$d \colon \mathcal{O}_{\mathbb{P}_X(Y)} \to \Omega^1_{Y/S} \otimes \mathcal{O}_{\mathbb{P}_X(Y)}$$

on $\mathcal{O}_{\mathbb{P}_X(Y)}$ is the F-transform (4.1) of the canonical p-connection

$$d' \colon \mathcal{O}_{\mathbb{\Delta}_{X'}(Y')} \to \Omega^1_{Y'/S} \otimes \mathcal{O}_{\mathbb{\Delta}_{X'}(Y')}$$

on $\mathcal{O}_{\mathbb{A}_{X'}(Y')}$. *Furthermore, the F-transform defines an equivalence of categories:*

$$\mathrm{MICP}(X'/Y'/S) \to \mathrm{MIC}(X/Y/S),$$

Proof. Since the square in the diagram is Cartesian, the map

$$\tau\colon \phi_{Y/S}^*(\mathcal{O}_{\mathbb{A}_{X'}(Y')}) \to \mathcal{O}_{\mathbb{A}_{X^\phi(Y)}} \cong \mathcal{O}_{\mathbb{P}_X(Y)}$$

is an isomorphism. We claim that this isomorphism identifies the F-transform connection with the standard connection d. We must show that if e' is a local section of $\mathcal{O}_{\mathbb{A}_{X'}(Y')}$, then

$$(\phi_{Y/S}^* \otimes \mathrm{id})(d'e') = pd(\tau(e'))$$

The construction of $\mathbb{A}_{X'}(Y')$ shows that $\mathcal{O}_{\mathbb{A}_{X'}(Y')}$ is topologically generated by elements e' such that $p^m e'$ belongs to $\mathcal{O}_{Y'}$ for some $m > 0$, and so it suffices to check this equality for such e'. Since the target is p-torsion free, in fact it suffices to check the result for elements of $\mathcal{O}_{Y'}$. Since the p-connection on $\mathcal{O}_{\mathbb{A}_{X'}(Y')}$ is compatible with the p-connection pd on $\mathcal{O}_{Y'}$ and the connection on $\mathcal{O}_{\mathbb{P}_X(Y)}$ is compatible with the connection d on $\mathcal{O}_Y$, the result is obvious for such elements.

To prove the last statement of the theorem, observe first that Shiho's theorem already tells us that the F-transform is an equivalence from the category of modules with quasi-nilpotent p-connection on Y'/S to the category of modules with quasi-nilpotent connection on Y/S. The p-complete faithful flatness of $\phi_{Y/S}$ implies that this functor preserves p-complete quasi-coherence. It remains only to check the compatibility condition. Suppose that E' is an object of $\mathrm{MICP}(X'/Y'/S)$ and that E is its F-transform. The $\mathcal{O}_{\mathbb{A}_{X'/S}(Y')}$-module structure of E' is a horizontal map:

$$\mathcal{O}_{\mathbb{A}_{X'}(Y')} \otimes_{\mathcal{O}_{Y'}} E' \to E'$$

which induces a horizontal map

$$\phi_{Y/S}^*(\mathcal{O}_{\mathbb{A}_{X'}(Y')} \otimes_{\mathcal{O}_{Y'}} E') \to \phi_{Y/S}^*(E')$$

Since the F-transform is compatible with tensor products, this gives us a horizontal map

$$(\mathcal{O}_{\mathbb{P}_X(Y)} \otimes_{\mathcal{O}_Y} \phi_{Y/S}^*(E') \to \phi_{Y/S}^*(E'),$$

endowing $\phi_{Y/S}^*(E')$ with the structure of an object of $\mathrm{MIC}(X/Y/S)$. This construction works in the opposite direction, proving the theorem. □

Remark 4.9. As we saw in Theorem 3.11, the category $\mathrm{MICP}(X'/Y'/S)$ is, in a suitable sense, independent of the choice of Y, and the same is true for the category $\mathrm{MIC}(X/Y/S)$. One can check that the F-transform is aso independent of this choice. Namely, if $g\colon Y \to Z$ is a morphism of p-completely smooth formal ϕ-schemes such that $g \circ i$ is again a closed immersion, then one finds a 2-commutative diagram

$$\begin{tikzcd} \mathrm{MICP}(X'/Z'/S) \arrow[r] \arrow[d] & \mathrm{MIC}(X/Z/S) \arrow[d] \\ MICP(X'/Y'/S) \arrow[r] & \mathrm{MIC}(X/Y/S) \end{tikzcd}$$

In section 6.5 we shall give a topos-theoretic version of the F-tranform that does not refer to any embedding.

4.2 A kernel for the F-transform

Let Y/S be a p-completely smooth morphism of ϕ-schemes. Shiho's Theorem 4.4, gives an equivalence of categories

$$\mathrm{MICP}(Y'/S) \to \mathrm{MIC}(Y/S).$$

The description of the functor in this direction is explicit and simple, but that of the inverse is less so. Here we shall describe a more symmetric construction of this equivalence, as a geometric transform. Namely, we will see that there is a formal ϕ-scheme T with affine morphisms $\pi_Y \colon T \to Y$ and $\pi_{Y'} \colon T \to Y'$, and that the F-transform and its inverse are given by (suitably defined) functors $\pi_{Y*}\pi^*_{Y'}$ and $\pi_{Y'*}\pi^*_Y$. This construction is the prismatic analog of the "torsor of Frobenius liftings" used in [24, §2.4] and its p-adic versions in [25] and [28].

If Y/S is a p-completely smooth morphism of formal ϕ-schemes, we let $Z := Y \times_S Y'$ and let $\Gamma \colon Y \to Z$ be the graph of the relative Frobenius map $\phi_{Y/S} \colon Y \to Y'$. If $\mathbb{\Delta}_\Gamma(Z)$ is the prismatic envelope of this immersion, there are morphisms

$$\begin{array}{rcl} \pi_Z \colon \mathbb{\Delta}_\Gamma(Z) & \to & Z \\ \pi_Y \colon \mathbb{\Delta}_\Gamma(Z) & \to & Z \to Y \\ \pi_{Y'} \colon \mathbb{\Delta}_\Gamma(Z) & \to & Z \to Y'. \end{array}$$

Note that these morphisms are affine, because $\overline{\mathbb{\Delta}}_\Gamma(Z)$ is affine over X. We also have isomorphisms:

$$\pi^*_Z(\Omega^i_{Z/Y}) \cong \pi^*_{Y'}(\Omega^i_{Y'/S})$$

$$\pi^*_Z(\Omega^i_{Z/Y'}) \cong \pi^*_Y(\Omega^i_{Y/S}),$$

which we may use without further mention in what follows.

Proposition 4.10. *With the notation above, let $\mathcal{A}_Y := \mathcal{O}_{\mathbb{\Delta}_\Gamma(Z)}$, which we view variously as a a sheaf of $\mathcal{O}_Z$ -algebras, a sheaf of left $\mathcal{O}_Y$-modules or a sheaf of right $\mathcal{O}_{Y'}$-modules.*

1. *The p-differential $d'\colon \mathcal{O}_Z \to \Omega^1_{Z/Y} \cong \mathcal{O}_Z \otimes \Omega^1_{Y'/S}$ extends uniquely to a quasi-nilpotent integrable p-connection on $\mathcal{A}_Y$ viewed as a right $\mathcal{O}_{Y'}$-module:*
$$d'\colon \mathcal{A}_Y \to \mathcal{A}_Y \otimes \Omega^1_{Y'/S}.$$
2. *The differential $d\colon \mathcal{O}_Z \to \Omega^1_{Z/Y'} \cong \Omega^1_{Y/S} \otimes \mathcal{O}_Z$ extends uniquely to an integrable and quasi-nilpotent connection on $\mathcal{A}_Y$ viewed as a left $\mathcal{O}_Y$-module:*
$$d\colon \mathcal{A}_Y \to \Omega^1_{Y/S} \otimes \mathcal{A}_Y.$$
3. *The object $(\mathcal{A}_Y, d)$ of $\mathrm{MIC}(Y/S)$ is the F-transform of the object $(\mathcal{O}_{\mathbb{A}_{Y'}(1)}, d')$ of $\mathrm{MICP}(Y'/S)$ (viewed as on $\mathcal{O}_{Y'}$-module via the first projection).*
4. *The map d' in (1) is $\mathcal{O}_Y$-linear, the map d in (2) is $\mathcal{O}_{Y'}$-linear, and the following diagram commutes:*

$$\begin{tikzcd}
\mathcal{A}_{Y/Z} \arrow[r, "d"] \arrow[d, "d'"'] & \Omega^1_{Y/S} \otimes \mathcal{A}_{Y/Z} \arrow[d, "\mathrm{id} \otimes d'"] \\
\mathcal{A}_{Y/Z} \otimes \Omega^1_{Y'/S} \arrow[r, "d \otimes \mathrm{id}"] & \Omega^1_{Y/S} \otimes \mathcal{A}_{Y/Z} \otimes \Omega^1_{Y'/S}
\end{tikzcd}$$

Proof. Proposition 3.6 tells us that the $\mathcal{O}_Z$-algebra $\mathcal{A}_Y$ admits a natural p-connection $d'\colon \mathcal{A}_Y \to \mathcal{A}_Y \otimes \Omega^1_{Z/S}$. Composing this with the projection $\Omega^1_{Z/S} \to \Omega^1_{Z/Y} \cong \Omega^1_{Y'/S} \otimes \mathcal{O}_Z$, we find the p-connection of statement (1).

The uniqueness of the connection d in (2) is clear, and we will deduce its existence from statement (3). The map $\phi_{Y/S} \times \mathrm{id}\colon Z \to Z'$ is flat, and the Cartesian diagram below is Cartesian:

$$\begin{tikzcd}
Y \arrow[r, "\Gamma"] \arrow[d, "\phi_{Y/S}"'] & Z \arrow[d, "\phi_{Y/S} \times \mathrm{id}"] \\
Y' \arrow[r, "(\mathrm{id}, \mathrm{id})"] & Y'(1).
\end{tikzcd}$$

It follows from Theorem 2.19 that the induced map
$$\mathbb{A}_\Gamma(Z) \to Z \times_{Y'(1)} \mathbb{A}_{Y'}(1) \cong Y \times_{Y'} \mathbb{A}_{Y'}(1)$$
is an isomorphism. Thus we have found an isomorphism of left $\mathcal{O}_Y$-modules:
$$\phi^*_{Y/S}(\mathcal{O}_{\mathbb{A}_{Y'}(1)}) \to \mathcal{A}_Y,$$
and we can endow $\mathcal{A}_Y$ with the F-transform of the p-connection of $\mathcal{O}_{\mathbb{A}_{Y'}(1)}$.

The linearity assertions in (4) are immediate from the definitions, and the diagram commutes when restricted to $\mathcal{O}_Z$. It follows that it commutes on all of $\mathcal{A}_Y$. □

We can now describe the F-transform in the following way. Recall that $\mathcal{A}_Y$ is regarded as a left $\mathcal{O}_Y$-module and a right $\mathcal{O}_{Y'}$-module, and, if E (resp. E') is an $\mathcal{O}_Y$-module (resp., $\mathcal{O}_{Y'}$-module), make identifications:

$$E \otimes \mathcal{A}_Y \cong \pi_Y^*(E), \quad \mathcal{A}_{Y/S} \otimes E' \cong \pi_{Y'}^*(E').$$

Suppose (E', ∇') is an object of $\mathrm{MICP}(Y'/S)$. Endow $\mathcal{A}_Y \otimes E'$ with the tensor product p-connection coming from the p-connections on E' and on $\mathcal{A}_Y$:

$$\nabla' \colon \mathcal{A}_Y \otimes E' \to \mathcal{A}_Y \otimes E' \otimes \Omega^1_{Y'/S}.$$

Since the connection d on $\mathcal{A}_Y$ is $\mathcal{O}_{Y'}$-linear, it induces a connection

$$\nabla := d \otimes \mathrm{id} \colon \mathcal{A}_Y \otimes E' \to \Omega^1_{Y/S} \otimes \mathcal{A}_Y \otimes E'$$

which annihilates E'. It follows that ∇ and ∇' commute and that ∇ induces a connection on the cohomology sheaves of the complex $(\mathcal{A}_Y \otimes E' \otimes \Omega^{\cdot}_{Y'/S}, d')$ and in particular on $(\mathcal{A}_Y \otimes E')^{\nabla'}$.

Similarly, if (E, ∇) is an object of $\mathrm{MIC}(Y/S)$, then we endow $E \otimes \mathcal{A}_Y$ with the tensor product connection

$$\nabla \colon E \otimes \mathcal{A}_Y \to \Omega^1_{Y/S} \otimes E \otimes \mathcal{A}_Y.$$

coming from the connections on $\mathcal{A}_Y$ and on E. Since the p-connection on $\mathcal{A}_Y$ is $\mathcal{O}_Y$-linear, we also find a p-connection:

$$\nabla' := \mathrm{id} \otimes d' \colon E \otimes \mathcal{A}_Y \to E \otimes \mathcal{A}_Y \otimes \Omega^1_{Y'/S},$$

which commutes with ∇.

Theorem 4.11. *With the constructions described in the previous paragraph, the following results hold.*

1. *If $(E', \nabla') \in \mathrm{MICP}(Y'/S)$, let*

$$(E, \nabla) := ((\mathcal{A}_Y \otimes E')^{\nabla'}, d \otimes \mathrm{id}).$$

Then the map

$$E \to (\mathcal{A}_Y \otimes E' \otimes \Omega^{\cdot}_{Y'/S}, d')$$

is a strict quasi-isomorphism, and the natural $\mathcal{O}_Y$-linear map

$$E \otimes \mathcal{A}_Y \to \mathcal{A}_Y \otimes E'$$

is an isomorphism. Furthermore, (E, ∇) is isomorphic to the F-transform of (E', ∇').

2. *If $(E, \nabla) \in \mathrm{MIC}(Y/S)$, let*

$$(E', \nabla') := (E \otimes \mathcal{A}_Y)^{\nabla}, \mathrm{id} \otimes d').$$

Then the map

$$E' \to (E \otimes \mathcal{A}_Y \otimes \Omega^{\cdot}_{Y/S}, d)$$

is a strict quasi-isomorphism, and the natural $\mathcal{O}_{Y'}$-linear map

$$\mathcal{A}_Y \otimes E' \to E \otimes \mathcal{A}_Y$$

is an isomorphism.

3. *The functor* $\mathrm{MICP}(Y'/S) \to \mathrm{MIC}(Y/S)$

$$(E', \nabla') \mapsto ((\mathcal{A}_Y \otimes E')^{\nabla'}, \nabla)$$

is an equivalence of categories, with quasi-inverse

$$(E, \nabla) \mapsto ((E \otimes \mathcal{A}_Y)^{\nabla}, \nabla')$$

Proof. The map $\pi_Y : Z \to Y$ is p-completely smooth, with a section $\Gamma \colon Y \to Z$, Thus we are in the situation of Lemma 3.12, with a change of notation: $S \mapsto Y$, $s \mapsto \Gamma$, and $E := \pi^*_{Y'}(E')$. The first part of statement (1) follows.

To see that $(\mathcal{A}_Y \otimes E')^{\nabla'}, \nabla)$ is the F-transform of (E', ∇'), let $\Gamma' \colon Y \to \mathbb{A}_\Gamma(Z)$ be the canonical section of Γ. Lemma 3.12 tells us that the map

$$E := (\mathcal{A}_Y \otimes E')^{\nabla'} \to \Gamma'^*(\mathcal{A}_Y \otimes E')$$

is an isomorphism. Since $\Gamma'^*(\mathcal{A}_Y \otimes E') \cong \phi^*_{Y/S}(E')$, we find an isomorphism $E \to \phi^*_{Y/S}(E')$. It remains to show that the connection on $(\mathcal{A}_Y \otimes E')^{\nabla'}$ agrees with the F-transform connection. We begin with the following result (really a special case).

Lemma 4.12. *If e'' is a local section of the kernel E'' of $\mathcal{A}_Y \otimes E' \to \Gamma'^*(\mathcal{A}_Y \otimes E')$, then*

$$\Gamma'^*(\nabla(e'')) = -(\zeta \otimes \mathrm{id})(\Gamma'^*(\nabla'(e'')))$$

Proof. Recall first that, since $\Gamma \colon Y \to Z$ is a regular immersion of p-completely smooth formal ϕ-schemes, Theorem 2.27 implies that $\mathbb{A}_\Gamma(Z)$ is $\mathbb{P}_{\tilde{\Gamma}}(\mathbb{D}_\Gamma(Z)$, where $\tilde{\Gamma}$ is the canonical section $Y \to \mathbb{D}_\Gamma(Z)$. If f is a local section of $\mathcal{O}_Y$, then

$$\epsilon(f) := \phi^*_{Y/S}(f) \otimes 1 - 1 \otimes \pi^*(f)$$

belongs to the ideal I_Γ of $\Gamma \colon Y \to Z$ and in fact this ideal is generated by the family of such elements. Then the ideal of $\tilde{\Gamma} \colon Y \to \mathbb{D}_Y(Z)$ is topoloogically generated by elements $p^{-1}\epsilon(f)$ for all such f, and hence the ideal of $\Gamma' \colon Y \to \mathbb{A}_Y(Z)$ is topologically generated as a PD-ideal by such elements. Hence every element of E'' is a p-adic limit of a sum of elements of the form $g^{[n]}e'$ for some $e' \in \mathcal{A}_Y \otimes E'$ and some $g = p^{-1}\epsilon(f)$. and it will suffice to prove the result for every such $g^{[n]}e'$.

Note first that

$$\nabla(g^{[n]}e') = g^{[n-1]}dg \otimes e' + g^{[n]}\nabla(e'),$$

which again belongs to E'' if $n > 1$, and the same holds for $\nabla'(e'')$. Thus $\nabla(g^{[n]}e')$ and $\nabla'(g^{[n]}e')$ both vanish when pulled back via Γ', so it suffices to treat the case $n = 1$.

Compute:

$$\begin{aligned} d(p^{-1}\epsilon(f)) &= p^{-1}\phi^*_{Y/S}(df) \\ d'(p^{-1}\epsilon(f))) &= -\pi^*(df) \end{aligned}$$

Then:

$$\begin{aligned} \Gamma'^*(\nabla(e'')) &= \Gamma'^*(\nabla(p^{-1}\epsilon(f)e')) \\ &= \Gamma'^*((p^{-1}\epsilon(f) \otimes e' + (p^{-1}\epsilon(f)\nabla(e')) \\ &= p^{-1}\phi^*_{Y/S}(df) \otimes e' \end{aligned}$$

$$\begin{aligned} \Gamma'^*(\nabla'(e'') &= \Gamma'^*(\nabla'(p^{-1}\epsilon(f)e') \\ &= \Gamma'^*('(p^{-1}\epsilon(f))e' + p^{-1}\epsilon(f)\nabla'(e') \\ &= -\pi^*(df) \otimes e' \end{aligned}$$

Since $\zeta(\pi^*(df)) = p^{-1}\phi^*(df)$, this proves the lemma. □

Now suppose that $e' \in E'$. We can uniquely write $1 \otimes e' = e + e''$, with $\nabla'(e) = 0$ and $e'' \in E''$. Since $\nabla(1 \otimes e') = 0$, it follows that $\nabla(e) = -\nabla(e'')$, which, by the lemma, is $(\zeta \otimes \mathrm{id})\nabla'(e'')$. But $\nabla'(e'') = \nabla'(1 \otimes e') - \nabla'(e) = \nabla'(e')$. We conclude that

$$\nabla(e) = (\zeta \otimes \mathrm{id})\nabla'(e'),$$

as claimed.

The following result, which is a special case of statement (2), is worth stating separately.

Lemma 4.13. *The de Rham complex $(\mathcal{A}_Y \otimes \Omega^{\cdot}_{Y/S}, d)$ of the $\mathcal{O}_Y$-module with connection $(\mathcal{A}_Y, d)$, viewed as a complex of $\mathcal{O}_{Y'}$-modules, is a resolution of $\mathcal{O}_{Y'}$, each term of which is p-completely flat.*

Proof. By statement (3) of Proposition 4.10, we know that $(\mathcal{A}_Y, \nabla)$ is the F-transform of $(\mathcal{O}_{\mathbb{A}_{Y'}(1)}, d')$. Theorem 4.6 implies that the map

$$(\mathcal{A}_Y \otimes \Omega^{\cdot}_{Y/S}, d) \to \phi_{Y/S*}(\mathcal{O}_{\mathbb{A}_{Y'}(1)} \otimes \Omega^{\cdot}_{Y'/S}, d')$$

is a strict quasi-isomorphism. Theorem 3.11 tells us that the complex on the right is a strict resolution of $\mathcal{O}_{Y'}$, and hence the same is true of the complex on the left. It is clear that each of its terms is p-completely flat over $\mathcal{O}_Y$ and hence also over $\mathcal{O}_{Y'}$. □

Now to prove statement (2), we use Shiho's Theorem 4.4, which tells us that $(E, \nabla) \in \mathrm{MIC}(Y/S)$ is the F-transform of some $(E', \nabla') \in \mathrm{MICP}(Y'/S)$. Then, by statement (1), we have isomorphisms:

$$(E, \nabla) \cong (\mathcal{A}_Y \otimes E')^{\nabla'}, d \otimes \mathrm{id})$$

$$E \otimes \mathcal{A}_Y \cong \mathcal{A}_Y \otimes E'$$
$$(E \otimes \mathcal{A}_Y \otimes \Omega^{\cdot}_{Y/S}, d) \cong (\mathcal{A}_Y \otimes E' \otimes \Omega^{\cdot}_{Y/S}, d \otimes \mathrm{id}_{E'}),$$

The lemma tells us that $(\mathcal{A}_Y \otimes \Omega^{\cdot}_{Y/S}, d)$ is a strict resolution of $\mathcal{O}_{Y'}$ all of whose terms are p-completely flat, and hence $(\mathcal{A}_Y \otimes E' \otimes \Omega^{\cdot}_{Y/S}, d \otimes \mathrm{id}_{E'})$ is a strict resolution of E'. Statement (2) follows, and Theorem 4.4 now also implies statement (3). $\square$

4.3 F-transform and p-curvature

Our goal here is to give an explicit formula for the p-curvature of the reduction modulo p of the F-transform of a module with integrable p-connection. This explicit formula is not needed for the remainder of this paper; all that is needed here is that quasi-nilpotent p-connections give rise to quasi-nilpotent connections, which has already been proved in [26].

Since everything takes place in characteristic p, we change the notation. Let $X \to S$ be a smooth morphism of schemes in characteristic $p > 0$, let $F_{X/S} \colon X \to X'$ be the relative Frobenius morphism, and let $\pi_{X/S} \colon X' \to X$ be the projection mapping. We can generalize the F-transform construction as follows. Suppose that we are given a splitting ζ' of the inverse Cartier isomorphism, as in the following diagram.

$$\begin{tikzcd}
& F_{X/S*}\Omega^1_{X/S} \\
\Omega^1_{X'/S} \arrow[ur, "\zeta"] \arrow[r, "\zeta'"] \arrow[dr, "C^{-1}_{X/S}"'] & F_{X/S*}\mathcal{Z}^1_{X/S} \arrow[u] \arrow[d] \\
& F_{X/S*}\mathcal{H}^1(\Omega^{\cdot}_{X/S})
\end{tikzcd}$$

(Recall, e.g. from Proposition 3.5, that a lifting of the Frobenius morphism to a smooth morphism of p-torsion free schemes provides such a splitting.) If $\theta' \colon E' \to \Omega^1_{X'/S}$ is a Higgs field on a sheaf of $\mathcal{O}_{X'}$-modules E', there is a unique connection ∇ on $E := F^*_{X/S*}(E')$ such that $\nabla(1 \otimes e') = (\zeta' \otimes \mathrm{id})\theta'(e')$ for every local section e' of E'. We call this (E, ∇) the *ζ-transform* of (E', θ'). This construction generalizes the F-transform (4.1) in the obvious way. Our goal is to compute the p-curvature of (E, ∇).

The splitting ζ induces, by adjunction, duality, and pullback, maps:

$$\begin{array}{rcl}
\tilde{\zeta} \colon F^*_{X/S}(\Omega^1_{X'/S}) & \to & \Omega^1_{X/S} \\
\hat{\zeta} \colon \quad T_{X/S} & \to & F^*_{X/S}(T_{X'/S})
\end{array}$$

It follows from the definition that there is a commutative diagram:

$$\begin{tikzcd} E \arrow[r, "F^*_{X/S}(\theta')"] \arrow[dr, "\nabla"'] & F^*_{X/S}(\Omega^1_{X'/S}) \otimes E \arrow[d, "F^*_{X/S}(\tilde{\zeta})"] \\ & \Omega^1_{X/S} \otimes E \end{tikzcd}$$

For the computations, we shall use the following standard notations for the Higgs field and connection:

$$\theta' \colon E' \to \Omega^1_{X'/S} \otimes E' \quad T_{X'/S} \to \mathrm{End}_{\mathcal{O}_{X'}}(E') : D' \mapsto \theta'_{D'}$$

$$\nabla \colon E \to \Omega^1_{X/S} \otimes E \quad T_{X/S} \to \mathrm{End}_{\mathcal{O}_{X'}}(E') : D \mapsto \nabla_D$$

We also have a map:

$$\Theta \colon T_{X/S} \to \mathrm{End}_{\mathcal{O}_X}(E) : D \mapsto F^*_{X/S}(\theta')_{\hat{\zeta}(D)}$$

and the connection ∇ corresponds to the map

$$\nabla \colon T_{X/S} \to \mathrm{End}_{\mathcal{O}_{X'}}(E) \colon D \mapsto D \otimes \mathrm{id} + \Theta_D.$$

Recall that the p-curvature of (E, ∇) is by definition the map

$$\psi \colon E \to F^*_{X/S}(\Omega^1_{X'/S}) \otimes E$$

such that

$$\langle \psi(e), \pi^*_{X/S}(D) \rangle = \nabla^p_D(e) - \nabla_{D^{(p)}}(e)$$

for all $e \in E$ and $D \in T_{X/S}$, where $D^{(p)} \in T_{X/S}$ is the pth iterate of D. That is, if $D \in T_{X/S}$ and $D' = \pi^*_{X/S}(D) \in T_{X'/S}$, we have

$$\psi_{D'} = \nabla^p_D - \nabla_{D^{(p)}} \in \mathrm{End}_{\mathcal{O}_X}(E).$$

Theorem 4.14. *With notations described in the previous paragraphs, for every local section D of $T_{X'/S}$, we have:*

$$\psi_{D'} = \Theta^p_D - F^*_{X/S}(\theta'_{D'}),$$

where $D' := \pi^*_{X/S}(D)$

Proof. Let us first remark that, when $E' = \mathcal{O}_X$, Theorem 4.14 is equivalent to a formula of Katz [17, 7.1.2], and our proof begins the same way. From the definition of the p-curvature and a formula of Jacobson, we find:

$$\begin{aligned} \psi_{D'} &:= \nabla^p_D - \nabla_{D^{(p)}} \\ &= (D \otimes \mathrm{id} + \Theta_D)^p - (D^{(p)} \otimes \mathrm{id} - \Theta_{D^{(p)}}) \\ &= (D \otimes \mathrm{id})^p + \mathrm{ad}^{p-1}_{D \otimes \mathrm{id}}(\Theta_D) + \Theta^p_D - (D^{(p)} \otimes \mathrm{id} - \Theta_{D^{(p)}}) \\ &= \mathrm{ad}^{p-1}_{D \otimes \mathrm{id}}(\Theta_D) + \Theta^p_D - \Theta_{D^{(p)}} \end{aligned}$$

Then the following lemma proves the theorem.

Lemma 4.15. *If D is any section $T_{X/S}$ and $D' := \pi^*_{X/S}(D)$, then as endomorphisms of $F^*_{X/S}(E')$, we have*

$$\operatorname{ad}^{p-1}_{D\otimes \mathrm{id}}(\Theta_D) - \Theta_{D^{(p)}} = -F^*_{X/S}(\theta'_{D'})$$

Proof. We work locally, with the aid of a system of coordinates $(t_1, \ldots, t_n)$. Let $(D_1, \ldots, D_n)$ be the basis for $T_{X/S}$ dual to $(dt_1, \ldots, dt_n)$, let $\omega_i := \zeta(dt'_i)$, let $D'_i = \pi^*_{X/S}(D_i) \in T_{X'/S}$, and let $\theta'_i := \theta'_{D'_i}$. The integrability condition implies that $\theta'_1, \ldots, \theta'_n$ is a family of commuting endomorphisms of E'; they also commute with each Θ_D because $\psi_{D'}$ is horizontal. Then for each $e' \in E'$, we have

$$\begin{aligned}\theta'(e') &= \sum_i dt'_i \otimes \theta'_i(e') \\ \nabla(1 \otimes e') &= \sum_i \omega_i \otimes \theta'_i(e').\end{aligned}$$

For $D \in T_{X/S}$, we have:

$$\begin{aligned}\Theta_D &= F^*_{X/S}(\theta')_{\hat{\zeta}(D)} \in \operatorname{End}_{\mathcal{O}_X} E \\ &= \sum_i \langle 1 \otimes dt'_i, \hat{\zeta}(D)\rangle \otimes \theta'_i \\ &= \sum_i \langle \zeta(dt'_i), D\rangle \otimes \theta'_i \\ &= \sum_i \langle \omega_i, D\rangle \otimes \theta'_i\end{aligned}$$

Recall from [17, 7.1.2.6] that the Cartier operator $C\colon F_*(\mathcal{Z}^1_{X/S}) \to \Omega^1_{X'/S}$ satisfies the following formula

$$F^*_{X/S}(\langle C(\omega), \pi^*_{X/S} D\rangle) = \langle \omega, D^{(p)}\rangle - D^{p-1}(\langle \omega, D\rangle)$$

for $\omega \in F_{X/S*}\mathcal{Z}^1_{X/S}$ and $D \in T_{X/S}$. Using the fact that Θ_D commutes with each θ'_i, we compute:

$$\begin{aligned}\operatorname{ad}^{p-1}_{D\otimes \mathrm{id}}(\Theta_D) - \Theta_{D^{(p)}} &= \operatorname{ad}^{p-1}_{D\otimes \mathrm{id}}\left(\sum_i \langle \omega_i, D\rangle \otimes \theta'_i\right) - \sum_i \langle \omega_i, D^{(p)}\rangle \otimes \theta'_i \\ &= \sum D^{p-1}\langle \omega_i, D\rangle \otimes \theta'_i - \langle \omega_i, D^{(p)}\rangle \otimes \theta'_i \\ &= -\sum F^*_{X/S}(\langle C(\omega_i), D'\rangle) \otimes \theta'_i\end{aligned}$$

Since $\omega_i = \zeta(dt'_i)$ and ζ is a splitting of the inverse Cartier isomorphism, in fact $C(\omega_i) = dt'_i$, and we find that

$$\begin{aligned}\operatorname{ad}^{p-1}_{D\otimes \mathrm{id}}(\Theta_D) - \Theta_{D^{(p)}} &= -\sum F^*_{X/S}\langle dt'_i, D'\rangle \otimes \theta'_i \\ &= -F^*_{X/S}(\theta'_{D'})\end{aligned}$$

as claimed. ∎

□

Remark 4.16. We should also point out that the ζ-transform was also discussed in [24, 2.11.1]. For quasi-nilpotent Higgs fields, [24, 2.13] gives another proof of the p-curvature formula. In the discussion there, this formula is presented in a more geometric way, which the reader may find more appealing. It used to define a twisted version of the ζ-transform whose p-curvature is Frobenius pull back of the Higgs field θ'.

5 Groupoids, stratifications, and differential operators

In this section adapt some standard constructions relating groupoids, stratifications, and differential operators to prismatic crystals. To facilitate the comparison to other theories, we let $\mathbb{T}$ stand for any one of $\mathbb{P}, \mathbb{D}$, or $\mathbb{\Delta}$, and sometimes also for $\mathbb{F}$. We work over a p-torsion free p-adic formal scheme S, endowed with a ϕ-scheme structure in the last case. If X is smooth $\overline{S}$-scheme, we let $\mathbb{T}(X/S)$ be the category of PD-enlargements of X/S, of p-adic enlargements of X/S, or of X/S-prisms, respectively. If $X \subseteq Y$ is a closed embedding, we write $\mathbb{T}_X(Y)$ for the appropriate envelope of X in Y. Section 8 in the appendix reviews (and reformulates slightly) the general theory of groupoid actions, stratifications, and crystals in the context of fibered categories. The reader is invited to consult or ignore this treatment as his/her convenience.

5.1 Prismatic stratifications and differential operators

Let $f\colon Y \to S$ be a p-completely smooth morphism of p-adic formal schemes, or, in the prismatic context, of formal ϕ-schemes. If $n \in \mathbf{N}$, we write $Y(n)$ for the $n+1$-fold product $Y \times_S Y \times_S \cdots Y$, noting that this product is again p-torsion free and that it inherits a Frobenius lifting in the prismatic case. If $X \to Y$ is a closed immersion, we let $\mathbb{T}_{X/Y}(n)$ denote the $\mathbb{T}$-envelope of X in $Y(n)$, embedded via the diagonal. In the important case in which $X = \overline{Y}$, we abbreviate this to $\mathbb{T}_Y(n)$.

Proposition 5.1. *Let $f\colon Y \to S$ and $g\colon Z \to S$ be p-completely smooth morphisms of p-torsion free p-adic formal schemes (resp. of formal ϕ-schemes if $\mathbb{T} = \mathbb{\Delta}$), and let $i\colon X \to \overline{Y}$ and $j\colon X \to \overline{Z}$ be closed $\overline{S}$-immersions.*

1. *The envelope $\mathbb{T}_X(Y \times_S Z)$ represents the product $\mathbb{T}_X(Y) \times \mathbb{T}_X(Z)$ in the category $\mathbb{T}(X/S)$. Similarly, for each n, $\mathbb{T}_{X/Y}(n)$ represents the $n+1$-fold product of $\mathbb{T}_X(Y)$ with itself in the category $\mathbb{T}(X/S)$.*

2. *For every $n \in \mathbf{N}$ and for $0 \leq i \leq n$, the diagram*

$$\begin{tikzcd} \mathbb{T}_{X/Y}(n)) \arrow[r] \arrow[d, "\mathbb{T}_X(p_i)"'] & \mathbb{T}_Y(n) \arrow[d, "p_i"] \\ \mathbb{T}_X(Y) \arrow[r, "\pi"] & Y \end{tikzcd}$$

is Cartesian.

Proof. The proof of statement (1) is purely formal. The maps

$$\mathbb{T}_X(Y \times_S Z) \to \mathbb{T}_X(Y) \text{ and } \mathbb{T}_X(Y \times_S Z) \to \mathbb{T}_X(Z)$$

define a map from the functor represented by $\mathbb{T}_X(Y \times_S Z)$ to the product of the functors represented by $\mathbb{T}_X(Y)$ and $\mathbb{T}_X(Z)$. To construct the inverse, suppose that (T, z_T) is an object of $\mathbb{T}(X/S)$ and that $f\colon T \to \mathbb{T}_X(Y)$ and $g\colon T \to \mathbb{T}_X(Z)$ are maps in $\mathbb{T}(X/S)$. Then $\pi_{\mathbb{T}} \circ f\colon T \to Y$ and $\pi_{\mathbb{T}} \circ g\colon T \to Z$ define a map $h\colon T \to Y \times_S Z$, and $\overline{h} = (i, j) \circ z_T\colon \overline{T} \to \overline{Y} \times_S \overline{Z}$. Thus h factors uniquely through $\mathbb{T}_X(Y \times_S Z)$.

We focus on the prismatic context in the proof of statement (2). First note that the map

$$\mathbb{T}_Y(n) \longrightarrow Y(n) \xrightarrow{pr_i} Y$$

is p-completely flat, by statement (3) of Proposition 2.36. It follows that the product $\mathbb{T}_X(Y) \times_Y \mathbb{T}_Y(n)$ is p-torsion free, and hence that the maps

$$z_Y \times z_{Y(n)}\colon \overline{\mathbb{T}}_X(Y) \times_Y \overline{\mathbb{T}}_Y(n) \to X \times_Y \overline{Y} = X$$

and

$$\pi_Y \times \pi_{Y(n)}\colon \mathbb{T}_X(Y) \times_Y \mathbb{T}_Y(n) \to Y \times_Y Y(n) = Y(n)$$

endow $\mathbb{T}_X(Y) \times_Y \mathbb{T}_Y Y(n)$ with the structure of an X-tube over $Y(n)$. Hence there is a unique map

$$\mathbb{T}_X(Y) \times_Y \mathbb{T}_Y(n) \to \mathbb{T}_{X/Y}(n) \tag{5.1}$$

of X-tubes over $Y(n)$. On the other hand, the map of pairs

$$(\mathrm{id}, i)\colon (Y(n), X) \to (Y(n), \overline{Y})$$

defines a Y-morphism $\mathbb{T}_X(Y(n)) \to \mathbb{T}_Y(n)$, and the map

$$(pr_1, i)\colon (Y(n), X) \to (Y, X)$$

defines a Y-morphism $\mathbb{T}_{X/Y}(n) \to \mathbb{T}_X(Y)$. These assemble to define a morphism

$$\mathbb{T}_{X/Y}(n) \to \mathbb{T}_X(Y) \times_Y \mathbb{T}_Y(n)$$

which is inverse to the mapping 5.1. One constructs the isomorphism

$$\mathbb{T}_{X/Y}(n) \to \mathbb{T}_Y(n) \times_Y \mathbb{T}_X(Y)$$

in the same way. □

We find morphisms

$$\begin{array}{rcl} t, s : \mathbb{T}_{X/Y}(1) & \longrightarrow & \mathbb{T}_X(Y) \\ \iota : Y & \longrightarrow & \mathbb{T}_Y(1) \\ c : \mathbb{T}_{X/Y}(1) \times_{\mathbb{T}_X(Y)} \mathbb{T}_{X/Y}(1) & \longrightarrow & \mathbb{T}_{X/Y}(1) \end{array}$$

covering the corresponding structural morphisms of the "indiscrete groupoid" $\mathcal{G}_Y$ defined by Y, as described in Example 8.2. (The morphism c is obtained as the composition of the isomorphism

$$\mathbb{T}_{X/Y}(1) \times_{\mathbb{T}_{X/Y}} \mathbb{T}_{X/Y}(1) \cong \mathbb{T}_{X/Y}(2)$$

of statement (1) of Proposition 5.1 with the map $\mathbb{T}_{X/Y}(2) \to \mathbb{T}_{X/Y}(1)$ induced by functoriality from the composition law of $\mathcal{G}_Y$.) These assemble to define a groupoid $\mathcal{G}_{\mathbb{T}_{X/Y}}$ over Y, with $\mathbb{T}_{X/Y}(1)$ representing the class of arrows. In fact, statement (1) of Proposition 5.1 tells us that $\mathbb{T}_{X/Y}(1)$ is the fiber product of Y with itself over S in the category $\mathbb{T}(Y/S)$, so $\mathcal{G}_{\mathbb{T}_{X/Y}}$ can be viewed as the indiscrete groupoid on the object $\mathbb{T}_X(Y)$ when viewed in this category. In the important case when $X = \overline{Y}$, we just write $\mathcal{G}_{\mathbb{T}_Y}$ for this indiscrete groupoid on the object Y of the category $\mathbb{A}(\overline{Y}/S)$.

If X is a closed subscheme of Y, there is a unique morphism ϵ making the following diagram commute:

$$\begin{tikzcd} \mathbb{T}_X(Y) \times_Y \mathbb{T}_Y(1) \arrow[r, "\epsilon"] \arrow[d, "\cong"'] & \mathbb{T}_Y(1) \times_Y \mathbb{T}_X(Y) \arrow[dl, "\cong"] \\ \mathbb{T}_X(Y(1)), & \end{tikzcd}$$

where the isomorphisms come from Statement (2) of Proposition 5.1. On T-values points, the morphism ϵ takes (y_1, y_1, y_2) to (y_1, y_2, y_2); its nontrivial content is that if y_1 is in $\mathbb{T}_X(Y)$ and (y_1, y_2) in in $\mathbb{T}_Y(1)$, then y_2 is also in $\mathbb{T}_X(Y)$. It is easy to see from this that ϵ defines a $\mathcal{G}_{\mathbb{T}_Y}$-stratification on $\mathbb{T}_X(Y)$, or, equivalently, a right action:

$$r \colon \mathbb{T}_X(Y) \times_Y \mathbb{T}_Y(1) \to \mathbb{T}_X(Y). \tag{5.2}$$

In every case we are considering, the morphisms s and $t : \mathbb{T}_{X/Y}(1) \to Y$ are affine. Let $\mathcal{A}_{\mathbb{T}_{X/Y}} := t_*(\mathcal{O}_{\mathbb{T}_{X/Y}(1)})$. Then the groupoid $\mathcal{G}_{\mathbb{T}_Y}$ comes from a Hopf algebra structure:

$$\begin{array}{rcl} c^\sharp \colon \mathcal{A}_{\mathbb{T}_{X/Y}} & \to & \mathcal{A}_{\mathbb{T}_{X/Y}} \hat{\otimes}_{\mathcal{O}_{\mathbb{T}_X(Y)}} \mathcal{A}_{\mathbb{T}_{X/Y}} \\ \iota^\sharp \colon \mathcal{A}_{\mathbb{T}_{X/Y}} & \to & \mathcal{O}_{\mathbb{T}_X(Y)} \\ \tau^\sharp \colon \mathcal{A}_{\mathbb{T}_{X/Y}} & \to & \mathcal{A}_{\mathbb{T}_{X/Y}}. \end{array}$$

If X is closed in Y, the right action of $\mathcal{G}_{\mathbb{T}_Y}$ on $\mathbb{T}_X(Y)$ corresponds to a coaction of $\mathcal{A}_{\mathbb{T}_Y}$ on $\mathcal{O}_{\mathbb{T}_X(Y)}$:

$$r^\sharp\colon \mathcal{O}_{\mathbb{T}_X(Y)} \to \mathcal{O}_{\mathbb{T}_X(Y)}\hat{\otimes}_{\mathcal{O}_Y}\mathcal{A}_{\mathbb{T}_Y}.$$

The ring of *(hyper-)$\mathbb{T}$-differential operators* is defined by taking the $\mathcal{O}_{\mathbb{T}_X(Y)}$-linear dual:

$$\mathcal{HD}_{\mathbb{T}_{X/Y}} := \mathrm{Hom}(\mathcal{A}_{\mathbb{T}_{X/Y}}, \mathcal{O}_{\mathbb{T}_X(Y)},$$

with ring structure induced by the dual of $c^\sharp$. The right action of $\mathcal{G}_{\mathbb{T}_Y}$ on $\mathbb{T}_X(Y)$ corresponds to a left action of $\mathcal{HD}_{\mathbb{T}_Y}$, given by

$$D(e) := (\mathrm{id}\hat{\otimes}D)(r^\sharp(e)),$$

for $e \in \mathcal{O}_{\mathbb{T}_X(Y)}$ and $D \in \mathcal{HD}_{\mathbb{T}_Y}$. If $\mathbb{T} = \mathbb{P}$, the ring $\mathcal{HD}_{\mathbb{T}_Y}$ is topologically generated by derivations, and if $\mathbb{T} = \mathbb{\Delta}$, by p-derivations; see [26], which we shall review below.

Statement (2) of Proposition 5.1 tells us that $\mathbb{T}_{X/Y}(1) \cong \mathbb{T}_X(Y) \times_Y \mathbb{T}_Y(1)$, hence that

$$\mathcal{A}_{\mathbb{T}_{X/Y}} \cong \mathcal{O}_{\mathbb{T}_X(Y)}\hat{\otimes}_{\mathcal{O}_Y}\mathcal{A}_{\mathbb{T}_Y}$$

and that

$$\mathcal{HD}_{\mathbb{T}_{X/Y}} \cong \mathcal{O}_{\mathbb{T}_X(Y)}\hat{\otimes}_{\mathcal{O}_Y}\mathcal{HD}_{\mathbb{T}_Y} \tag{5.3}$$

where the completion here means the inverse limit over the duals of the formal neighborhoods of the diagonal ideal in $\mathcal{A}_{\mathbb{T}_Y}$.

Let us make these constructions explicit in a local situation with the aid of a system of local coordinates. We suppose that X/S and Y/S are smooth, that $S = \mathrm{Spf}\, R$, and that we have a system of local coordinates $(x_1, \dots, x_n)$ for Y such that the ideal of X in Y is generated by $(p, x_1, \dots, x_r)$. The following straightforward computations appear in the literature, for example in [6], [26], [25], [28].

First of all, there are p-completely étale maps:

$$\begin{array}{rcl} Y & \to & \mathrm{Spf}\, R[x_1, \dots, x_n]\hat{} \\ \mathbb{F}_X(Y) & \to & \mathrm{Spf}\, R[[x_1, \dots, x_r]][x_{r+1}, \dots, x_n]\hat{} \\ \mathbb{P}_X(Y) & \to & \mathrm{Spf}\, R\langle x_1, \dots, x_r\rangle[x_{r+1}, \dots, x_n]\hat{} \\ \mathbb{D}_X(Y) & \to & \mathrm{Spf}\, R[t_1 \dots, t_r][x_{r+1}, \dots, x_n]\hat{}, \quad \text{where } x_i = pt_i, \text{ for } 1 \leq r \end{array} \tag{5.4}$$

The prismatic case is not so familiar or explicit; we can only say that, as a consequence of Proposition 2.21, there is a p-completely étale map:

$$\mathbb{\Delta}_X(Y) \to \mathrm{Spf}\, B^\infty[x_{r+1}, \dots, x_n]\hat{}, \quad \text{where}$$

$$B^\infty := R[t_{i,j}]\hat{}/(t_{i,0}, pt_{i,j+1} - \delta^j(x_i) + t_{i,j}^p) : i = 1, \dots, r, j \in \mathbf{N}.$$

Let $\xi_i := 1 \otimes x_i - x_i \otimes 1$ in $\mathcal{O}_{Y(1)}$. Viewing $\mathcal{O}_{Y(1)}$ as an $\mathcal{O}_Y$-module via the first projection t, we identify $x_i \in \mathcal{O}_Y$ with $x_i \otimes 1 \in \mathcal{O}_{Y(1)}$. Then we have coordinates $x_1, \ldots, x_n, \xi_1, \ldots, \xi_n$ for $Y(1)$, and $(p, \xi_1, \ldots, \xi_n)$ is a sequence of generators for the ideal of $\overline{Y}$ in $Y(1)$. Thus:

$$\begin{array}{rcll} \mathbb{F}_Y(1) & = & \operatorname{Spf} \mathcal{O}_Y[[\xi_1, \ldots, \xi_n]] & \\ \mathbb{P}_Y(1) & = & \operatorname{Spf} \mathcal{O}_Y\langle \xi_1, \ldots, \xi_n \rangle\hat{} & \\ \mathbb{D}_Y(1) & = & \operatorname{Spf} \mathcal{O}_Y[\eta_1, \ldots, \eta_n]\hat{}, & \text{where } \xi_i = p\eta_i \\ \mathbb{A}_Y(1) & = & \operatorname{Spf} \mathcal{O}_Y\langle \eta_1 \ldots, \eta_n \rangle\hat{}, & \text{where } \xi_i = p\eta_i \end{array} \tag{5.5}$$

Recall that the composition law $c\colon Y(1) \times_Y Y(1) \to Y(1)$ corresponds to the map $p_{13}\colon Y \times_S Y \times_S Y \to Y \times_S Y$. One then gets the following formulas for the Hopf algebra structures:

$$\begin{array}{rcl} c^\sharp\colon \mathcal{O}_{\mathbb{F}_Y(1)} \to \mathcal{O}_{\mathbb{F}_Y(2)} & : & \xi_i \mapsto \xi_i \otimes 1 + 1 \otimes \xi_i \\ c^\sharp\colon \mathcal{O}_{\mathbb{P}_Y(1)} \to \mathcal{O}_{\mathbb{P}_Y(2)} & : & \xi_i \mapsto \xi_i \otimes 1 + 1 \otimes \xi_i \\ c^\sharp\colon \mathcal{O}_{\mathbb{D}_Y(1)} \to \mathcal{O}_{\mathbb{D}_Y(2)} & : & \eta_i \mapsto \eta_i \otimes 1 + 1 \otimes \eta_i \\ c^\sharp\colon \mathcal{O}_{\mathbb{A}_Y(1)} \to \mathcal{O}_{\mathbb{A}_Y(2)} & : & \eta_i \mapsto \eta_i \otimes 1 + 1 \otimes \eta_i, \end{array} \tag{5.6}$$

For each multi-index $I := (I_1, \ldots, I_n)$, define differential operators:

$$\begin{array}{rcl} \partial_I \in \mathcal{HD}_{\mathbb{P}_Y}\colon t_*\mathcal{O}_{\mathbb{P}_Y(1)} \to \mathcal{O}_Y & : & \xi_J \mapsto \delta_{I,J} \\ \partial_I \in \mathcal{HD}_{\mathbb{D}_Y}\colon t_*\mathcal{O}_{\mathbb{D}_Y(1)} \to \mathcal{O}_Y & : & \eta_J \mapsto \delta_{I,J} \\ \partial_I \in \mathcal{HD}_{\mathbb{A}_Y}\colon t_*\mathcal{O}_{\mathbb{A}_Y(1)} \to \mathcal{O}_Y & : & \eta_J \mapsto \delta_{I,J} \end{array} \tag{5.7}$$

The set of these operators forms a formal basis for each $\mathcal{HD}_{\mathbb{T}_Y}$, in that every operator can be written uniquely as a formal sum $\sum_I a_I \partial_I$ with $a_I \in \mathcal{O}_Y$ (with no convergence conditions). Operators in $\mathcal{HD}_{\mathbb{T}_{X/Y}}$ can be a sum of the same form, now with coefficients in $\mathcal{O}_{\mathbb{T}_X(Y)}$. Furthermore we have the following composition laws:

$$\begin{array}{rcll} \partial_I \partial_J & = & \partial_{I+J} & \in \mathcal{HD}_{\mathbb{P}_Y} \\ \partial_I \partial_J & = & \dfrac{(I+J)!}{I!J!} \partial_{I+J} & \in \mathcal{HD}_{\mathbb{D}_Y} \\ \partial_I \partial_J & = & \partial_{I+J} & \in \mathcal{HD}_{\mathbb{A}_Y} \end{array} \tag{5.8}$$

In particular, the rings $\mathcal{HD}_{\mathbb{P}_Y}$ and $\mathcal{HD}_{\mathbb{A}_Y}$ are formally generated by operators of degree at most one, but this is not the case for $\mathcal{HD}_{\mathbb{D}_Y}$.

The right actions of $\mathcal{G}_{\mathbb{T}_Y}$ on $\mathbb{T}_X(Y)$ cover the action of $\mathcal{G}_Y$ on Y. These actions are given by the following formulas:

$$\begin{array}{rcl} r^\sharp\colon \mathcal{O}_Y \to \mathcal{O}_{\mathbb{F}_Y(1)} & : & x_i \mapsto x_i + \xi_i \\ r^\sharp\colon \mathcal{O}_{\mathbb{P}_X(Y)} \to \mathcal{O}_{\mathbb{P}_X(Y)} \hat{\otimes} \mathcal{O}_{\mathbb{P}_Y(1)} & : & x_i \mapsto x_i + \xi_i \\ r^\sharp\colon \mathcal{O}_{\mathbb{D}_X(Y)} \to \mathcal{O}_{\mathbb{D}_X(Y)} \hat{\otimes} \mathcal{O}_{\mathbb{D}_Y(1)} & : & x_i \mapsto x_i + p\eta_i, \quad t_i \mapsto t_i + \eta_i \\ r^\sharp\colon \mathcal{O}_{\mathbb{A}_X(Y)} \to \mathcal{O}_{\mathbb{A}_X(Y)} \hat{\otimes} \mathcal{O}_{\mathbb{A}_Y(1)} & : & x_i \mapsto x_i + p\eta_i, \quad t_i \mapsto t_i + \eta_i \end{array} \tag{5.9}$$

These actions give rise to corresponding stratifications and consequently to actions of rings of differential operators. These in turn correspond to connections on $\mathcal{O}_Y$ and on $\mathcal{O}_{\mathbb{P}_X(Y)}$ and to p-connections on $\mathcal{O}_{\mathbb{D}_X(Y)}$ and $\mathcal{O}_{\mathbb{A}_X(Y)}$:

$$\nabla\colon \mathcal{O}_Y \to \Omega^1_Y \quad : \quad x_i^n \mapsto n x^{n-1} dx_i$$

$$
\begin{array}{rcl}
\nabla\colon \mathcal{O}_{\mathbb{P}_X(Y)} \to \mathcal{O}_{\mathbb{P}_X(Y)} \otimes \Omega^1_{Y/S} & : & x_i^{[n]} \mapsto x_i^{[n-1]} dx_i \\
\nabla'\colon \mathcal{O}_{\mathbb{D}_X(Y)} \to \mathcal{O}_{\mathbb{D}_X(Y)} \otimes \Omega^1_{Y/S} & : & t_i^n \mapsto n t_i^{n-1} dx_i \\
\nabla'\colon \mathcal{O}_{\mathbb{A}_X(Y)} \to \mathcal{O}_{\mathbb{A}_X(Y)} \otimes \Omega^1_{Y/S} & : & t_{i,j} \mapsto d\delta^j(x_i) - t_{i,j-1}^{p-1} d' t_{i,j-1}
\end{array}
\quad (5.10)
$$

This last formula follows from the formula (3.8) for d' in the proof of Proposition 3.6.

The *right* regular representations of $\mathcal{G}_Y$ on $Y(1)_s$ and $Y(1)_t$ described in Example 8.9 give rise to stratifications and connections which we claim are given by the following formulas:

$$
\begin{array}{rcll}
\nabla\colon \mathcal{O}_{\mathbb{F}_{Y(1)_t}} \to \mathcal{O}_{\mathbb{F}_{Y(1)_t}} \otimes \Omega^1_{Y/S} & : & x_i \otimes 1 \mapsto 0 & \\
 & & \xi_i \mapsto \xi_i + 1 \otimes 1 \otimes dx_i & (5.11) \\
\nabla\colon \mathcal{O}_{\mathbb{F}_{Y(1)_s}} \to \mathcal{O}_{\mathbb{F}_{Y(1)_s}} \otimes \Omega^1_{Y/S} & : & 1 \otimes x_i \mapsto 0 & \\
 & & \xi_i \mapsto -1 \otimes 1 \otimes dx_i & (5.12) \\
\nabla'\colon \mathcal{O}_{\mathbb{A}_{Y(1)_t}} \to \mathcal{O}_{\mathbb{A}_{Y(1)_t}} \otimes \Omega^1_{Y/S} & : & x_i \otimes 1 \mapsto 0 & \\
 & & \eta_i \mapsto \eta_i + 1 \otimes 1 \otimes dx_i & (5.13) \\
\nabla'\colon \mathcal{O}_{\mathbb{A}_{Y(1)_s}} \to \mathcal{O}_{\mathbb{A}_{Y(1)_s}} \otimes \Omega^1_{Y/S} & : & 1 \otimes x_i \mapsto 0 & \\
 & & \eta_i \mapsto -1 \otimes 1 \otimes dx_i & (5.14)
\end{array}
$$

It suffices to check these formulas when $\mathbb{T} = \mathbb{F}$. We begin with the following formulas for the right actions of $\mathcal{G}_Y$ on $\mathbb{F}_Y(1)_s$ and $\mathbb{F}_Y(1)_t$, which are consequences of the descriptions in equation (8.8).

$$
\begin{array}{rcl}
r_s^*\colon \mathcal{O}_{\mathbb{F}_{Y(1)_s}} \to \mathcal{O}_{\mathbb{F}_{Y(1)_s}} \otimes \mathcal{O}_{\mathbb{F}_{Y(1)}} & : & a \otimes b \mapsto a \otimes 1 \otimes 1 \otimes b \\
r_t^*\colon \mathcal{O}_{\mathbb{F}_{Y(1)_t}} \to \mathcal{O}_{\mathbb{F}_{Y(1)_t}} \otimes \mathcal{O}_{\mathbb{F}_{Y(1)}} & : & a \otimes b \mapsto 1 \otimes b \otimes 1 \otimes a
\end{array}
$$

Recalling that $1 \otimes x_i = x_i \otimes 1 + \xi_i$, we find

$$
\begin{array}{rcl}
r_s^*\colon x_i \otimes 1 & \mapsto & x_i \otimes 1 \otimes 1 \otimes 1 \\
1 \otimes x_i & \mapsto & 1 \otimes 1 \otimes 1 \otimes x_i \\
\xi_i & \mapsto & 1 \otimes 1 \otimes 1 \otimes x_i - x_i \otimes 1 \otimes 1 \otimes 1 \\
 & = & 1 \otimes 1 \otimes x_i \otimes 1 + 1 \otimes 1 \otimes \xi_i - x_i \otimes 1 \otimes 1 \otimes 1 \\
 & = & 1 \otimes x_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \xi_i - x_i \otimes 1 \otimes 1 \otimes 1 \\
 & = & \xi_i \otimes 1 \otimes 1 + 1 \otimes 1 \otimes \xi_i \\
r_t\colon x_i \otimes 1 & \mapsto & 1 \otimes 1 \otimes 1 \otimes x_i \\
1 \otimes x_i & \mapsto & 1 \otimes x_i \otimes 1 \otimes 1 \\
\xi_i & \mapsto & 1 \otimes x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes 1 \otimes x_i \\
 & = & 1 \otimes x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes x_i \otimes 1 - 1 \otimes 1 \otimes \xi_i \\
 & = & 1 \otimes x_i \otimes 1 \otimes 1 - 1 \otimes x_i \otimes 1 \otimes 1 - 1 \otimes 1 \otimes \xi_i \\
 & = & -1 \otimes 1 \otimes \xi_i
\end{array}
$$

We deduce that:

$$
\nabla_s(x_i \otimes 1) = 0
$$

$$
\begin{aligned}
\nabla_s(\xi_i) &= 1 \otimes 1 \otimes dx_i \\
\nabla_t(1 \otimes x_i) &= 0 \\
\nabla_t(\xi_i) &= -1 \otimes 1 \otimes dx_i,
\end{aligned}
$$

as claimed.

The following proposition, a consequence of the above discussion, summarizes what we shall need.

Proposition 5.2. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes.*

1. *The reduction of $\mathcal{HD}_{\Delta(Y)}$ modulo p is canonically isomorphic to the completion of $S^{\cdot}T_{Y_1/S}$ along the ideal $S^{+}T_{Y_1/S}$ of the zero section. Thus there are ring homomorphisms:*
$$\mathcal{HD}_{\Delta(Y)} \to \hat{S}^{\cdot}T_{Y_1/S} \to \mathcal{O}_Y.$$

2. *If X is a closed subscheme of Y_1 which is smooth over S_1, there is a natural isomorphism*
$$\mathcal{O}_{\Delta_X(Y)}\hat{\otimes}_{\mathcal{O}_Y}\mathcal{HD}_{\Delta(Y)} \to \mathcal{HD}_{\Delta_X(Y)},$$
where the completion is taken with respect to the kernel of the homomorphism $\mathcal{HD}_{\Delta(Y)} \to \mathcal{O}_Y$. $\square$

To prepare for a discussion of prismatic crystals, we let $\mathbf{F}\colon \mathbf{V} \to \mathbb{T}(X/S)$ be the fibered category whose objects are pairs (T, E_T), where T is an object of $\mathbb{T}(X/S)$, where E_T is a p-completely quasi-coherent sheaf of $\mathcal{O}_T$-modules on T (7.20), and where $\mathbf{F}(T, E_T) := T$. Since we may also want to work geometrically, we let $\mathbf{V}E_T = \operatorname{Spec} S^{\cdot}E_T$.

The following result, whose (omitted) proof follows from the above discussions and the methods described, for example in [6] and [26], summarizes various ways of understanding stratifications. For clarity, we state it just for prismatic crystals.

Theorem 5.3. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes and let $X \to Y$ be a closed immersion, where $X/\overline{S}$ is smooth. Suppose that $E \in \mathbf{V}_{\mathbb{T}_Y}$ is a p-completely quasi-coherent sheaf of $\mathcal{O}_{\mathbb{T}_X Y}$-modules on Y. Then the following sets of data are equivalent:*

1. *A right action of $\mathcal{G}_{\Delta X/Y}$ on $\mathbf{V}E$.*

2. *A $\mathcal{G}_{\Delta X/Y}$-stratification on $\mathbf{V}E$.*

3. *A quasi-nilpotent left action of the ring $\mathcal{HD}_{\Delta X/Y}$ on E,*

4. *A quasi-nilpotent left action of the ring $\mathcal{HD}_{\Delta Y}$ on E compatible with its action on $\mathcal{O}_{\mathbb{T}_X(Y)}$.*

5. *A quasi-nilpotent p-connection ∇' on E, compatible with the p-connection on $\mathcal{O}_{\Delta_X(Y)}$.*

□

Remark 5.4. Let $Y = \operatorname{Spf} W[x]\hat{}$, with $\psi(x) = x^p + x$, and let X be the closed subscheme defined by (p, x). Although Y is not a ϕ-scheme, Theorem 2.19 constructs a prismatic envelope $\mathbb{\Delta}_X(Y)$. We saw in Remark 3.7 that $\mathcal{O}_{\mathbb{\Delta}_X(Y)}$ does not inherit a p-connection relative on Y, despite the fact that the arguments here would seem to give an isomorphism $\mathbb{\Delta}_X(Y) \times_Y \mathbb{\Delta}_Y(Y(1)) \cong \mathbb{\Delta}_Y Y(1) \times_Y \mathbb{\Delta}_X(Y)$. The issue is that $\mathbb{\Delta}_Y(Y(1))$ itself is not well behaved, although it too exists. To see this, write $Y(1)$ as the formal spectrum of $W[x_1, x_2]\hat{}$, with $\psi(x_i) = x_i^p + x_i$. Then the ideal of the diagonal is defined by $\xi := x_2 - x_1$. Write x for x_1, so $W[x_1, x_2]\hat{} = W[x, , \xi]\hat{}$, with

$$\psi(\xi) = x_2^p - x_1^p + x_2 - x_1 \equiv \xi^p + \xi \pmod{p}.$$

Then the prismatic envelope of the diagonal is the formal spectrum of the p-adic completion of $W[x, \xi, \eta_1, \eta_2, \ldots]/(p\eta_1 = \xi, p\eta_2 = \eta_1^p - \eta_1, p\eta_3 = \eta_2^p + \eta_2 \cdots]$, and the ideal of the diagonal section is defined by the p-adic closure of the ideal generated by $(\eta_1, \eta_2 \cdots)$. In the quotient by the square of this ideal, we see that $\xi = p\eta_1, \eta_1 = p\eta_2, \eta_2 = p\eta_3 \cdots$, so ξ vanishes in the p-adic completion, and the first infinitesimal neighborhood of the diagonal is just the diagonal itself.

Recall that the ring $\mathcal{HD}_{\mathbb{\Delta}}$ of prismatic differential is $\mathcal{H}om(t_*(\mathcal{O}_{\mathbb{\Delta}_Y(1)}), \mathcal{O}_Y)$. We discuss general differential operators in §8.4 in a geometric context. In the prismatic context, this boils down to the following notions. We refer to the treatment in §8.4 for details and proofs.

Definition 5.5. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes, and let $t, s\colon \mathbb{\Delta}_Y(1) \to Y$ be the two projections.*

1. *If Ω is a sheaf of $\mathcal{O}_Y$-modules, let*

$$\mathcal{L}_{\mathbb{\Delta}}(\Omega) := t_*s^*(\Omega) = \mathcal{O}_{\mathbb{\Delta}_Y(1)}\hat{\otimes}\Omega.$$

2. *If Ω and Ω' are sheaves of $\mathcal{O}_Y$-modules, a (hyper) prismatic differential operator from Ω to Ω' is an $\mathcal{O}_Y$-linear map $t_*s^*(\Omega) \to \Omega'$, i.e., a map*

$$D\colon \mathcal{L}_{\mathbb{\Delta}}(\Omega) = \mathcal{O}_{\mathbb{\Delta}_Y(1)}\hat{\otimes}_{\mathcal{O}_Y}\Omega \to \Omega'.$$

3. *If D is a prismatic differential operator from Ω to Ω', then $\mathcal{L}_{\mathbb{\Delta}}(D)$ is the $\mathcal{O}_Y$-linear map:*

$$\mathcal{L}_{\mathbb{\Delta}}(D) := \mathcal{O}_{\mathbb{\Delta}_Y(1)}\hat{\otimes}\Omega \xrightarrow{\delta\otimes\mathrm{id}} \mathcal{O}_{\mathbb{\Delta}_Y(1)}\hat{\otimes}\mathcal{O}_{\mathbb{\Delta}_Y(1)}\hat{\otimes}\Omega \xrightarrow{\mathrm{id}\otimes D} \mathcal{O}_{\mathbb{\Delta}_Y(1)}\hat{\otimes}\Omega'.$$

4. *If E is an $\mathcal{O}_Y$-module with prismatic stratification ϵ and D is a prismatic differential operator from Ω to Ω', then $\epsilon(D)$ is the prismatic differential*

operator from $E\hat{\otimes}\Omega$ *to* $E\hat{\otimes}\Omega'$ *defined by the following diagram:*

$$
\begin{tikzcd}
\mathcal{L}_{\mathbb{\Delta}}(E\hat{\otimes}\Omega) \arrow[r, "\epsilon(D)"] \arrow[d, "\cong"'] & E\hat{\otimes}\Omega' \\
\mathcal{O}_{\mathbb{\Delta}_Y(1)}\hat{\otimes}E\otimes\Omega \arrow[r, "\epsilon\hat{\otimes}\mathrm{id}"] & E\hat{\otimes}\mathcal{O}_{\mathbb{\Delta}_Y(1)}\hat{\otimes}\Omega. \arrow[u, "\mathrm{id}\hat{\otimes}D"']
\end{tikzcd}
$$

Since $\mathcal{L}_{\mathbb{\Delta}}(\Omega)$ is computed using s but is endowed with the $\mathcal{O}_Y$-module structure coming from t, the prismatic connection on $t_*\mathcal{O}_{\mathbb{\Delta}_Y(1)}$ described in Proposition 3.6 carries over to $\mathcal{L}_{\mathbb{\Delta}}(\Omega)$ and $\mathcal{L}_{\mathbb{\Delta}}(\Omega')$, and if D is a prismatic differential operator from Ω to Ω', the corresponding $\mathcal{O}_Y$-linear map $\mathcal{L}_{\mathbb{\Delta}}(D)$ is horizontal.

There is a commutative diagram:

$$
\begin{tikzcd}
\Omega \arrow[d, "s^*"'] \arrow[dr, "\mathrm{id}"] & \\
\mathcal{L}_{\mathbb{\Delta}}(\Omega) \arrow[r, "\iota^*"] & \Omega,
\end{tikzcd}
\tag{5.15}
$$

where ι^* is induced by the diagonal mapping and $s^*(\omega) := 1\otimes\omega$. When $\Omega = \mathcal{O}_Y$, the map s is compatible with the prismatic connections. (See diagram (8.5).

The following prismatic variant of the crystalline construction [6, 6.15] then follows from Proposition 8.16.

Proposition 5.6. *Let* E *be an* $\mathcal{O}_Y$*-module with prismatic stratification* ϵ*, let* Ω *be an* $\mathcal{O}_Y$*-module, and define*

$$\beta\colon E\otimes\mathcal{L}_{\mathbb{\Delta}}(\Omega) \cong \mathcal{L}_{\mathbb{\Delta}}(E\otimes\Omega)$$

to be $\epsilon\otimes\mathrm{id}$.

1. *The map* β *is a horizontal isomorphism.*
2. *If* D *is a prismatic differential operator from* Ω *to* Ω'*, the following diagram commutes:*

$$
\begin{tikzcd}
E\otimes\mathcal{L}_{\mathbb{\Delta}}(\Omega) \arrow[r, "\beta"] \arrow[d, "\mathrm{id}\otimes\mathcal{L}_{\mathbb{\Delta}}(D)"'] & \mathcal{L}_{\mathbb{\Delta}}(E\otimes\Omega) \arrow[d, "\mathcal{L}_{\mathbb{\Delta}}(\epsilon(D))"] \\
E\otimes\mathcal{L}_{\mathbb{\Delta}}(\Omega') \arrow[r, "\beta"] & \mathcal{L}_{\mathbb{\Delta}}(E\otimes\Omega').
\end{tikzcd}
$$

The differential in the p-de Rham complex of Y/S are prismatic differential operators. As in the crystalline model explained in [6, 6.11], we have the following description of their linearization.

Proposition 5.7. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes. View $\mathbb{A}_Y(1)$ as a formal scheme over Y via the morphism t. Then there is a natural isomorphism of complexes:*

$$(\Omega^{\cdot}_{\mathbb{A}_{Y(1)/Y}}, d') \to \mathcal{L}_{\mathbb{A}}(\Omega^{\cdot}_{Y/S}, d')$$

In particular, if Y/S admits a system of local coordinates $(x_1, \ldots, x_n)$ and $\eta_1, \ldots, \eta_n) \in \mathcal{O}_{\mathbb{A}_{Y(1)}}$ are as in Equation (5.5), then the homomorphism

$$\mathcal{L}_{\mathbb{A}}(d') \colon \mathcal{L}_{\mathbb{A}}(\Omega^i_{Y/S}) \to \mathcal{L}_{\mathbb{A}}(\Omega^{i+1}_{Y/S})$$

is given by

$$\eta_1^{[m_1]} \cdots \eta_n^{[m_n]} \otimes \omega \mapsto \sum_k \eta_1^{[m_1]} \cdots \eta_k^{[m_k-1]} \eta_n^{[m_n]} \otimes dx_k \wedge \omega + \eta_1^{[m_1]} \cdots \eta_n^{[m_n]} \otimes d\omega$$

Proof. As noted in diagram 8.9 in the discussion of the the general construction, there is a commutative diagram:

$$\begin{tikzcd}
\mathcal{O}_{\mathbb{A}_{Y(1)}} \arrow[r, "\mathcal{L}(d')"] & \mathcal{O}_{\mathbb{A}_Y} \otimes \Omega^1_{Y(1)/Y} \arrow[r, "\mathcal{L}(d')"] & \mathcal{O}_{\mathbb{A}_Y} \otimes \Omega^2_{Y(1)/Y} \arrow[r, "d'"] & \cdots \\
\mathcal{O}_{Y(1)} \arrow[u] \arrow[r, "d'"] & \Omega^1_{Y(1)/Y} \arrow[u] \arrow[r, "d'"] & \Omega^2_{Y(1)/Y} \arrow[u] \arrow[r, "d'"] & \cdots \\
\mathcal{O}_Y \arrow[u, "t^\sharp"] \arrow[r, "d'"] & \Omega^1_{Y/S} \arrow[u, "t^*"] \arrow[r, "d'"] & \Omega^2_{Y/S} \arrow[u, "t^*"] \arrow[r, "d'"] & \cdots
\end{tikzcd}$$

As we have argued before, $\mathcal{O}_{\mathbb{A}_{Y(1)}}$ is locally topologically generated by elements a such that $p^n a$ lies in $\mathcal{O}_Y$ for some n, and since the terms of the top complex are p-torsion free and p-adically separated, its differentials $\mathcal{L}(d')$ are uniquely determined by the commutativity of the diagram. Thus they must agree with the differentials of the complex $(\Omega^{\cdot}_{\mathbb{A}_{Y(1)/Y}}, d')$. In the presence of local coordinates, we have $p\eta_i = s^\sharp(x_i) - t^\sharp(x_i)$, so $d'\eta_i = pd\eta_i = dx_i$. The formula in the proposition follows.

∎

5.2 Morphisms among tubular groupoids

Let us explain the relationships among the various groupoids we have now constructed, as well as how they relate to the relative Frobenius morphism. As we shall see, these constructions allow for a geometric framework for Shiho's theory of the F-transform.

Let Y/S be a p-completely smooth morphism of formal ϕ-schemes and $X \to Y$ a closed subscheme, with $X/\overline{S}$ smooth The morphisms of tubes, some of which were illustrated in diagram (2.4), prolong to morphisms of groupoids:

$$\mathcal{G}_{\mathbb{A}X/Y} \to \mathcal{G}_{\mathbb{D}X/Y} \to \mathcal{G}_{\mathbb{P}X/Y} \to \mathcal{G}_{\mathbb{F}X/Y} \tag{5.16}$$

In particular, an action of $\mathcal{G}_{\mathbb{P}X/Y}$ restricts to an action of $\mathcal{G}_{\mathbb{D}X/Y}$, which in turn restricts to an action of $\mathcal{G}_{\mathbb{A}X/Y}$. When $X = \overline{Y}$ and we are working with actions on $\mathcal{O}_Y$-modules, this amounts to restricting an action of the ring $\mathcal{H}_{\mathbb{D}Y}$ to the actions of the p-derivations, *i.e.*, to the underlying p-connection. The following result, inspired by an argument in [18], gives a criterion for extending actions of $\mathcal{G}_{\mathbb{A}}$ to $\mathcal{G}_{\mathbb{D}}$.

Proposition 5.8. *Suppose that E is a p-torsion free sheaf of $\mathcal{O}_Y$-modules endowed with an integrable p-connection ∇' whose action on E/pE has level less than $p-1$ (resp. $p-2$). Then ∇' extends uniquely to an action of $\mathcal{D}_{\mathbb{D}}(Y)$ (resp.to $\mathcal{HD}_{\mathbb{D}}(Y)$).*

Proof. The uniqueness follows from the assumption that E is p-torsion free, and it implies that we can work locally. Choose local coordinates $(t_1, \ldots, t_n)$ on Y/S, let $(\partial_1, \ldots, \partial_n)$ be the basis for $T_{Y/S}$ dual to $(dt_1, \ldots, dt_n)$, and write ∇_i for the corresponding endomorphism of E'. To say that $(E/pE, \nabla')$ has level less than ℓ means that the ideal $\{\oplus S^i T_{Y/S} : i \geq \ell\}$ annihilates E/pE, *i.e.*, that $\nabla_1^{I_1} \cdots \nabla_n^{I_n} E \subseteq pE$ whenever $I_1 + \cdots + I_n \geq \ell$. In particular, if the level is less than $p-1$, then $\nabla_i^{p-1} E \subseteq pE$. It follows that for each $m > 0$

$$\nabla_i^{p^m} E = (\nabla_i^{p-1})^{p^{m-1}} (\nabla_i^{p-1})^{p^{m-2}} \cdots (\nabla_i^{p-1}) E \subseteq p^{p^{m-1}} p^{p^{m-2}} \cdots pE.$$

Recalling that $\mathrm{ord}_p(p^m!) = p^{m-1} + \cdots + 1$, we see that $\nabla_i^{p^m} E \subseteq p^m! E$ and then that $\nabla_i^n \subseteq n! E$ for all n. It then follows from the formulas in (5.8 that the ring $\mathcal{D}_{\mathbb{D}}$ operates on E. If the level is less than $p-2$, we have $\nabla_i^{p-2} E \subseteq pE$, and then

$$\begin{aligned} \nabla_i^{p^m} E &= (\nabla_i^{p-2} \nabla_i)^{p^{m-1}} (\nabla_i^{p-2} \nabla_i)^{p^{m-2}} \cdots (\nabla_i^{p-2}) \nabla_i E \\ &\subseteq p^{p^{m-1}} p^{p^{m-2}} \cdots p \nabla_i^{p^{m-1}} E \\ &\subseteq p^{m!} p^{m-2} E. \end{aligned}$$

This implies that the operation of $\mathcal{D}_{\mathbb{D}}(Y)$ is quasi-nilpotent, and hence that it extends to $\mathcal{HD}_{\mathbb{D}}(Y)$. □

In the prismatic context, the Frobenius lifting induces additional morphisms of groupoids. This gives another approach to the F-transform and Shiho's theorem. The main point is to see how a quasi-nilpotent connection gives rise to descent data for the relative Frobenius morphism, as explained by the morphism of groupoids u in the following proposition.

Proposition 5.9. *Let Y/S be a p-completely smooth morphism of ϕ-schemes, let $\phi_{Y/S} \colon Y \to Y'$ be the associated relative Frobenius morphism, and let $\mathcal{G}_{\Phi Y}$*

be the groupoid over Y corresponding to $\phi_{Y/S}$ as explained in Example 8.9. There are corresponding morphism of groupoids:

$$\begin{tikzcd}
 & \mathcal{G}_{\mathbb{\Phi}_Y} \arrow[d, "u"'] & \\
\mathcal{G}_{\mathbb{\Delta}_Y} \arrow[r, "\Psi"] \arrow[dr, "\mathbb{\Delta}(\phi_{Y/S})"'] & \mathcal{G}_{\mathbb{P}_Y} \arrow[d, "\Phi"] \arrow[dr, "\mathbb{P}(\phi_{Y/S})"] & \\
 & \mathcal{G}_{\mathbb{\Delta}_{Y'}} \arrow[r, "\Psi'"] & \mathcal{G}_{\mathbb{P}_{Y'}}
\end{tikzcd}$$

The morphisms defining Φ are p-completely faithfully flat. Moreover, the composite $\Phi \circ u$ factors through the identity section.

Proof. The existence of the lower rhombus is a consequence of Corollary 2.34, applied with Y replaced by $Y(1)$ and X by the diagonal embedding of $\overline{Y}$. That corollary also explains why Φ is p-completely faithfully flat. Using the same construction with $Y(2)$ in place of $Y(1)$, one can check that Φ and Ψ are compatible with the composition laws and identity sections.

To construct the morphism u, we use the following lemmas.

Lemma 5.10. *Let $X/\overline{S}$ be a morphism of schemes in characteristic p, and let $F_{X/\overline{S}}\colon X \to X'$ be the relative Frobenius morphism, and $\pi\colon X' \to X$ the base change map. Then the scheme theoretic image of the Frobenius endomorphism of $X \times_{X'} X$ is contained in the diagonal.*

Proof. We have a commutative diagram:

$$\begin{tikzcd}
 & X' \arrow[r, "\pi"] \arrow[d, "\Delta_{X'}"] & X \arrow[d, "\Delta_X"] \\
X \times_{X'} X \arrow[ur, dashed] \arrow[r, "F_{X/S} \times F_{X/S}"] & X' \times_{X'} X' \arrow[r, "\pi \times \pi"] & X \times_{X'} X
\end{tikzcd}$$

The composition along the bottom is the absolute Frobenius endomorphism of $X \times_{X'} X$. □

Lemma 5.11. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes, with relative Frobenius morphism $\phi\colon Y \to Y'$. Then $Y \times_{Y'} Y$ inherits the structure of a formal ϕ-scheme, and the diagonal embedding $Y \to Y \times_{Y'} Y$ is defined by a PD ideal.*

Proof. Let $(Z, \phi_Z) := (Y \times_{Y'} Y, \phi_Y \times_{\phi_{Y'}} \phi_Y)$, Since $\phi_{Y/S}$ is p-completely flat, this fiber product is again p-torsion free, by Proposition 7.14, and ϕ_Z is a Frobenius lift. By Proposition 2.4, the scheme theoretic image of $\overline{Z}$ by ϕ_Z is $Z_{\mathbb{P}}$, the smallest PD-subscheme of Z. Applying the previous lemma with $X = \overline{Y}$, we see that this image is contained in the diagonal embedding of Y in Z. Thus the ideal of Y is contained in the maximal PD-ideal $I_{\mathbb{P}}$ of $\mathcal{O}_Z$. Since Y is p-torsion free, it follows that its ideal is also a PD-ideal. $\square$

The map $Y \times_{Y'} Y \to Y \times_S Y$ sends the diagonal of $Y \times_{Y'} Y$ to the diagonal of $Y \times_S Y$, and since the former is defined by a PD ideal, this map factors uniquely through a map u as claimed. Repeating this argument with $Y \times_{Y'} Y \times_{Y'} Y$ in place of $Y \times_{Y'} Y$ shows that u is compatible with the composition laws.

The composite $\Phi \circ u$ is given by the following diagram,

$$\begin{tikzcd}
 & Y' \arrow[dr, "\iota"] & \\
Y \times_{Y'} Y \arrow[ur, dashed] \arrow[r, "u"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathbb{P}_Y(1) \arrow[r, "\Phi"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathbb{A}_{Y'}(1) \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
Y \arrow[r, "\mathrm{id}"] & Y \arrow[r, "\phi_{Y/S}"] & Y'
\end{tikzcd}$$

which shows that $\Phi \circ u$ factors through the identity section.

It may be enlightening to write formulas for these morphisms In terms of the local coordinate description of these groupoids given in equations (5.5). The formals for Ψ and Φ are straightforward:

$$\Psi^\sharp \colon \mathcal{O}_Y\langle \xi_1, \ldots, \xi_n\rangle^\wedge \to \mathcal{O}_Y\langle \eta_1, \ldots, \eta_n\rangle^\wedge \quad : \quad \xi_i \mapsto p\eta_i \tag{5.17}$$
$$\Phi^\sharp \colon \mathcal{O}_{Y'}\langle \eta'_1, \ldots \eta'_n\rangle^\wedge \to \mathcal{O}_Y\langle \xi_1, \ldots, \xi_n\rangle^\wedge \quad : \quad \eta'_i \mapsto \phi_{Y/S}(\xi_i) \tag{5.18}$$

It is apparent from the formulas that $\Psi^\sharp$ is well-defined, but this is less apparent for $\Phi^\sharp$. The point is that $\phi_{Y/S}(\xi'_i) \equiv \xi_i^p$ modulo pI_Y, not just modulo p. This will imply that $\phi_{Y/S}(\eta'_i) \equiv (p-1)!\xi_i^{[p]}$ modulo I_Y, which belongs to the PD ideal I_Y. To check the claimed congruence, we compute as follows:

$$\begin{array}{rcl}
\phi_{Y/S}(\xi') & = & \phi_{Y/S}(1 \otimes x' - x' \otimes 1) \\
 & = & 1 \otimes \phi_{Y/S}(x') - \phi_{Y/S}(x') \otimes 1 \\
 & = & 1 \otimes x^p - x^p \otimes 1 + p \otimes \delta(x) - \delta(x) \otimes p \\
 & = & (x \otimes 1 + \xi)^p - x^p \otimes 1 + p \otimes \delta(x) - \delta(x) \otimes p \\
 & = & \displaystyle\sum_0^p \binom{p}{i} (x^i \otimes 1)\xi^{p-i} - x^p \otimes 1 + p \otimes \delta(x) - \delta(x) \otimes p \\
 & = & \displaystyle\xi^p + \sum_1^{p-1} \binom{p}{i} (x^i \otimes 1)\xi^{p-i} + p(1 \otimes \delta(x) - \delta(x) \otimes 1)
\end{array}$$

Thus $\phi_{Y/S}(\xi') \equiv \xi^p \pmod{pI_Y}$, as claimed. The morphism u is more difficult to write explicitly, as we explain in the remark below. $\square$

Remark 5.12. The morphism u does not factor through the PD-completion of the diagonal of Y in $\mathbb{P}_Y(1)$ and seems to be difficult to compute explicitly. For example, let $Y := \operatorname{Spf} W[t]\hat{}$ and let $\phi(t) = t^p$. Then $Y \times_S Y = \operatorname{Spf} W[t_1, t_2]\hat{}$, where $t_1 := t \otimes 1, t_2 := 1 \otimes t$. Let $\xi := t_2 - t_1$ and view $Y(1)$ as a Y-scheme via p_1. Thus t identifies with t_1, and the ideal of $Y \times_{Y'} Y$ in $Y \times_S Y$ is generated by

$$t_2^p - t_1^p = (t+\xi)^p - t^p = \xi^p + \sum_{i=1}^{p-1} \binom{p}{i} \xi^i t^{p-i} = \xi^p - p\xi y,$$

where

$$y := -(p-1)! \sum_{i=1}^{p-1} \frac{\xi^{i-1} t^{p-i}}{i!(p-i)!} \equiv -t^{p-1} \pmod{\xi}.$$

Then $Y \times_{Y'} Y = \operatorname{Spf} W[t, \xi]/(\xi^p - p\xi y)$, and in this quotient, we have

$$\begin{array}{rcl} \xi^p & = & p\xi y \\ \xi^{pn} & = & p^n \xi^n y^n \\ \xi^{[pn]} & = & \dfrac{p^n n! \xi^{[n]} y^n}{(pn)!} \end{array}$$

Since $\operatorname{ord}_p(pn!) = n + \operatorname{ord}_p(n!)$ and y^n is not divisible by p, we see that $\xi^{[pn]}$ and $\xi^{[n]}$ share the same divisibility by p. In particular, if n is a power of p, $\xi^{[n]}$ is not divisible by p.

Before explaining our proof of Shiho's theorem, we should make explicit the relationship between the F-transform and the morphism Φ.

Lemma 5.13. *Let Y/S be p-completely smooth and (E', ∇') an object of $\mathrm{MICP}(Y'/S)$, with corresponding action r' of $\mathcal{G}_{\mathbb{A}Y'}$. Then the F-transform of (E', ∇') corresponds to the pullback of (E', r') via the morphism Φ of Proposition 5.9. Furthermore, the action of $\mathcal{G}_{\Phi Y}$ on $\phi^*_{Y/S}(E')$ corresponding to its descent data is the pullback of $\Phi^*(r')$ by u.*

Proof. Let (E, ∇) be the F-transform of (E', ∇'). By definition, $E = \phi^*_{Y/S}(E')$, and it remains only to prove that the connection ∇ corresponds to the action $\Phi^*(r')$. This can be checked after restricting to the first infinitesimal neighborhood of the diagonal, where the requisite compatibility is proved in Proposition 4.2. The last statement follows from the fact that $\Phi \circ u$ factors through the identity section. $\square$

We can now explain how Shiho's proof of theorem 4.4 fits into our current context. His proof is quite detailed, so here we give only a sketch.

Proof of Theorem 4.4. Let (E, ∇) be a p-completely quasi-coherent sheaf of $\mathcal{O}_Y$-modules with integrable and quasi-nilpotent connection, and let

$$r\colon \mathbf{V}E \times_Y \mathcal{G}_{\mathbb{P}_Y} \to \mathbf{V}E$$

be the corresponding action of $\mathcal{G}_{\mathbb{P}_Y}$ on $\mathbf{V}E$. (Here we abusively use the same notation for the groupoid $\mathcal{G}_{\mathbb{P}_Y}$ and its object of arrows $\mathbb{P}_Y(1)$.) We claim that there is an $(E', \nabla') \in \mathrm{MICP}(Y'/S)$ whose F-tranform is (E, ∇). Thanks to Lemma 5.13, it will suffice to show that there is a sheaf of $\mathcal{O}_{Y'}$-modules E' with an action r' of $\mathcal{G}_{\mathbb{A}_{Y'}}$ such that $\Phi^*(E', r') \cong (E, r)$.

The pullback of r by the morphism $u\colon \mathcal{G}_{\Phi_Y} \to \mathcal{G}_{\mathbb{P}_Y}$ of Proposition 5.9 induces an action $u^*(r)$ of $\mathcal{G}_{\Phi_Y}$ on $\mathbf{V}E$. This action is in fact descent data for the p-completely faithfully flat morphism $\phi_{Y/S}$. Thanks to Proposition 7.19, these data give rise to a p-completely quasi-coherent sheaf of $\mathcal{O}_{Y'}$-modules E' and an isomorphism $\phi^*_{Y/S}(E') \cong E$, which we view as a linear scheme $\mathbf{V}E'$ over Y' together with an isomorphism $Y \times_{Y'} \mathbf{V}E' \cong \mathbf{V}E$. We claim that r induces an action r' of $\mathcal{G}_{\mathbb{A}_{Y'}}$ on $\mathbf{V}E'$, *i.e.*, that there is a commutative diagram:

$$\begin{tikzcd}
\mathbf{V}E \times_Y \mathcal{G}_{\mathbb{P}_Y} \arrow[r, "r"] \arrow[d, "\pi \times \Phi"'] & \mathbf{V}E \arrow[d, "\pi"] \\
\mathbf{V}E' \times_{Y'} \mathcal{G}_{\mathbb{A}_{Y'}} \arrow[r, dashed, "r'"] & \mathbf{V}E',
\end{tikzcd}$$

Such an r' will automatically satisfy the cocycle conditions in statement (1) of Definition 8.4), since r does.

Applying Theorem 4.8 to the diagonal embedding of $\overline{Y}$ in $Y(1)$, we see that $\mathcal{G}_{\mathbb{P}_Y} := \mathbb{P}_Y(1)$ is the F-transform of $\mathcal{G}_{\mathbb{A}_{Y'}} := \mathbb{A}_{Y'}(1)$, viewed over the formal ϕ-scheme $Y(1)$, via the morphisms t and s. Thus the morphism Φ presents $\mathcal{G}_{\mathbb{A}_{Y'}}$ as the quotient of $\mathcal{G}_{\mathbb{P}_Y}$ by an action of the groupoid $\mathcal{G}_{\Phi Y}(1)$. By Lemma 5.13, this action is the restriction by u of the action of $\mathcal{G}_{\mathbb{P}_{Y(1)}}$ corresponding to the canonical connection on $\mathbb{P}_{Y(1)}$. By construction, this action corresponds to the tautological action of $\mathcal{G}_{\mathbb{P}_{Y(1)}}$ on $\mathbb{P}_Y(1)$. As explained in Proposition 8.10, there is an isomorphism of groupoids $\mathcal{G}_{\mathbb{P}_{Y(1)}} \cong \mathcal{G}_{\mathbb{P}_Y}(1)$, and the tautological action of the former corresponds to the conjugation action by the latter. Thus the action of $\mathcal{G}_{\Phi Y}(1)$ on $\mathcal{G}_{\mathbb{P}_Y}$ is the restriction by u of the action of $\mathcal{G}_{\mathbb{P}_Y}(1)$ given by $h(g_1, g_2) = g_1^{-1} h g_2$. Similarly, by its very definition, the action of $\mathcal{G}_{\Phi Y}$ on $\mathbf{V}E$ is given by the restriction of the action of $\mathcal{G}_{\mathbb{P}_Y}$ via the morphism u, *i.e.*, $vg = r(vu(g))$ for $g \in \mathcal{G}_{\mathbb{P}_\Phi}$.

Since π is the quotient morphism by an action of $\mathcal{G}_{\Phi Y}$ and Φ is the quotient morphism by an action of $\mathcal{G}_{\Phi Y}(1)$, we find that $\pi \times \Phi$ is a quotient morphism by an action of $\mathcal{G}_{\Phi Y}(1)$, where $(v, h)(g_1, g_2) := (vg_1, h(g_1, g_2))$. Thus, to show that r descends, it will suffice to check that $\pi(r(v, h)) = \pi(r(v, h)(g_1, g_2))$ for all $(g_1 . g_2) \in \mathcal{G}_{\Phi Y}(1)$. In fact, since the action of $\mathcal{G}_{\Phi Y}$ on $\mathbf{V}E$ is the restriction of the action r of $\mathcal{G}_{\mathbb{P}_Y}$, we compute:

$$\begin{aligned} r((v,h)(g_1,g_2)) &= r(vg_1, h(g_1,g_2)) \\ &= r(vg_1, g_1^{-1}hg_2) \\ &= r(v, hg_2) \\ &= r(r(v,h)g_2) \\ &= r(v,h)g_2 \end{aligned}$$

Thus, if $g_2 \in \mathcal{G}_{\Phi G}$, we see that $\pi(r(v,h)(g_1,g_2)) = \pi(r(v,h))$, as required. □

6 The prismatic topos and its cohomology

We finally turn to study of the prismatic topos per se. For technical reasons, we shall need to introduce an additional topos, based on small prisms, which is easier to work with, but does not seem to be adequately functorial. In order to focus ideas, we repeat the main definitions of [7], in the special case we are studying here. Throughout this section, we let S be a formal ϕ-scheme. If $S = \operatorname{Spf} A$, then $(A,(p))$ is a bounded prism in the sense of [7].

6.1 The prismatic topos

Let S be a formal ϕ-scheme and let X/S be a morphism of formal schemes; typically X will in fact be a scheme over $\overline{S} := S_1$. Recall from Definition 2.17 that an X/S-prism (resp. a small X-prism) is a pair (T, z_T), where T/S is a formal ϕ-scheme over S and where $z_T \colon \overline{T} \to X$ is an S-morphism (resp. an S-morphism such that the induced map $\overline{T} \to \overline{X}$ is flat. (The category of X/S-prisms is the same as the category of $\overline{X}/S$ prisms, but we will find the additional notational flexibility convenient.)

Definition 6.1. *Let S be a formal ϕ-scheme and X/S a morphism of p-adic formal schemes. Then $\mathbb{\Delta}(X/S)$ is the category of X/S-prisms, and $\mathbb{\Delta}_s(X/S)$ is the full subcategory of small X/S-prisms. We endow these categories with the topology in which the coverings are the ϕ-morphisms $f \colon (\widetilde{T}, z_{\widetilde{T}}) \to (T, z_T)$ where f is quasi-compact and p-completely faithfully flat (7.12). We may also sometimes consider $\mathbb{\Delta}(X/S)$ and $\mathbb{\Delta}_s(X/S)$ with the Zariski or p-completely étale topology.*

The following result explains the relation between these two sites and establishes functoriality of the prismatic topos. Note that we have not established functoriality of the small prismatic topos.

Proposition 6.2. *Let $X/\overline{S}$ be a smooth morphism of schemes. Then $\mathbb{\Delta}(X/S)$ and $\mathbb{\Delta}_s(X/S)$, endowed with any of the topologies above, form sites. Furthermore, the inclusion functor $u \colon \mathbb{\Delta}_s(X/S) \to \mathbb{\Delta}(X/S)$ is continuous and cocontinuous and hence induces a morphism of topoi:*

$$u \colon (X/S)_{\mathbb{\Delta}_s} \to (X/S)_{\mathbb{\Delta}}.$$

A morphism $f\colon \overline{X} \to \overline{X}'$ of smooth schemes over $\overline{S}$ induces a continuous and cocontinuous map $\mathbb{\Delta}(f)\colon \mathbb{\Delta}(X/S) \to \mathbb{\Delta}(X'/S)$ and hence a morphism of topoi

$$f_{\mathbb{\Delta}}\colon (X/S)_{\mathbb{\Delta}} \to (X'/S)_{\mathbb{\Delta}}.$$

Proof. It is verified in [7, 3.12] that $\mathbb{\Delta}(X/S)$ does in fact form a site with the p-completely flat topology. The Zariski topology and étale topologies are even easier. Let us check that the same is true for $\mathbb{\Delta}_s(X/S)$. It is clear that isomorphisms are coverings and that the composition of coverings is a covering (see Proposition 7.14). Furthermore, if $T' \to T$ is a morphism and $\widetilde{T} \to T$ is a covering in $\mathbb{\Delta}_s(X/S)$, then Proposition 7.14 implies that the fiber product $T' \times_T \widetilde{T}$ is p-torsion free. Furthermore, the map $\widetilde{T}_1 \to T_1$ is flat, hence so is the map $T'_1 \times_T \widetilde{T}_1 \to T'_1$, and since $T'_1 \to X$ is flat, so is the map $(T' \times_T \widetilde{T})_1 \to X$. Thus $T' \times_T \widetilde{T}$ is again small.

To prove that u is cocontinuous, let $\widetilde{T} \to T$ be a cover in $\mathbb{\Delta}_s(X/S)$ and let $\widetilde{T}' \to u(\widetilde{T})$ be a cover in $\mathbb{\Delta}(X/S)$. Then $\widetilde{T}' \to u(\widetilde{T})$ is p-completely flat, hence $\widetilde{T}'_1 \to T_1$ is flat, hence $\widetilde{T}'_1 \to X$ is flat, so T' is again small, hence in the image of u. To check that u is continuous, observe that if $\widetilde{T} \to T$ is a covering in $\mathbb{\Delta}_s(X/S)$, then $u(\widetilde{T}) \to u(T)$ is a covering in $\mathbb{\Delta}(X/S)$, and if $T' \to T$ is a morphism in $\mathbb{\Delta}_s(X/S)$, then the fiber product $\widetilde{T} \times_T T'$ is again small, hence the map $u(\widetilde{T} \times_T T') \to u(\widetilde{T}) \times_{u(T)} u(T')$ is an isomorphism.

It is also checked in [7, 4.3] that a morphism $f\colon X \to X'$ of S-schemes induces a morphism of topoi. The argument there is rather abstract; here is another. If (T, z_T) is an X-prism, then

$$\mathbb{\Delta}(f)(T, z_T) := (T, f \circ z_T)$$

is an X'-prism. Thus we find a functor

$$\mathbb{\Delta}(f)\colon \mathbb{\Delta}(X/S) \to \mathbb{\Delta}(X'/S) : (T, z_T) \mapsto (T, f \circ z_T)$$

To check that this functor is cocontinuous, let $g\colon (\widetilde{T}, z_{\widetilde{T}}) \to (T, f \circ z_T)$ be a covering of X'-prisms. The diagram

$$\begin{tikzcd}
T_1 \arrow[d, "z_T"'] & \widetilde{T}_1 \arrow[l, "g_1"'] \arrow[d, "z_{\widetilde{T}}"] \\
X \arrow[r, "f"] & X'
\end{tikzcd}$$

commutes. Then $(\widetilde{T}, z_T \circ g_1)$ is an X-prism, and the map g defines a covering of X-prisms $(\widetilde{T}, z_T \circ g_1) \to (T, z_T)$ with $\mathbb{\Delta}(f)(\widetilde{T}, z_{\widetilde{T}} \circ g_1) = (\widetilde{T}, z_{\widetilde{T}})$. To check that $\mathbb{\Delta}(f)$ is continuous, suppose instead that $(\widetilde{T}, z_{\widetilde{T}}) \to (T, z_T)$ is a covering of X-prisms. Then $(\widetilde{T}, f \circ z_{\widetilde{T}}) \to (T, f \circ z_T)$ is a covering of X'-prisms. Furthermore, if $T' \to T$ is a morphism in $\mathbb{\Delta}(X/S)$, then from the description of fiber products we saw in Proposition 2.36, it is clear that the map

$$\mathbb{\Delta}(f)\left((\widetilde{T}, z_{\widetilde{T}}) \times_{(T,z_T)} (T', z_{T'})\right) \to (\widetilde{T}, f \circ z_{\widetilde{T}}) \times_{(T, f\circ z_T)} (T', f \circ z_{T'})$$

is an isomorphism. $\square$

The following result illustrates an important advantage of working with small prisms. Before stating it, we introduce a useful, if somewhat abusive, abbreviated notation. Suppose that $f\colon X \to X'$ is a closed immersion of S-schemes and $(T', z_{T'})$ is an X'-prism. Then, $z_{T'}^{-1}(X) \to T'$ is also a closed immersion, and we can form the prismatic envelope $\mathbb{\Delta}_{z_{T'}^{-1}(X)}(T')$. The map $\overline{\mathbb{\Delta}}_{z_{T'}^{-1}(X)}(T') \to z_{T'}^{-1}(X) \to X$ endows $\mathbb{\Delta}_X(T')$ with the structure of an X-prism, and if no confusion is likely, we write

$$\mathbb{\Delta}_X(T') := f_{\mathbb{\Delta}}^{-1}(T') := \mathbb{\Delta}_{z_{T'}^{-1}(X)}(T') \tag{6.1}$$

Proposition 6.3. *Suppose that Y/S is a p-completely smooth morphism of formal ϕ-schemes and $i\colon X \to \overline{Y}$ is a regular closed immersion.*

1. *If T is a small Y-prism, the prismatic envelope $\mathbb{\Delta}_X(T)$ of $z_T^{-1}(X)$ in T is a small X-prism. Furthermore, if $T' \to T$ is a morphism in $\mathbb{\Delta}_s(Y/S)$, the natural map*
$$\mathbb{\Delta}_X(T') \to T' \times_T \mathbb{\Delta}_X(T)$$
is an isomorphism.

2. *The prismatic envelope $\mathbb{\Delta}_X(Y(1))$ of the composition $X \to Y \to Y(1)$ of i with the diagonal embedding in $Y(1)$ is small. Furthermore, if $T \to Y$ is any morphism in $\mathbb{\Delta}_s(Y/S)$, the natural map*
$$\mathbb{\Delta}_X(T \times_S Y) \to T \times_Y \mathbb{\Delta}_X(Y(1))$$
is an isomorphism.

Proof. If T is a small Y-prism, the map $z_T\colon \overline{T} \to \overline{Y}$ is flat, and since $X \to \overline{Y}$ is a regular immersion, the same is true of the map $z_T^{-1}(X) \to \overline{T}$. Then it follows from statement (2) of Theorem 2.19 that the map $\overline{\mathbb{\Delta}}_X(T) \to X$ is flat, so $\mathbb{\Delta}_X(T)$ is a small X-prism. Suppose $T' \to T$ is a morphism in $\mathbb{\Delta}_s(Y/S)$. Working locally, choose an $\mathcal{O}_{\overline{Y}}$-regular sequence which generates the ideal of X in $\overline{Y}$. Since z_T and $z_{T'}$ are flat, this sequence remains regular in $\overline{T}$ and $\overline{T}'$. Then statement (1) of Proposition 2.38 implies that the map $\mathbb{\Delta}_X(T') \to T' \times_T \mathbb{\Delta}_X(T)$ is an isomorphism.

Since Y/S is smooth, the diagonal $\overline{Y} \to \overline{Y} \times_S \overline{Y}$ is a regular immersion, and hence so is the the map $X \to \overline{Y}(1)$. Then statement (1) implies that $\mathbb{\Delta}_X(Y)(1)$ is small. Furthermore, if $T \to Y$ is a morphism of small Y/S-prisms, the map

$\overline{T} \to \overline{Y}$ is flat, hence $\overline{T} \times_S \overline{Y} \to \overline{Y} \times_S \overline{Y}$ is also flat, and hence $T \times_S Y \to Y(1)$ is p-completely flat. Thus it follows from Theorem 2.19 that the natural map

$$\mathbb{\Delta}_X(T \times_S Y) \to (T \times_S Y) \times_{Y(1)} \mathbb{\Delta}_X(Y)(1) \cong T \times_Y \mathbb{\Delta}_X(Y)(1)$$

is an isomorphism. $\square$

Note that Propositions 6.3 is not true if we work with full prismatic sites. For example, let $S = \operatorname{Spf} W$, let $X = \operatorname{Spec} k$, and let $Y = \operatorname{Spf} W[x]^{\hat{}}$, with $\phi(x) := x^p$ and the inclusion $X \to Y$ defined by (p, x). Then the prismatic envelope $\mathbb{\Delta}_X(Y) = \operatorname{Spf} W\langle x/p\rangle^{\hat{}}$ of X in Y can also be viewed as a Y-prism T' via the (not flat) map $\overline{\mathbb{\Delta}}_X(Y) \to X \to \overline{Y}$, and the natural map $f \colon T' \to Y$ is a morphism in $\mathbb{\Delta}(Y/S)$. However, the map

$$\mathbb{\Delta}_X(T') \to T' \times_T \mathbb{\Delta}_X(Y) \tag{6.2}$$

is not an isomorphism. Indeed, $\mathbb{\Delta}_X(T') \cong \mathbb{\Delta}_X(Y) = \operatorname{Spf} W\langle x/p\rangle$, but

$$T' \times_Y \mathbb{\Delta}_X(Y) \cong \operatorname{Spf}\left(W\langle x/p\rangle^{\hat{}} \hat{\otimes}_{W[x]^{\hat{}}} W\langle x/p\rangle^{\hat{}}\right),$$

because of all the p-torsion in the tensor product. It could be argued that (6.2) is an isomorphism if the fiber product appearing is taken in the category of formal ϕ-schemes, but this would involve killing an incomputable amount of p-torsion.

Proposition 6.4. *Let $i \colon X \to X'$ be a closed immersion of smooth schemes over $\overline{S}$. If $T' \in \mathbb{\Delta}(X'/S)$, the sheaf $i^{-1}(T')$ in $(X/S)_{\mathbb{\Delta}}$ is represented by $\mathbb{\Delta}_X(T')$, which is small if T' is small.*

Proof. For any morphism $f \colon X \to X'$, the functor $f^{-1}(T')$ is by definition the sheaf on $\mathbb{\Delta}(X/S)$ sending an object $T \in \mathbb{\Delta}(X/S)$ to the set of ϕ-morphisms $v \colon T \to T'$ such that $z_{T'} \circ \overline{v} = f \circ z_T$, *i.e.*, such that $(z_T, \overline{v})$ factors through $X \times_{X'} \overline{T}'$. If $f = i$ is a closed immersion, then this functor is represented by $\mathbb{\Delta}_X(T')$. Furthermore, since X and X' are smooth, i is a regular immersion, and since $\overline{T}' \to X'$ is flat, the immersion $X \times_{X'} \overline{T}' \to \overline{T}'$ is also regular. Then Theorem 2.19 tells us that $\mathbb{\Delta}_X(T')$ is also small. $\square$

6.2 Coverings of prismatic final objects

Let S be a formal ϕ-scheme and $X/\overline{S}$ a smooth morphism. We shall describe two constructions of coverings of the final object of the topos $(X/S)_{\mathbb{\Delta}}$. The first of these, based on [7, 4.16, 4.17] and a conversation with A. Shiho, is in fact a covering in the Zariski topology. However, it seems to be quite unwieldy in practice.

Proposition 6.5. *Let S be a formal ϕ-scheme, let $Y \to S$ be a p-completely smooth morphism of p-adic formal schemes, and let $r \colon Y_\phi \to Y$ be the universal morphism from a formal ϕ-S-scheme to Y described in Example 1.2. If $X \to \overline{Y}$*

is a closed immersion, let $\tilde{X} := r^{-1}(X)$, and let $(\mathbb{\Delta}_{\tilde{X}}(Y_\phi), z_{\tilde{X}})$ be the prismatic envelope of $\tilde{X}$ in Y_ϕ. Then

$$(\mathbb{\Delta}_X(Y_\phi), r_{|_{\tilde{X}}} \circ z_{\tilde{X}})$$

is a small X-prism and is a covering of the final objects of $(X/S)_{\mathbb{\Delta}}$ and $(X/S)_{\mathbb{\Delta}_s}$ respectively, if these are endowed with the Zariski topology.

Proof. We first claim that $r\colon Y_\phi \to Y$ is p-completely flat. This can be checked étale locally on Y, so we may assume that Y and S are affine and that Y is étale over an affine space over S. Then as we saw in the explicit construction in Example 1.2, the space Y_ϕ is étale over an affine space over Y, hence p-completely flat. Since $X \to \overline{Y}$ is a closed immersion of smooth $]ovS$-schemes, it is a regular immersion, and since $\overline{Y}_\phi \to \overline{Y}$ is flat, $\tilde{X} \to \overline{Y}_\phi$ is also a regular immersion. Then it follows from Theorem 2.19 that $\overline{Y}_\phi \to \tilde{X}$ is flat. Since $r_{|_{\tilde{X}}}$ is also flat, we can conclude that $\mathbb{\Delta}_X(Y_\phi)$ is small.

We claim that every affine X-prism admits a morphism to $\mathbb{\Delta}_X(Y_\phi)$. Since T is p-adically complete and affine and Y/S is formally smooth, the map $\overline{T} \to X \to Y$ extends to an S-morphism $T \to Y$, which in turn lifts to a unique morphism of formal ϕ-schemes $T \to Y_\phi$. This morphism necessarily maps $\overline{T}$ to $r^{-1}(X)$ and hence factors through $\mathbb{\Delta}_X(Y_\phi)$. □

We should explain our characterization of $\mathbb{\Delta}_X(Y_\phi)$ as "unwieldy." This is partly because Y_ϕ is itself somewhat unwieldy, but there is more to the story. For example, suppose that $X = \operatorname{Spec} k[x]$ and $Y = \operatorname{Spf} W[x]^\wedge$. Then $Y_\phi = \operatorname{Spf} W[x_0, x_1, \ldots]^\wedge$ with $\phi(x_i) = x_i^p + px_{i+1}$, as we saw in Example 1.2. Here $\tilde{X} := r^{-1}(X) = \overline{Y}_\phi$, which does not seem so unwieldy. However to do cohomology calculations, one needs to understand $\mathbb{\Delta}_X(Y_\phi(1))$, the prismatic envelope of $\tilde{X} \times_X \tilde{X}$ in $Y_\phi \times Y_\phi$, which seems very difficult to describe explicitly.

Fortunately, it turns out that, if one is willing to use the p-completely flat topology, the construction of coverings of the final object becomes much simpler. The following proposition was inspired by a result [20, 3.4] of Morrow and Tsuji, which it generalizes.

Proposition 6.6. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes and let $X \to \overline{Y}$ be a closed immersion, where $X/\overline{S}$ is smooth. Then the prismatic envelope $\mathbb{\Delta}_X(Y)$ of X in Y is small and is a covering of the final objects of $(X/S)_{\mathbb{\Delta}}$ and $(X/S)_{\mathbb{\Delta}_s}$ respectively, if these are endowed with the p-completely flat topology.*

Proof. Since $X \to \overline{Y}$ is a regular immersion, statement (2) of Theorem 2.19 shows that $\mathbb{\Delta}_X(Y)$ is small. To show that it covers the final object of $(X/S)_{\mathbb{\Delta}}$, we shall show that if T is an affine X-prism, then there exists a p-completely faithfully flat morphism of ϕ-schemes $\widetilde{T} \to T$ and a morphism of X-prisms $\widetilde{T} \to \mathbb{\Delta}_X(Y)$. As in the previous proposition, the formal smoothness of the underlying morphism $Y \to S$ implies that the map $f_1\colon T_1 \to X \to Y$ lifts to a morphism $T \to Y$. This map may not be compatible with the Frobenius liftings, but Proposition 1.12 shows that we may find a p-completely faithfully flat $u\colon \widetilde{T} \to T$ and a morphism of ϕ-schemes $\tilde{f}\colon \widetilde{T} \to Y$ such that $\tilde{f}_1 = u_1 \circ f_1$. Then $\widetilde{T}$ becomes an X-prism over Y and hence $\tilde{f}$ factors through $\Delta_X(Y)$. $\square$

6.3 Prismatic crystals

Continuing to follow [7], we endow $(X/S)_{\Delta}$ with the structure of a ringed topos, with structure sheaf $\mathcal{O}_{X/S}$ given by $T \mapsto \mathcal{O}_T$. This sheaf is p-completely quasi-coherent, in the sense of Definition 7.20.

Definition 6.7. *If S is a formal ϕ-scheme and X/S is a scheme, a crystal of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$ (resp. $\Delta_s(X/S)$) is a p-completely quasi-coherent sheaf E of $\mathcal{O}_{X/S}$-modules such that for each morphism $f\colon T' \to T$ in $\Delta(X/S)$ (resp. $\Delta_s(X/S)$), the corresponding transition map $f^*(E_T) \to E_{T'}$ is an isomorphism.*

A crystal E of $\mathcal{O}_{X/S}$-modules can also be viewed geometrically. If T is an object of $\Delta(X/S)$, then E_T is a p-completely quasi-coherent sheaf of $\mathcal{O}_T$-modules, and we let $\mathbf{V}E_T$ denote the corresponding formal scheme, as described in Remark 7.22. Then $T \mapsto \mathbf{V}E_T$ is a crystal of formal schemes. More precisely let $\mathbf{V}\Delta(X/S)$ be the category of morphisms of p-adic formal schemes $V \to T$, where T is an object of $\Delta(X/S)$, and let $\mathbf{F}\colon \mathbf{V}\Delta(X/S) \to \Delta(X/S)$ be the functor taking a morphism to its target. Then $\mathbf{V}\Delta(X/S)$ is fibered over $\Delta(X/S)$, and the assignment $T \mapsto (\mathbf{V}E_T \to T)$ defines a crystal in $\mathbf{V}\Delta(X/S)$ in the sense of Definition 8.13.

Proposition 6.8. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes and $X \to \overline{Y}$ a closed immersion, where $X/\overline{S}$ is smooth.*

1. *The morphism $u\colon \Delta_s(X/S) \to \Delta(X/S)$ induces an equivalence between the corresponding categories of crystals of $\mathcal{O}_{X/S}$-modules.*

2. *The functor*

$$u^* \circ i_{\Delta *}\colon (X/S)_{\Delta} \to (Y/S)_{\Delta} \to (Y/S)_{\Delta_s}$$

takes crystals of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$ to crystals of $\mathcal{O}_{Y/S}$-modules on $\Delta_s(Y/S)$. In particular, $\mathcal{A}_{X/Y/S} := u^* i_{\Delta *}(\mathcal{O}_{\Delta})$ is a crystal of $\mathcal{O}_{Y/S}$-algebras on $\Delta_s(Y/S)$. If E is a crystal of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$, then $u^* i_{\Delta *}(E)$ inherits the structure of a p-completely quasi-coherent $\mathcal{A}_{X/Y/S}$-module.

3. *The functor $E \mapsto u^* i_{\Delta *}(E)$ is an equivalence from the category of crystals of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$ to the category of crystals of p-completely quasi-coherent crystals of $\mathcal{A}_{X/Y/S}$-modules on $\Delta_s(Y/S)$.*

Proof. The general formalism of crystals, reviewed in §8.3, tells us that if T is a covering of the final object, the category of crystals of $\mathcal{O}_{X/S}$-modules is equivalent to the category of p-completely quasi-coherent $\mathcal{O}_T$-modules endowed with a stratification; this holds both on $\Delta(X/S)$ and on $\Delta_s(X/S)$. Proposition 6.6

tells us that $\Delta_X(Y)$ is such a covering, and Proposition 6.3 tells us that $\Delta_X(Y)$ and $\Delta_X(Y(1))$ are small. Statement (1) follows.

Recall from Proposition 6.4 that if T is a Y-prism, the sheaf $i_{\Delta}^{-1}(T)$ is represented by the prismatic envelope $(\Delta_X(T), z_{\Delta}, \pi_T)$ of $z_T^{-1}(X)$ in T. If E is a sheaf in $(X/S)_{\Delta}$, it follows that

$$(i_{\Delta*}(E))_T = \pi_{T*}(E_{\Delta_X(T)}). \tag{6.3}$$

The morphism π_T is affine, and we can conclude that, if E is a sheaf of p-completely quasi-coherent $\mathcal{O}_{X/S}$-modules, then $i_{\Delta*}(E)$ becomes a p-completely quasi-coherent sheaf of $\mathcal{O}_{Y/S}$-modules. Moreover, $(\mathcal{A}_{X/Y/S})_T = \pi_{T*}(\mathcal{O}_{\Delta_X(T)})$, and thus $(i_{\Delta*}(E))_T$ inherits the structure of a p-completely quasi-coherent $\mathcal{A}_{X/Y/S}$-module.

Suppose that E is a crystal of $\mathcal{O}_{X/S}$-modules in $(X/S)_{\Delta}$ and $f\colon T' \to T$ is a morphism in $\Delta(Y/S)$. This map induces a morphism $\Delta(f)\colon \Delta_X(T') \to \Delta_X(T)$, fitting into a commutative diagram:

$$\begin{tikzcd}
\Delta_X(T') \arrow[r, "\Delta(f)"] \arrow[d, "\pi_{T'}"'] & \Delta_X(T) \arrow[d, "\pi_T"] \\
T' \arrow[r, "f"] & T
\end{tikzcd}$$

Proposition 6.3 tells us that this diagram is Cartesian if T' and T are small. Since π_T and $\pi_{T'}$ are affine, this implies that the map

$$f^*\pi_{T*}(E_T) \to \pi_{T'*}\Delta(f^*)(E_T)$$

is an isomorphism. Since E is a crystal on $\Delta(X/S)$, the map

$$\Delta(f)^*(E_{\Delta_X(T)}) \to E_{\Delta_{X(T')}}$$

is also an isomorphism. As we have seen, $i_{\Delta*}(E)_T = \pi_{T*}(E_{\Delta_X(T)})$, and similarly for T'. Thus $u^*i_{\Delta*}(E)$ forms a crystal of $\mathcal{O}_{Y/S}$-modules on $\Delta_s(Y/S)$. This proves statement (2).

To prove statement (3), suppose that E is a crystal of p-completely quasi-coherent $\mathcal{A}_{X/Y/S}$-modules on $\Delta_s(Y/S)$. Then E_Y and $(\mathcal{A}_{X/Y/S})_Y$ are sheaves of $\mathcal{O}_Y$-modules endowed with prismatic stratifications; in fact E_Y has a structure of a p-completely quasi-coherent $(\mathcal{A}_{X/Y/S})_Y$-module, compatible with the given stratifications. But $(\mathcal{A}_{X/Y/S})_Y = \pi_{T*}(\mathcal{O}_{\Delta_X(Y)})$, and so E_Y can be viewed as a sheaf of $\mathcal{O}_{\Delta_X(Y)}$-modules on $\Delta_X(Y)$, and is endowed with prismatic stratification. As we recalled in the proof of statement (1), this gives rise to a crystal of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$. It is clear that this construction gives a quasi-inverse to the functor $u^*i_{\Delta*}$. $\square$

Note that Proposition 6.8 is not true if we work with full prismatic sites, as the discussion after Proposition 6.3 shows.

The construction of coverings in Proposition 6.6 enables us to establish the speculated relationship between prismatic crystals and p-connections sketched in the introduction.

Theorem 6.9. *With the hypothesis of Proposition 6.8, there is a natural equivalence, made explicit below, from the category* $\mathrm{MICP}(X/Y/S)$ *given in Definition 3.10, to the category of crystals of* $\mathcal{O}_{X/S}$*-modules on* $\mathbb{\Delta}(X/S)$.

Proof. Recall than an object (E, ∇') of $\mathrm{MICP}(X/Y/S)$ is a p-completely quasi-coherent sheaf of $\pi_{Y*}(\mathcal{O}_{\mathbb{\Delta}_X(Y)})$-modules E together with a quasi-nilpotent p-connection $\nabla' \colon E' \to \Omega^1_{Y/S} \otimes E'$ which is compatible with the canonical p-connection on $\pi_{Y*}(\mathcal{O}_{\mathbb{\Delta}_X(Y)})$. As we saw in Theorem 5.3, the connection ∇' induces a prismatic stratification on E', which is compatible with the prismatic stratification on $\pi_{Y*}(\mathcal{O}_{\mathbb{\Delta}_X(Y)})$. These data lead to a crystal of p-completely quasi-coherent $\mathcal{A}_{X/Y/S}$-modules on Y/S, and so the theorem follows from statement (3) of Proposition 6.8.

$\square$

The following special case is worth stating.

Corollary 6.10. *If* Y/S *is a* p*-completely smooth morphism of formal* ϕ*-schemes, the category of* $\mathcal{O}_{Y/S}$*-modules on* $\mathbb{\Delta}(Y/S)$ *is equivalent to the category of* p*-completely quasi-coherent* $\mathcal{O}_Y$*-modules with quasi-nilpotent* p*-connection.* $\square$

6.4 Cohomology of the prismatic topos

We are now prepared to prove the motivating results of this project. Let S be a formal ϕ-scheme and $X/\overline{S}$ a smooth morphism. A p-completely quasi-coherent sheaf of $\mathcal{O}_{X/S}$-modules on $\mathbb{\Delta}(X/S)$ assigns to each X-prism T a p-completely quasi-coherent sheaf of $\mathcal{O}_T$-modules E_T. In particular, each of these is a "p-adic sheaf" in the sense of Definition 7.2, and we shall identify it with the inverse system $E_{T,n} : n \in \mathbf{N}$ of its reductions modulo powers of p. As explained in §7.2, the category of p-adic sheaves is not abelian, but is an exact subcategory of the abelian category of inverse systems of p-torsion sheaves. In particular, in forming the derived category $D^+_{\mathbb{\Delta}}(X/S)$ of sheaves of $\mathcal{O}_{X/S}$-modules on $\mathbb{\Delta}(X/S)$, we localize by morphisms of complexes which are strict quasi-isomorphisms, *i.e.*, are quasi-isomorphisms modulo every power of p; (see Definition 7.5). We have a morphism of topoi

$$v_{X/S} \colon (X/S)_{\mathbb{\Delta}} \to X_{\acute{e}t},$$

and hence derived functors

$$Rv_{X/S*} \colon D^+_{\mathbb{\Delta}}(X/S) \to D^+(X_{\acute{e}t}),$$

where the latter is the derived category of inverse systems of abelian sheaves on $X_{\acute{e}t}$.

If T is an X/S-prism, let $\mathbb{\Delta}(X/S)_{|T}$ denote the localized site whose objects are morphisms $T' \to T$ in $\mathbb{\Delta}(X/S)$, and let $s\colon \mathbb{\Delta}(X/S)_{|T} \to \mathbb{\Delta}(X/S)$ be the morphism taking $T' \to T$ to T. As explained in [6, 5.23], there is a corresponding morphism a corresponding morphism of topoi

$$j_T\colon ((X/S)_{\mathbb{\Delta}})_{|T} \to (X/S)_{\mathbb{\Delta}},$$

where j_T^{-1} takes the (sheaf represented by) T' to (the sheaf represented by)

$$\mathbb{\Delta}_X(T' \times_S T) := \mathbb{\Delta}_{X'}(T' \times_S T),$$

where $X' := (z_{T'} \times z_T)^{-1}(X)$. Note that if T is small, then T/S is p-completely flat, hence $T' \times_S T$ is p-torsion free, and the product is computed in the category of formal schemes.

We shall need the following analog of [6, 5.26]. We state it for $\mathbb{\Delta}(X/S)$, but the same arguments show it also holds for $\mathbb{\Delta}_s(X/S)$.

Proposition 6.11. *If T is an object of $\mathbb{\Delta}(X/S)$, let T_{pcf} denote the topos of sheaves on the site defined by the p-completely flat coverings of T. Then there is a 2-commutative diagram of morphisms of topoi:*

$$\begin{CD}
((X/S)_{\mathbb{\Delta}_s})_{|T} @>{\phi}>> T_{\mathrm{pcf}} \\
@V{j_T}VV @VV{\lambda}V \\
(X/S)_{\mathbb{\Delta}_s} @>{v_{X/S}}>> X_{\acute{e}t}.
\end{CD}$$

Furthermore, the following statements are verified.

1. *The functor ϕ_* is exact. If E is a sheaf on T_{pcf}, then*

$$v_{X/S*} j_{T*} \phi^*(E) \cong \lambda_*(E).$$

2. *If E_T is a p-completely quasi-coherent sheaf of $\mathcal{O}_T$-modules on T_{pcf}, then $\phi^* E$ is a crystal of $\mathcal{O}_{X/S}$-modules on $((X/S)_{\mathbb{\Delta}})_{|T}$.*

3. *If $z_T\colon \overline{T} \to X$ is affine and E is a crystal of $\mathcal{O}_{X/S}$-modules on $((X/S)_{\mathbb{\Delta}})_{|T}$, then $R^q j_{T*}(E)$ and $R^q v_{X/S*}(j_{T*}(E))$ vanish for $q > 0$.*

Proof. If E is a sheaf in $((Y/S)_{\mathbb{\Delta}})_{|T}$, then $\phi_*(E) := E_{(T,\mathrm{id})}$, and if E_T is a sheaf on T and $(T', h') \in (\mathbb{\Delta}(X/S))_{|T}$, then $\phi^{-1}(E_{T'}) = h'^{-1}(E_T)$. It is clear that ϕ^{-1} is left adjoint to ϕ_*, that ϕ^{-1} and ϕ_* are exact, and that $\phi^*(E)$ is a crystal if E is p-completely quasi-coherent. The morphism λ is defined as the composition

$$\lambda := T_{\mathrm{pcf}} \longrightarrow T_{\acute{e}t} \xrightarrow{z_T} Z_{\acute{e}t}.$$

The 2-commutativity of the diagram is easy to check from the definitions. Then if E is a sheaf on T_{pcf},

$$v_{X/S*} j_{T*} \phi^*(E) \cong \lambda_* \phi_* \phi^*(E) = \lambda_*(\phi^*(E)_{(T,\mathrm{id})}) = \lambda_*(E).$$

This completes the proof of statements (1) and (2).

To prove (3), let T' be an object of $\Delta(X/S)$, let $T'' := \Delta_X(T' \times_S T)$, with projections $h\colon T'' \to T$, and $h'\colon T'' \to T'$. Then $(T'', h) \in (\Delta(X/S))_{|T}$, and if E is a sheaf in $(\Delta(X/S))_{|T}$, then $(j_{T*}E)_{T'} = h'_*(E_{(T'',h)})$. There is a Cartesian diagram:

$$\begin{tikzcd} X' \arrow[r] \arrow[d] & \overline{T} \arrow[d, "z_T"] \\ \overline{T}' \arrow[r, "z_{T'}"] & X \end{tikzcd}$$

and, since $z_T\colon \overline{T} \to X$ is by assumption affine. the map $X' \to \overline{T}'$ is also affine. Since the map $\overline{T}'' \to X'$ is affine by the construction in Theorem (2.19), we conclude that the composition $\overline{h}'\colon \overline{T}'' \to X' \to \overline{T}$ is also affine. If E is a p-completely quasi-coherent sheaf in $((X/S)_{\Delta})_{|T}$, then $E_{(T'',h)}$ is p-completely quasi-coherent on T'', and it follows from Proposition 7.21 that $R^q h'_* E_{(T'',h)} = 0$ vanishes for $q > 0$. Since this holds for all $T' \in \Delta(X/S)$, we can conclude that $R^q j_{T*}E$ vanishes for $q > 0$. Since $\phi_*(E)$ is also p-completely quasi-coherent and z_T is affine, it follows also that $R^q\lambda_*\phi_*E$ also vanishes for $q > 0$. Since ϕ_* is exact, we conclude that $R^q(\lambda \circ \phi)_*E = 0$ for $q > 0$. Then the vanishing of $R^q j_{T*}E$ implies that $R^q v_{X/S*}(j_{T*}E)$ also vanishes for $q > 0$. $\square$

Proposition 6.11 will allow us to use Čech-Alexander complexes to compute prismatic cohomology. Let T be a small X/S-prism, which we can view as an object of $\Delta(X/S)$ or of $\Delta_s(X/S)$. Since T is small, it is p-completely flat over S, so by Proposition 7.14, the $n{+}1$-fold product $T(n) := T \times_S T \times_S \cdots T$, computed in the category of p-adic formal-schemes is p-torsion free. We endow it with its a natural structure of a formal ϕ-scheme. The map $z_{T(n)}\colon \overline{T}(n) \to X(n)$ is again flat, and since $X/\overline{S}$ is smooth, the diagonal embedding $X \to X(n)$ is a regular immersion. Then $z^{-1}_{T(n)}(X) \to \overline{T}(n)$ is also a regular immersion, so the prismatic envelope $\Delta_X(T(n))$ of $z^{-1}_{T(n)}(X)$ in $T(n)$ is again small, by Theorem 2.19. Now if E is a sheaf of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$, let

$$C^n_T(E) := j_{T(n)*} j^*_{T(n)}(E),$$

and let $C^{\cdot}_T(E)$ be the Čech-Alexander complex:

$$C^{\cdot}_T(E) := C^0_T(E) \to C^1_T(E) \to \cdots \to C^n_T(E) \to \cdots$$

with the usual boundary maps.

Proposition 6.12. *With the notations above, suppose that E is a p-completely quasi-coherent sheaf of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$ or $\Delta_s(X/S)$.*

1. *For each n, the sheaf $C^n_T(E)$ is acyclic for the functors*

$$v_{X/S}\colon D^+(X/S)_{\Delta} \to X_{\acute{e}t} \text{ and } D^+(X/S)_{\Delta_s} \to X_{\acute{e}t}.$$

2. *If T is a covering of the final object of $(X/S)_{\Delta}$, then the natural maps $E \to C^{\cdot}_T(E)$ and $Rv_{X/S*}E \to v_{X/S*}C^{\cdot}_T(E)$ are strict quasi-isomorphisms.*

Proof. Statement (1) of this proposition follows from statement (3) of Proposition 6.11, applied to the prism $T(n)$. If T is a covering of the final object of $(X/S)_{\Delta}$, then, as explained for example in [6, 5.29], the natural map $E \to C^{\cdot}_T(E)$ is a quasi-isomorphism, and statement (2) follows. □

Corollary 6.13. *Let E be a p-completely quasi-coherent sheaf of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$ and let $u^*(E)$ be its restriction to $\Delta_s(X/S)$. Then the natural map*

$$Rv_{X/S*}(E) \to Rv_{X/S*}(u^*(E))$$

is an isomorphism.

Proof. Without loss of generality we may assume that X is affine and choose a lifting Y of X along with its Frobenius. Then Y is a covering of the final object of $(X/S)_{\Delta}$ and of $(X/S)_{\Delta_s}$, and $Rv_{X/S*}(E)$ and $Rv_{X/S*}(u^*(E))$ are both computed by the same Čech-Alexander complex. □

Recall from Definition 5.5 the functor $\mathcal{L}_{\Delta}$ from the category of $\mathcal{O}_Y$-modules to the category of $\mathcal{O}_Y$-modules with prismatic stratification. If E is an $\mathcal{O}_Y$-module, we write $L(E)$ for the crystal of $\mathcal{O}_{Y/S}$-modules on $\Delta_s(Y/S)$ corresponding to $\mathcal{L}_{\Delta}(E)$. The following result is the prismatic analog of the crystalline [6, 6.10]. Here it seems to be important to work in the small site.

Proposition 6.14. *Let $j_Y : ((Y/S)_{\Delta_s})_{|Y} \to (Y/S)_{\Delta_s}$ and $\phi : ((Y/S)_{\Delta_s})_{|Y} \to Y_{\mathrm{pcf}}$ be the morphisms as in Proposition 6.11, and let E be a sheaf of $\mathcal{O}_Y$-modules on Y_{pcf}.*

1. *There is a natural isomorphism*

$$L(E) \cong j_{Y*}(\phi^*(E))$$

of crystals of $\mathcal{O}_{Y/S}$-modules on $\Delta_s(Y/S)$.

2. *If $\lambda : Y_{\mathrm{pcf}} \to Y_{\acute{e}t}$ is the natural map, there is a natural isomorphism:*

$$v_{Y/S*}(L(E)) \cong \lambda_*(E).$$

If $D : \mathcal{L}_{\Delta}(E) \to E'$ is a prismatic differential operator, let $\overline{D} : E \to E'$ be the composition of D with s^*. Then the diagram

$$\begin{tikzcd} v_{Y/S*}(L(E)) \arrow[r, "v_{Y/S*}(L(D))"] \arrow[d, "\cong"'] & v_{Y/S*}(L(E')) \arrow[d, "\cong"] \\ \lambda_*(E) \arrow[r, "\overline{D}"] & \lambda_*(E') \end{tikzcd}$$

commutes.

3. *If E is p-completely quasi-coherent, then $R^q v_{Y/S*}(L(E)) = 0$ for $q > 0$.*

Proof. We first define a morphism $j_Y^*(L(E)) \to \phi^*(E)$ as follows. For $(T', h) \in \mathbb{\Delta}_s(X/S)_{|Y}$, the definitions tell us that

$$(j_Y^* L(E))_{(T',h)} := L(E)_{T'} = h^*(\mathcal{L}_{\mathbb{\Delta}}(E)).$$

Composing with the pullback of the map 5.15) $\mathcal{L}_{\mathbb{\Delta}}(E) \to E$, we find the desired:

$$(j_Y^* L(E))_{(T',h)} = h^*(\mathcal{L}_{\mathbb{\Delta}}(E)) \to h^*(E) = (\phi^*(E))_{(T',h)}.$$

To prove the proposition, it will be enough to check that the adjoint to this construction is an isomorphism. Our claim is that for each $T \in \mathbb{\Delta}_s(Y/S)$, the map $L(E)_T \to j_{T*}(\phi^* E)$ is an isomorphism. We can check this p-completely flat locally on T, and so by Proposition 6.6, we may assume that there is a morphism of prisms $h \colon T \to Y$. We find a commutative diagram:

$$\begin{tikzcd}
\mathbb{\Delta}_Y(T \times_S Y) \arrow[r, "f"] \arrow[d, "p_T"'] & \mathbb{\Delta}_Y(1) \arrow[r, "p_2"] \arrow[d, "p_1"] & Y \\
T \arrow[r, "h"] & Y,
\end{tikzcd}$$

in which the square is Cartesian by Proposition 6.3. Then, using the definitions and the fact that $p_1 \colon \mathbb{\Delta}_Y(1) \to Y$ is affine, we get that:

$$\begin{aligned}
L(E)_T &= h^*(\mathcal{L}_{\mathbb{\Delta}}(E)) \\
&= h^*(\mathcal{O}_{\mathbb{\Delta}_Y(1)} \hat{\otimes} E) \\
&= h^*(p_{1*}(p_2^*(E))) \\
&= p_{T*}(f^*(p_2^*(E))),
\end{aligned}$$

On the other hand, since $\mathbb{\Delta}_Y(T \times_S Y) \to Y$ represents $j_Y^{-1}(T)$ we have

$$j_{Y*}(\phi^*(E))_T = p_{T*}((\phi^*(E))_{\mathbb{\Delta}_Y(T \times_S Y)}) = p_{T*}(f^*(p_2^*(E))),$$

proving statement (1). The remaining statements then follow from Propositions 6.11 and 8.17. $\square$

We are now ready to prove the main motivating result of this project.

Theorem 6.15. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes and let $X \to Y$ be a closed immersion, where $X/\overline{S}$ is smooth. If E is a crystal of $\mathcal{O}_{X/S}$-modules on $\mathbb{\Delta}(X/S)$, let $(E_{\mathbb{\Delta}_X(Y)}, \nabla')$ be the corresponding object of* MICP$(X/Y/S)$*, as described in Corollary (6.10). Then there is a canonical strict quasi-isomorphism:*

$$Rv_{X/S*}E \cong (E_{\mathbb{\Delta}_X(Y)} \otimes \Omega^{\cdot}_{Y/S}, d').$$

Proof. In fact we shall deduce the theorem from the special case in which $X = \overline{Y}$, stated as Corollary 6.17 below. This deduction uses the following lemma.

Lemma 6.16. *If E is an abelian sheaf $\mathbb{\Delta}(X/S)$, then $R^q i_{\mathbb{\Delta}*}(E) = 0$ for $q > 0$.*

Proof. If T is an object of $\mathbb{\Delta}(Y/S)$ and E is a sheaf on $\mathbb{\Delta}(X/S)$, then the value of $i_{\mathbb{\Delta}*}(E)$ on T is $E_{\mathbb{\Delta}_X(T)}$ (using the notation in Equation 6.1). It follows that the functor $i_{\mathbb{\Delta}*}$ is exact, and hence that $R^q i_{\mathbb{\Delta}*}$ vanishes if $q > 0$. □

Now let E be a crystal of $\mathcal{O}_{X/S}$-modules. Since $v_{X/S} = v_{Y/S} \circ i_{\mathbb{\Delta}}$, the lemma implies that $Rv_{X/S*}(E) \cong Rv_{Y/S*}(i_{\mathbb{\Delta}*}(E))$. Unfortunately $i_{\mathbb{\Delta}*}(E)$ is not a crystal of $\mathcal{O}_{Y/S}$-modules, but, by statement (1) of Proposition 6.8, its restriction $\tilde{E} := u^* i_{\mathbb{\Delta}*} E$ to $\mathbb{\Delta}_s(Y/S)$ is. Corollary 6.17 will tells us that $Rv_{Y/S*}(\tilde{E})$ is represented by the p-de Rham complex of $(\tilde{E}_Y, \nabla')$. Since $\tilde{E}_Y = E_{\mathbb{\Delta}_X(Y)}$, this will prove the theorem. □

Corollary 6.17. *Let Y/S be a p-completely smooth morphism of formal ϕ-schemes, let E be a crystal of $\mathcal{O}_{Y/S}$-modules on $\mathbb{\Delta}(Y/S)$ (resp., on $\mathbb{\Delta}_s(Y/S)$), and let (E_Y, ∇') be the corresponding $\mathcal{O}_Y$-module with integrable p-connection (6.9). Then there is a canonical strict quasi-isomorphism*

$$Rv_{Y/S*}(E) \cong (E_Y \otimes \Omega^{\cdot}_{Y/S}, d').$$

Proof. Our proof will follow the method of proof of its crystalline analog as carried out in [6]. We first explain the case when $E = \mathcal{O}_Y$, with its canonical prismatic connection. The map s^* in diagram (5.15) defines a horizontal morphism $\mathcal{O}_Y \to \mathcal{L}_{\mathbb{\Delta}}(\mathcal{O}_Y)$, which we will see extends to a morphism of complexes of modules with prismatic connection:

$$(\mathcal{O}_Y, d') \longrightarrow \mathcal{L}_{\mathbb{\Delta}}(\Omega^{\cdot}_{Y/S}, d'). \tag{6.4}$$

Tensoring with E_Y and composing with the isomorphism β of Proposition 5.6, we get:

$$(E_Y, \nabla') \longrightarrow E_Y \hat{\otimes} \mathcal{L}_{\mathbb{\Delta}}(\Omega^{\cdot}_{Y/S}, d') \xrightarrow[\cong]{\beta} \mathcal{L}_{\mathbb{\Delta}}(E_Y \otimes \Omega^{\cdot}_{Y/S}, d'). \tag{6.5}$$

and hence a corresponding morphism of complexes of crystals on $\mathbb{\Delta}(Y/S)$:

$$E \to L(E_Y \otimes \Omega^{\cdot}_{Y/S}, d') \tag{6.6}$$

Lemma 6.18 below will tell us that this map is a strict quasi-isomorphism. By statement (3) of Proposition 6.14, each term of the complex $L(E \otimes \Omega^{\cdot}_{Y/S})$ is acyclic for $v_{Y/S*}$, and by statement (2) of that proposition, $v_{Y/S*}(L(E \otimes \Omega^{\cdot}, d')$ identifies with $(E_Y \otimes \Omega^{\cdot}_{Y/S}, d')$. Thus the following lemma will complete the proof of the corollary. □

Lemma 6.18. *let (E_Y, ∇') be an $\mathcal{O}_Y$-module with integrable and quasi-nilpotent p-connection, and let E be the corresponding crystal on $\mathbb{\Delta}_s(Y/T)$, Then the map (6.6) is a strict quasi-isomorphism.*

Proof. The isomorphism β induces an isomorphism of complexes of crystals of $\mathcal{O}_{Y/S}$-modules on $\mathbb{\Delta}_s(Y/S)$:

$$E \otimes L(\Omega^{\cdot}_{Y/S}, d') \cong L(E_Y \otimes \Omega^{\cdot}_{Y/S}, d')$$

Thus it will be enough to prove that the morphism $E \to E\hat{\otimes}L(\Omega^{\cdot}_{Y/S}, d')$ induced by the first arrow in equation (6.5) is a strict quasi-isomorphism. Since the terms of the complex $L(\Omega^{\cdot}_{Y/S}, d')$ are p-completely flat, it is enough to prove this statement when $E = \mathcal{O}_{Y/S}$.

The claim is that for every $T \in \mathbb{\Delta}_s(Y/S)$, the map $\mathcal{O}_T \to L(\Omega^{\cdot}_{Y/S})_T$ is a strict quasi-isomorphism. Thanks to Proposition 7.16, this can be verified after replacing T by a p-completely flat cover, so by Proposition 6.6 we may without loss of generality that there is a morphism of Y-prisms $T \to Y$. Then, as we saw in Propositions 6.14 and 5.6

$$L(\Omega^{\cdot}_{Y/S})_T \cong \mathcal{O}_T \otimes_{\mathcal{O}_Y} \mathcal{L}_{\mathbb{\Delta}}(\Omega^{\cdot}_{Y/S}).$$

Since T is small, the map $T \to Y$ is p-completely flat, so again by Proposition 7.16,, we are reduced to checking our claim when $T = Y$. Moroever, it follows from Proposition 5.7 that the complex $\mathcal{L}_{\mathbb{\Delta}}(\Omega^{\cdot}_{Y/S})$ identifies with the complex $(\Omega^{\cdot}_{\mathbb{\Delta}_Y(1)/Y}, d')$. If we write Z for $Y(1)$, then the first projection $Z \to Y$ is a p-completely smooth morphism of formal ϕ-schemes, with a section defined by the diagonal, and $\mathbb{\Delta}_Y(1)$ is the prismatic envelope of this section. Then the simplest form of prismatic Poincaré lemma, statement (3) of Lemma 3.12, implies that the map $\mathcal{O}_Y \to \Omega^{\cdot}_{Z/Y}$ is a strict quasi-isomorphism. This concludes the proof. $\square$

6.5 PD-prisms and the prismatic F-transform

In this section we introduce a variant of the prismatic site which lies between the prismatic and crystalline theories. Inspired by work of Oyama [25] and Xu [28], it will allow us to give a more geometric interpretation of the F-transform. It also provides a canonical factorization of the prismatic Frobenius endomorphism which clarifies why it is an isogeny. Although this construction is not formalized explicitly in [7], some of its key aspects are used in some of the comparison theorems there.

Recall that if X is a scheme in characteristic p, the scheme theoretic image $F_X(X)$ of its absolute Frobenius endomorphism is the closed subscheme defined by the ideal of sections of $\mathcal{O}_X$ whose pth power is zero. There is a canonical factorization

$$X \xrightarrow{F_X} X \quad = \quad X \xrightarrow{f_X} F_X(X) \xrightarrow{j_X} X. \tag{6.7}$$

Note that, since F_X is a homeomorphism and $f_X^\sharp$, the morphism $f_X$is an epimoorphismin the category of schemes.

If X is closed in Y and $X^\phi := F_Y^{-1}(X)$, then $F_{X^\phi}(X^\phi) \subseteq X \subseteq X^\phi$, and we have maps

$$X^\phi \xrightarrow{F_{X^\phi}} X^\phi \quad = \quad X^\phi \xrightarrow{f_{X^\phi}} F_{X^\phi}(X) \xrightarrow{k_{X/Y}} X' \xrightarrow{\pi} X \xrightarrow{inc} X^\phi. \tag{6.8}$$

Definition 6.19. *If S is a formal ϕ-scheme and $X/\overline{S}$ is a morphism of schemes, an X/S-ϕ-prism is a pair (T, y_T) where T is a formal ϕ-scheme over S and $y_T \colon \phi(\overline{T}) \to X$ is an $\overline{S}$-morphism. We denote by $\mathbb{\Delta}_\phi(X/S)$ the category of X/S-ϕ-prisms, and endow it with the p-completely flat topology. If $X \to Y$ is a closed immersion of X in a formal ϕ-scheme, then $(\mathbb{\Delta}_X^\phi(Y), y_{\mathbb{\Delta}}, \pi_{\mathbb{\Delta}})$ is the universal X/S-ϕ-prism endowed with a ϕ-morphism to Y.*

We omit the verifications that $\mathbb{\Delta}_\phi(X/S)$ forms a site and that its formation is functorial.

Remark 6.20. If T is a formal ϕ-scheme, then $\phi(\overline{T}) = F_{\overline{T}}(\overline{T})$, which Proposition 2.4 tells us is equal to $T_{\mathbb{P}}$, the smallest PD-subscheme of T. Thus we could just as well have defined $\mathbb{\Delta}_\phi(X/S)$ to be the site whose objects are "PD-prisms", *i.e.*, pairs (T, y_T), where T is a formal ϕ-scheme over S and y_T is an S-morphism $T_{\mathbb{P}} \to X$, and denoted the site by $\mathbb{\Delta}_{\mathbb{P}}(X/S)$. Theorem 6.24 below implies that, when S is perfect, the sites $\mathbb{\Delta}_\phi(X/S)$ and $\mathbb{\Delta}(X/S)$ give rise to equivalent topoi.

The key geometric construction we shall need is the following description of ϕ-prismatic neighborhoods. Recall that if $X \subseteq \overline{Y}$, then $X^\phi := F_{\overline{Y}}^{-1}(X)$, and, from Theorem 2.27, that $\mathbb{P}_X(Y)$ identifies with $\mathbb{\Delta}_{X^\phi}(Y)$. This will allow us to relate ϕ-prismatic envelopes to divided power envelopes.

Proposition 6.21. *If Y is a formal ϕ-scheme and $i \colon X \to Y$ s a closed immersion, let X^ϕ be the inverse image of X under $F_{\overline{Y}}$ and let $(\mathbb{\Delta}_{X^\phi}(Y), z_{\mathbb{\Delta}^\phi}, \pi_{\mathbb{\Delta}^\phi})$ be the prismatic neighborhood of X^ϕ in Y. Then there is a unique morphism $y_{\mathbb{\Delta}} \colon \phi(\overline{\mathbb{\Delta}}_{X^\phi}(Y)) \to X$ such that $z_{\mathbb{\Delta}} \circ j_{\mathbb{\Delta}} = i \circ y_{\mathbb{\Delta}}$, and the diagram:*

$$\begin{tikzcd}
\phi_{\mathbb{\Delta}}(\overline{\mathbb{\Delta}}_{X^\phi}(Y)) \arrow[r] \arrow[d, "y_{\mathbb{\Delta}}"'] & \mathbb{\Delta}_{X^\phi}(Y) \arrow[d, "\pi_{\mathbb{\Delta}^\phi}"] \\
X \arrow[r, "i"] & Y
\end{tikzcd}$$

represents $(\mathbb{\Delta}_X^\phi(Y), y_{\mathbb{\Delta}}, \pi_{\mathbb{\Delta}})$. If Y/S is p-completely smooth and $X/\overline{S}$ is smooth, then $(\mathbb{\Delta}_X^\phi(Y), y_{\mathbb{\Delta}})$ is small.

Proof. The factorization (6.7) of the Frobenius endomorphism of $\overline{\mathbb{\Delta}}_{X^\phi}(Y)$ gives the top row of the following commutative diagram:

$$\begin{tikzcd}
\overline{\mathbb{\Delta}}_{X^\phi}(Y) \arrow[r, "f_{\overline{\mathbb{\Delta}}}"] \arrow[d, "z_{\mathbb{\Delta}^\phi}"'] & \phi(\mathbb{\Delta}_{X^\phi}(Y)) \arrow[r, "j_{\overline{\mathbb{\Delta}}}"] \arrow[d] \arrow[dr, dashed, "y_{\mathbb{\Delta}}"] & \overline{\mathbb{\Delta}}_{X^\phi}(Y) \arrow[dr, "z_{\mathbb{\Delta}^\phi}"] & \\
X^\phi \arrow[r, "f_{X^\phi}"] & F_{X^\phi}(X^\phi) \arrow[r, "j_{X^\phi}"] & X \arrow[r, "inc"] & X^\phi.
\end{tikzcd} \tag{6.9}$$

The bottom row comes from the factorization of the Frobenius endomorphism of X^ϕ, and the remaining solid arrows exist because of the functoriality of this factorization. The dotted arrow $y_{\mathbb{A}}$ is defined to make the triangle (and hence the diagram) commute, and endows $\mathbb{A}_{X^\phi}(Y)$ with the structure of an X/S-ϕ-prism.

Conversely if (T, y_T, π_T) is an X/S-ϕ-prism over Y, the diagram

$$\begin{tikzcd}
\overline{T} \arrow[r, "y_T \circ f_T"] \arrow[d, "\overline{\pi}_T"'] & X \arrow[d, "\bar{i}"] \\
\overline{Y} \arrow[r, "F_{\overline{Y}}"] & \overline{Y},
\end{tikzcd}$$

shows that $F_{\overline{Y}} \circ \overline{\pi}_T$ factors through X^ϕ, hence that $T \to Y$ factors through $\mathbb{A}_{X^\phi}(Y)$. This proves that $(\mathbb{A}_{X^\phi}, y_{\mathbb{A}}, \pi_{\mathbb{A}^\phi})$ does enjoy the requisite universal property. If Y/S is p-completely smooth and $X/\overline{S}$ is smooth, then X' is regularly immersed in $\overline{Y}$, and since $F_{\overline{Y}/S}$ is flat and $X^\phi = F^{-1}_{\overline{Y}/S}(X')$, it is also regularly immersed in $\overline{Y}$. Then it follows from Theorem 2.19 that its prismatic neighborhood is small. □

Combining this result with Theorem 2.27, we see that ϕ-prismatic envelopes are essentially the same as PD-envelopes, at least in the smooth case.

Corollary 6.22. *In the situation of Proposition 6.21, suppose that Y/S is a p-completely smooth and that $X/\overline{S}$ is smooth. Then there is a natural Y-isomorphism $\mathbb{P}_X(Y) \to \mathbb{A}^\phi_X(Y)$ fitting into a commutative diagram*

$$\begin{tikzcd}
\mathbb{P}_X(Y) \arrow[r] & \mathbb{A}^\phi_X(Y) \arrow[r, "\cong"] & \mathbb{A}_{X^\phi}(Y) \\
\mathbb{A}^\phi_X(Y)_{\mathbb{P}} \arrow[u] \arrow[r] \arrow[dr, "\cong"'] & \overline{\mathbb{A}}^\phi_X(Y) \arrow[u] \arrow[r, "\cong"] \arrow[d, "y_{\mathbb{A}}"] & \overline{\mathbb{A}}_{X^\phi}(Y) \arrow[u] \arrow[d, "z_{\mathbb{A}^\phi}"] \\
& X \arrow[r, "inc"] & X^\phi
\end{tikzcd}$$

□

We are now ready to define a pair of functors A and B which allow us to factor the prismatic Frobenius morphism. We shall see later that B is a geometric incarnation of the F-transform.

Definition 6.23. *Suppose that S is a formal ϕ-scheme and that $X/\overline{S}$ is a smooth morphism.*

1. *$A\colon \mathbb{A}(X/S) \to \mathbb{A}_\phi(X/S)$ is the functor*

$$A\colon (T, z_T) \mapsto (T, z_T \circ j_T),$$

where $j_T\colon F_{\overline{T}}(\overline{T}) \to \overline{T}$ is the inclusion 6.7.

2. $B\colon \mathbb{\Delta}_\phi(X/S) \to \mathbb{\Delta}(X'/S)$ is the functor

$$B\colon (T, y_T) \mapsto (T, \tilde{z}_T),$$

where $\tilde{z}_T\colon \overline{T} \to X'$ is the unique S-morphism making the following diagram commute:

$$\begin{tikzcd}
\overline{T} \arrow[r, "f_{\overline{T}}"] \arrow[d, "\tilde{z}_T"'] & \phi(\overline{T}) \arrow[d, "y_T"] \\
X' \arrow[r, "\pi"] & X.
\end{tikzcd}$$

(Note: This morphism exists because $y_T \circ f_{\overline{T}}$ and π are $F_{\overline{S}}$-morphisms.)

Theorem 6.24. *Suppose that S is a formal ϕ-scheme and that $X/\overline{S}$ is a smooth morphism.*

1. *There is a commutative diagram of continuous and cocontinuous functors:*

$$\begin{tikzcd}
\mathbb{\Delta}(X/S) \arrow[r, "A"] \arrow[dr, "\mathbb{\Delta}(F_{X/S})"'] & \mathbb{\Delta}_\phi(X/S) \arrow[d, "B"] \arrow[dr, "\mathbb{\Delta}_\phi(F_{X/S})"] & \\
& \mathbb{\Delta}(X'/S) \arrow[r, "A'"] & \mathbb{\Delta}_\phi(X'/S).
\end{tikzcd}$$

2. *If $\mathcal{F}$ is a sheaf on $\mathbb{\Delta}_\phi(X/S)$ (resp. on $\mathbb{\Delta}(X'/S)$), then the presheaf*

$$(T, z_T) \mapsto \mathcal{F}(A(T, z_T)) \quad \text{resp.} \quad (T, y_T) \mapsto \mathcal{F}(B(T, y_T))$$

is the sheaf $A^{-1}(\mathcal{F})$ (resp. $B^{-1}(\mathcal{F})$). Moreover, there are isomorphisms:

$$\begin{aligned}
A^{-1}(\mathcal{O}_{X/S}) &\to \mathcal{O}_{X/S} \\
B^{-1}(\mathcal{O}_{X'/S}) &\to \mathcal{O}_{X/S},
\end{aligned}$$

and hence morphisms of ringed topoi:

$$\begin{aligned}
A_{\mathbb{\Delta}}\colon ((X/S)_{\mathbb{\Delta}}, \mathcal{O}_{X/S}) &\to ((X/S)_{\mathbb{\Delta}_\phi}, \mathcal{O}_{X/S}) \\
B_{\mathbb{\Delta}}\colon ((X'/S)_{\mathbb{\Delta}_\phi}, \mathcal{O}_{X/S}) &\to ((X'/S)_{\mathbb{\Delta}}, \mathcal{O}_{X'/S}).
\end{aligned}$$

3. *The morphism $B_{\mathbb{\Delta}}$ is an equivalence of ringed topo and induces an equivalence from the category of crystals of $\mathcal{O}_{X'/S}$ modules on $\mathbb{\Delta}(X'/S)$ to the category of crystals of $\mathcal{O}_{X'/S}$ modules on $\mathbb{\Delta}_\phi(X/S)$.*

Proof. To check the commutativity of the diagram, let (T, z_T) be an object of $\mathbb{\Delta}(X/S)$. Then $A(T, Z_T) = (T, j_T)$, where $y_T := z_T \circ j_T$ and $B(A(T, z_T)) = (T, \tilde{z}_T)$, where $\tilde{z}_T$ is the unique S-morphism such that $\pi \circ z_T = y_T \circ f_{\overline{T}}$. Since $\mathbb{\Delta}(F_{X/S})(T, z_T) = (T, F_{X/S} \circ z_T)$, we much check that the two maps $F_{X/S} \circ z_T$ and $\tilde{z}_T \colon \overline{T} \to X'$ from $\overline{T}$ to X' agree. Since $F_{X/S} \circ z_T$ is an S-morphism, it will suffice to check that $\pi \circ F_{X/S} \circ z_T = y_T \circ f_{\overline{T}}$. This follows from the following commutative diagram.

$$\begin{tikzcd}
\overline{T} \arrow[r, "f_T"] \arrow[d, "z_T"'] & \phi(\overline{T}) \arrow[r, "j_T"] \arrow[dr, "y_T"] & \overline{T} \arrow[d, "z_T"] \\
X \arrow[r, "F_{X/S}"] & X' \arrow[r, "\pi"] & X
\end{tikzcd}$$

This shows $B \circ A = \mathbb{\Delta}(F_{X/S})$. We leave the second triangle for the reader.

We omit the proof that A is continuous and cocontinuous. To see that B is continuous, observe first that it takes coverings to coverings. Furthermore, if $(\widetilde{T}, y_{\widetilde{T}}) \to (T, y_T)$ and $(T', y'_T) \to (T, y_T)$ are morphisms in $\mathbb{\Delta}_\phi(X/S)$ and $(\widetilde{T}, y_{\widetilde{T}}) \to (T, y_T)$ is p-completely flat, then it follows from the construction in Proposition 2.36 that the fiber product $(\widetilde{T}, y_{\widetilde{T}}) \times_{(T, y_T)} (T', y_{T'})$ in the category $\mathbb{\Delta}_\phi(X/S)$ is given by the usual fiber product. Since the same holds in the category $\mathbb{\Delta}(X'/S)$, we see that B preserves fiber products (at least) in this case. To see that B is cocontinuous, let (T, y_T) be an object of $\mathbb{\Delta}_\phi(X/S)$ and let $u \colon (\widetilde{T}, z_{\widetilde{T}}) \to (T, \tilde{z}_T) := B(T, y_T)$ be a covering. We shall see that this covering is induced by a covering of (T, y_T). In fact, $u \colon \widetilde{T} \to T$ is a p-completely faithfully flat morphism and $z_{\widetilde{T}} \colon \widetilde{T}_1 \to X'$ is a morphism such that $z_{\widetilde{T}} = \tilde{z}_T \circ u_1$. Let $y_{\widetilde{T}} := y_T \circ u_\phi \colon \phi(\widetilde{T}_1) \to X$. Then $(\widetilde{T}, y_{\widetilde{T}})$ is an object of $\mathbb{\Delta}_\phi(X/S)$, and u defines a p-completely flat covering $\tilde{u} \colon (\widetilde{T}, y_{\widetilde{T}}) \to (T, y_T)$. Let $(\widetilde{T}, \tilde{z}_{\widetilde{T}}) := B(\widetilde{T}, y_{\widetilde{T}})$, so that $\tilde{z}_{\widetilde{T}} \colon \widetilde{T}_1 \to X'$ is the unique S-map such that $\pi \circ \tilde{z}_{\widetilde{T}} = y_{\widetilde{T}} \circ f_{\widetilde{T}_1}$. But then

$$\begin{aligned}
\pi \circ \tilde{z}_{\widetilde{T}} &= y_{\widetilde{T}} \circ f_{\widetilde{T}_1} \\
&= y_T \circ u_\phi \circ f_{\widetilde{T}_1} \\
&= y_T \circ f_T \circ u_1 \\
&= \pi \circ \tilde{z}_T \circ u_1 \\
&= \pi \circ z_{\widetilde{T}}
\end{aligned}$$

This implies that $z_{\widetilde{T}} = z_{\widetilde{T}}$. so in fact $B(\tilde{u})$ coincides with the original covering $(\widetilde{T}, \tilde{z}_{\widetilde{T}}) \to (T, \tilde{z}_T)$.

If $\mathcal{F}$ is a sheaf on $\mathbb{\Delta}_\phi(X/S)$, recall that $A^{-1}(\mathcal{F})$ is the sheaf associated to the presheaf which takes an object (T, z_T) of $\mathbb{\Delta}(X/S)$ to $\mathcal{F}(A(T, z_T))$. Since A is continuous, this presheaf is in fact a sheaf. Note that $A(T, z_T) = (T, y_T)$, and so the sheaf $A^{-1}(\mathcal{F})_{(T, z_T)}$ on T is the same as the sheaf $\mathcal{F}_{(T, y_T)}$. Applying this to $\mathcal{O}_{X/S}$, we see that $A^{-1}(\mathcal{O}_{X/S}) = \mathcal{O}_{X/S}$. It follows that if $\mathcal{F}$ is a sheaf of $\mathcal{O}_{X/S}$-

modules on $\mathbb{\Delta}_\phi(X/S)$, then $A^{-1}(\mathcal{F}) \cong A^*(\mathcal{F})$, and that $A^*(\mathcal{F})$ is p-completely quasi-coherent if and only if $\mathcal{F}$ is. The same argument works for B.

Lemma 6.25. *In the situation of Theorem 6.24,*

1. *The functor B is fully faithful.*
2. *Every object of $\mathbb{\Delta}(X'/S)$ admits a cover by an object in the image of B.*

Proof. The proof is a modification of Oyama's argument, as explained by Xu [28, 9.8]. To see that B is fully faithful, suppose that (T, y_T) and $(\widetilde{T}, y_{\widetilde{T}})$ are objects of $\mathbb{\Delta}_\phi(X/S)$. Then a morphism $(\widetilde{T}, y_{\widetilde{T}}) \to (T, y_T)$ is a morphism $g\colon \widetilde{T} \to T$ such that

$$y_T \circ g_\phi = y_{\widetilde{T}}, \tag{6.10}$$

and a morphism $B(\widetilde{T}, y_{\widetilde{T}}) \to B(T, y_T)$ is a morphism $g\colon \widetilde{T} \to T$ such that

$$\tilde{z}_T \circ \overline{g} = \tilde{z}_{\widetilde{T}}. \tag{6.11}$$

Thus it is obvious that B is faithful. To see that it is full, suppose that $g\colon \widetilde{T} \to T$ defines a morphism $B(\widetilde{T}, y_{\widetilde{T}}) \to B(T, y_T)$. Then

$$\begin{aligned} \tilde{z}_T \circ \overline{g} &= \tilde{z}_{\widetilde{T}} \\ \pi \circ \tilde{z}_T \circ \overline{g} &= \pi \circ \tilde{z}_{\widetilde{T}} \\ y_T \circ f_T \circ \overline{g} &= y_{\widetilde{T}} \circ f_{\widetilde{T}} \\ y_T \circ g_\phi \circ f_{\widetilde{T}} &= y_{\widetilde{T}} \circ f_{\widetilde{T}} \end{aligned}$$

Since $f_{\widetilde{T}}$ is a scheme-theoretic epimorphism, it follows that $y_T \circ g_\phi = y_{\widetilde{T}}$, so g defines a morphism $(\widetilde{T}, y_{\widetilde{T}}) \to (T, y_T)$. This proves statement (1).

To prove (2), suppose that $(T', z_{T'})$ is an object of $\mathbb{\Delta}(X'/S)$. Without loss of generality, we assume that X' and T' are affine. Choose a p-completely smooth formal ϕ-scheme Y/S lifting $X/\overline{S}$. Then Y'/S is again p-completely smooth, and by Theorem 1.12, we may, after replacing T' by a p-completely flat covering, assume that there is a morphism $(T', z_{T'}) \to (Y', \mathrm{id}_{X'})$. The map $\phi_{Y/S}\colon Y \to Y'$ is p-completely flat, and hence so is the map $v\colon T := T' \times_{Y'} Y \to T'$. Note that $\overline{T} \cong \overline{T}' \times_{X'} X$, and that we have a commutative diagram:

$$\begin{tikzcd} \overline{T} \arrow[r, "f_{\overline{T}}"] \arrow[d, "p_X"'] & \phi(\overline{T}) \arrow[r, "j_{\overline{T}}"] \arrow[dr, dashed, "y_T"'] & \overline{T} \arrow[r, "\overline{v}"] \arrow[d, "p_X"'] \arrow[dr, "z_T"] & \overline{T}' \arrow[r, "\pi"] \arrow[d, "z_{T'}"] & \overline{T} \arrow[d, "p_X"] \\ X \arrow[r, "f_X"] & F_X(X) \arrow[r, "\cong"] & X \arrow[r, "F_{X/S}"] & X' \arrow[r, "\pi"] & X \end{tikzcd}$$

The map z_T defined by the diagram endows T with the structure of an X'/S-prism, the map y_T gives T the structure of ϕ-X/S prism, and v defines a p-completely flat cover $(T, z_T) \to (T', z_{T'})$. Note that $y_T \circ f_{\overline{T}} = F_X \circ p_X$ We claim that $B(T, y_T)$ is equal to (T, z_T), which will prove (2). We have:

$$\begin{aligned} \pi \circ z_T &= \pi \circ z_{T'} \circ \overline{v} \\ &= F_X \circ p_X \\ &= y_T \circ f_{\overline{T}} \end{aligned}$$

By the definition of $B(T, y_T)$ this proves the claim. □

Since the p-completely flat topology comes from a pre-topology, Lemma 6.25 and a general theorem of Oyama [25, 4.2.1] imply that B induces an equivalence of topoi. As we have seen, $B^{-1}(\mathcal{O}_{X'/S}) \cong \mathcal{O}_{X/S}$, so in fact $B_{\mathbb{\Delta}}$ is an equivalence of ringed topoi. Suppose that E' is a sheaf l of $\mathcal{O}_{X'/S}$-modules on $\mathbb{\Delta}(X'/S)$. of Theorem 6.24, it is easy to check that $B^*(E')$ is a crystal of $\mathcal{O}_{X/S}$-modules on $\mathbb{\Delta}_\phi(X/S)$. To prove the converse, one can argue locally, using Lemma 6.25.

□

The following result establishes the naturality of the functors B and A. Its proof is immediate from the definitions.

Proposition 6.26. *Let S be a formal ϕ-scheme and $f\colon X \to Y$ a morphism of smooth $\overline{S}$-schemes. Then there are commutative diagram:*

$$\begin{tikzcd} \mathbb{\Delta}_\phi(X/S) \arrow[r, "B_X"] \arrow[d, "\mathbb{\Delta}_\phi(f)"'] & \mathbb{\Delta}(X'/S) \arrow[d, "\mathbb{\Delta}(f')"] \\ \mathbb{\Delta}_\phi(Y/S) \arrow[r, "B_Y"] & \mathbb{\Delta}(Y'/S) \end{tikzcd}$$

$$\begin{tikzcd} \mathbb{\Delta}(X/S) \arrow[r, "A_X"] \arrow[d, "\mathbb{\Delta}(f)"'] & \mathbb{\Delta}_\phi(X/S) \arrow[d, "\mathbb{\Delta}_\phi(f)"] \\ \mathbb{\Delta}(Y/S) \arrow[r, "A_Y"] & \mathbb{\Delta}_\phi(Y/S) \end{tikzcd}$$

Remark 6.27. If (T, y_T) is an (X/S)-ϕ-prism, then $T_{\mathbb{P}} \to T$ is a PD-thickening, and thus by forgetting the ϕ-structure of T we can view (T, y_T) as an object of the site $\mathbb{P}(X/S)$ consisting of the PD-enlargements of X/S. This defines a functor

$$\mathbb{\Delta}_\phi(X/S) \to \mathbb{P}(X/S). \tag{6.12}$$

Although we have not written the details, it is clear that this functor will induce an equivalence on the category of crystals of modules, compatibly with cohomology. In particular, if E is a crystal of $\mathcal{O}_{X/S}$-modules on $\mathbb{\Delta}_\phi(X/S)$ and X is closed in a p-completely smooth formal ϕ-scheme Y/S, then $Rv_{X/S*}E$ is calculated by the de Rham complex of (E_Y, ∇).

There are similarly defined functors

$$\mathbb{\Delta}_\phi(X/S) \to \mathbb{D}_\phi(X/S) \quad \mathbb{\Delta}(X/S) \to \mathbb{D}(X/S). \tag{6.13}$$

Here $\mathbb{D}_\phi(X/S)$ and $\mathbb{D}(X/S)$ are the sites considered by Oyama and Xu. The objects of $\mathbb{D}_\phi(X/S)$ are pairs (T, y_T), where T is a p-torsion free p-adic formal scheme and $y_T : F_{\overline{T}}(T) \to X$ is a morphism from the scheme theoretic image of $F_{\overline{T}}$ to X. Then there is a 2-commutative diagram

$$\begin{tikzcd}
\mathbb{\Delta}_\phi(X/S) \arrow[r, "B"] \arrow[d] & \mathbb{\Delta}(X'/S) \arrow[d] \\
\mathbb{D}_\phi(X/S) \arrow[r, "C"] & \mathbb{D}(X'/S),
\end{tikzcd}$$

where C is the equivalence defined by Oyama and Xu.

Proposition 6.28. *Suppose that Y/S is a p-completely smooth morphism of formal ϕ-schemes.*

1. *Let $y_Y : \phi(\overline{Y}) \to \overline{Y}$ be the inclusion. Then $(Y, y_Y) \in \mathbb{\Delta}_\phi(Y/S)$ covers the final object of the topos $(Y/S)_{\mathbb{\Delta}_\phi}$. More generally, if X is closed in Y, then $(\mathbb{\Delta}^\phi_X(Y), y_{\mathbb{\Delta}_\phi})$ covers the final object of $(X/S)_{\mathbb{\Delta}_\phi}$.*

2. *If E is a crystal of $\mathcal{O}_{Y/S}$-modules on $\mathbb{\Delta}_\phi(Y/S)$, then its value on (Y, y_Y) is canonically endowed with a nilpotent connection, and this correspondence induces an equivalence from the category of such crystals to the category of p-completely quasi-coherent $\mathcal{O}_Y$-modules with nilpotent connection.*

Proof. Suppose that (T, y_T) is an affine object of $\mathbb{\Delta}_\phi(Y/S)$. Then $y_T : \phi(\overline{T}) \to \overline{Y}$ is an $\overline{S}$-morphism, and since $\overline{Y}/\overline{S}$ is smooth and $\phi(\overline{T}) \to \overline{T}$ is a nil immersion it can be extended to an S-morphism $\overline{T} \to \overline{Y}$. This morphism necessarily takes $\phi(\overline{T})$ to $\phi(\overline{Y})$, and hence in fact y_T factors through $\phi(\overline{Y})$. Now using the formal smoothness of Y/S, one can find a lifting $T \to Y$, which, after a p-completely faithfully flat cover, can be chosen to be compatible with the Frobenius liftings, thanks to Theorem 1.12. The generalization to the case of $X \subseteq Y$ is straightforward.

It follows from statement (1) that the category of crystals of $\mathcal{O}_{Y/S}$-modules on $\mathbb{\Delta}_\phi(Y/S)$ is equivalent to the category of p-completely quasi-coherent $\mathcal{O}_Y$-modules endowed with a right action of the groupoid $t, s : \mathbb{\Delta}^\phi_Y(1) \to Y$. Corollary 6.22 shows that there is an isomorphism of groupoids:

$$\begin{tikzcd}
\mathbb{\Delta}^\phi_Y(1) \arrow[r, "\cong"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathbb{P}_Y(1) \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\
Y \arrow[r, "\mathrm{id}"] & Y,
\end{tikzcd} \tag{6.14}$$

and hence an equivalence between the categories of crystals of $\mathcal{O}_{Y/S}$-modules on $\mathbb{P}(Y/S)$ and on $\mathbb{\Delta}_\phi(Y/S)$. Since crystals on $\mathbb{P}(Y/S)$ are given by modules with quasi-nilpotent connection, statement (2) follows. $\square$

We can now explain why B^* corresponds to the F-transform and why A^* corresponds to the p-transform. Since $B_{\mathbb{\Delta}}$ is an equivalence, this gives another proof of Shiho's theorem 4.4.

Theorem 6.29. *Suppose that X is embedded as a closed subscheme of a p-completely smooth ϕ-scheme Y/S.*

1. *If E is a crystal of $\mathcal{O}_{X/S}$-modules on $\mathbb{\Delta}_\phi(X)$, its value on $\mathbb{\Delta}^\phi_X(Y)$ is endowed with a canonical quasi-nilpotent connection.*
2. *If E' is a crystal of $\mathcal{O}_{X'/S}$-modules on $\mathbb{\Delta}(X'/S)$, then $B^*(E')$ is a crystal of $\mathcal{O}_{X/S}$-modules on $\mathbb{\Delta}_\phi(X/S)$ whose value on $\mathbb{\Delta}^\phi_X(Y) \cong \mathbb{P}_X(Y)$ is the F-transform of the value of E' on $\mathbb{\Delta}_{X'/S}(Y')$.*
3. *If E is a crystal of $\mathcal{O}_{Y/S}$-modules on $\mathbb{\Delta}_\phi(Y/S)$, then $A^*(E)$ is a crystal of $\mathcal{O}_{Y/S}$-modules on $\mathbb{\Delta}(Y/S)$, whose value on Y is the p-transform (Example 3.3) of its value on Y.*

Proof. The geometry behind this result comes from the following relationships between the functors A and B and the formation of envelopes.

Lemma 6.30. *In the situation of Theorem 6.29, the following statements hold.*

1. *Let $\Phi\colon \mathbb{\Delta}^\phi_X(Y) = \mathbb{\Delta}_{X^\phi}(Y) \longrightarrow \mathbb{\Delta}_{X'}(Y')$ be the morphism induced by*
$$(\phi_{Y/S}, k_{X/Y})\colon (Y, X^\phi) \to (Y', X'),$$
where $k_{X/Y}$ is the map defined in (6.8). Then Φ fits into a morphism of X'/S-prisms
$$\Phi\colon B(\mathbb{\Delta}^\phi_X(Y), y_{\mathbb{\Delta}}) \to (\mathbb{\Delta}_{X'}(Y'), z_{\mathbb{\Delta}'})$$
2. *The morphism $\Psi\colon \mathbb{\Delta}_X(Y) \to \mathbb{\Delta}^\phi_X(Y)$ induced by the map*
$$(\mathrm{id}, inc)\colon (Y, X) \to (Y, X^\phi)$$
fits into a morphism of X/S-ϕ-prisms:
$$\Psi\colon A(\mathbb{\Delta}_X(Y), z_{\mathbb{\Delta}})) \to (\mathbb{\Delta}^\phi_X(Y), y_{\mathbb{\Delta}})$$

Proof. Let $(\mathbb{\Delta}^\phi_X(Y), \tilde{z}_{\mathbb{\Delta}}) := B(\mathbb{\Delta}^\phi_X(Y), y_{\mathbb{\Delta}}))$. To prove that Φ fits into a morphism of prisms as shown, we must show that $z_{\mathbb{\Delta}'} \circ \overline{\Phi} = \tilde{z}_{\mathbb{\Delta}}$, *i.e.*, that the triangle in the diagram below commutes.

$$\begin{tikzcd}
\overline{\mathbb{\Delta}}_{X^\phi}(Y) \arrow[r, "\overline{\Phi}"] \arrow[dr, "\tilde{z}_{\mathbb{\Delta}}"] & \overline{\mathbb{\Delta}}_{X'}(Y') \arrow[r, "\mathbb{\Delta}(\pi)"] \arrow[d, "z_{\mathbb{\Delta}'}"] & \overline{\mathbb{\Delta}}_X(Y) \arrow[r, "\mathbb{\Delta}(inc)"] \arrow[d, "z_{\mathbb{\Delta}}"] & \overline{\mathbb{\Delta}}_{X^\phi}(Y) \arrow[d, "z_{\mathbb{\Delta}^\phi}"] \\
 & X' \arrow[r, "\pi"] & X \arrow[r, "inc"] & X^\phi
\end{tikzcd}$$

Recall that $inc \circ y_{\mathbb{\Delta}} = z_{\mathbb{\Delta}^\phi} \circ j_{\mathbb{\Delta}}$. Since the squares commute, we have:

$$\begin{aligned}
inc \circ y_{\mathbb{\Delta}} \circ f_{\overline{\mathbb{\Delta}}} &= z_{\mathbb{\Delta}^\phi} \circ j_{\mathbb{\Delta}} \circ f_{\overline{\mathbb{\Delta}}} \\
&= z_{\mathbb{\Delta}^\phi} \circ F_{\overline{\mathbb{\Delta}}} \\
&= z_{\mathbb{\Delta}^\phi} \circ \mathbb{\Delta}(inc) \circ \mathbb{\Delta}(\pi) \circ \overline{\Phi} \\
&= inc \circ \pi \circ z_{\mathbb{\Delta}'} \circ \overline{\Phi}
\end{aligned}$$

Since inc is a monomorphism, it follows that $y_{\mathbb{\Delta}} \circ f_{\overline{\mathbb{\Delta}}} = \pi \circ z_{\mathbb{\Delta}'} \circ \overline{\Phi}$. Since $\tilde{z}_{\mathbb{\Delta}}$ is the unique S-morphism such that $\pi \circ \tilde{z}_{\mathbb{\Delta}} = y_{\mathbb{\Delta}} \circ f_{\overline{\mathbb{\Delta}}}$, we concude that $\tilde{z}_{\mathbb{\Delta}} = z_{\mathbb{\Delta}'} \circ \overline{\Phi}$ as claimed.

Now suppose that $(T, y_T) \in \mathbb{\Delta}^\phi(X/S)$ and that

$$g\colon (T, \tilde{z}_T) = B(T, y_T)) \to (\mathbb{\Delta}_{X'}(Y'), z_{\mathbb{\Delta}'})$$

is a morphism of X'-prisms.

For (2), we recall that $\mathbb{\Delta}_{\phi_X}(Y) = \mathbb{\Delta}_{X^\phi}(Y)$ and note the diagrams:

$$\begin{tikzcd}
\overline{\mathbb{\Delta}}_X(Y) \arrow[r, "\overline{\Psi}"] \arrow[d, "z_{\mathbb{\Delta}}"'] & \overline{\mathbb{\Delta}}_{X^\phi}(Y) \arrow[d, "y_{\mathbb{\Delta}}"] \\
X \arrow[r] & X^\phi
\end{tikzcd}
\qquad
\begin{tikzcd}
\phi(\overline{\mathbb{\Delta}}_X(Y)) \arrow[r, "\phi(\overline{\Psi})"] \arrow[d, "\phi(z_{\mathbb{\Delta}})"'] & \phi(\overline{\mathbb{\Delta}}_{X^\phi}(Y)) \arrow[r] \arrow[d, "\phi(y_{\mathbb{\Delta}})"] & \overline{\mathbb{\Delta}}_{X^\phi}(Y) \arrow[d, "y_{\mathbb{\Delta}}"] \\
\phi(X) \arrow[r] & \phi(X^\phi) \arrow[r] & X
\end{tikzcd}$$

Thus we find a commutative diagram

$$\begin{tikzcd}
\overline{\mathbb{\Delta}}_X(Y) \arrow[r, "\overline{\Psi}"] \arrow[d, "z_{\mathbb{\Delta}} \circ j_{\mathbb{\Delta}}"'] & \overline{\mathbb{\Delta}}_{X^\phi}(Y) \arrow[d, "y_{\mathbb{\Delta}}"] \\
X \arrow[r, "\mathrm{id}"] & X,
\end{tikzcd}$$

which fills in to define the morphism described in statement (2). $\square$

Suppose that E' is a crystal of $\mathcal{O}_{X'/S}$-modules on $\mathbb{\Delta}_{X'}(Y')$. Then if (T, z_T) is an object of $\mathbb{\Delta}_{\phi_X}(Y)$, the definitions tell us that $B^*(E')_T = E'_{B(T)}$, and the lemma implies that

$$B^*(E')_{\mathbb{\Delta}^\phi_X(Y)} \cong \Phi^*(E'_{\mathbb{\Delta}_{X'}(Y')}). \tag{6.15}$$

Similarly, if E is a crystal of $\mathcal{O}_{X/S}$ crystals on $\mathbb{\Delta}_\phi(X/S)$, we find that

$$A^*(E)_{\mathbb{\Delta}_X(Y)} \cong \Psi^*(E_{\mathbb{\Delta}^\phi_X(Y)}) \tag{6.16}$$

Turning to the proof of the theorem, we first consider the case when $X = \overline{Y}$. Then a crystal of $\mathcal{O}_{X/S}$-modules on $\mathbb{\Delta}_\phi(Y/S)$ is given by a p-completely quasi-coherent $\mathcal{O}_Y$-module E_Y endowed with a $\mathbb{\Delta}^\phi_Y(1)$-stratification which, thanks to the isomorphism (6.14), is the same as a $\mathbb{P}_Y(1)$-stratification, which in turn is equivalent to the data of a quasi-nilpotent connection on E_Y. This explains (1).

For (2), we obtain from diagram (6.14) a commutative diagram of groupoids:

$$\begin{tikzcd} \mathbb{A}_Y^{\phi}(1) \arrow[r, "\Phi"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathbb{A}_{Y'}(1) \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ Y \arrow[r, "\phi_{Y/S}"] & Y', \end{tikzcd}$$

which in fact we already saw in Proposition 5.9. Equation (6.15) shows that

$$B^*(E')_Y \cong \phi_{Y/S}^*(E') \text{ and that } B^*(E')_{\mathbb{A}_{\phi_Y(1)}} \cong \Phi^*(E')_{\mathbb{A}_{Y'(1)}}.$$

Since these isomorphisms are compatible with pullback by t and s, we conclude that the $\mathbb{A}_\phi$-stratification of $B^*(E'_Y)$ is the Φ-pullback of the $\mathbb{A}$-stratification of E'. Thanks to Lemma 5.13, this shows that the corresponding connection on $\phi_{Y/S}^*(E)$ is the F-transform of the p-connection on E', proving (2) in this case.

The proof of (3) is similar. We have a morphism of groupoids (see again Proposition 5.9):

$$\begin{tikzcd} \mathbb{A}_Y(1) \arrow[r, "\Psi"] \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] & \mathbb{A}_Y^{\phi}(1) \arrow[d, shift right, "t"'] \arrow[d, shift left, "s"] \\ Y \arrow[r, "\mathrm{id}"] & Y, \end{tikzcd}$$

Equation 6.16 implies that

$$A^*(E)_Y \cong E_Y \text{ and that } A^*(E)_{\mathbb{A}_Y(1)} \cong \Psi^*(E_{\mathbb{A}_Y^{\phi}(1))})$$

It follows that the $\mathbb{A}$-stratification ϵ' on $A^*(E)_Y$ is the Ψ-pullback of the $\mathbb{A}_\phi$-stratification ϵ on E_Y. Using equation (5.17), it is easy to see that the p-connection corresponding to ϵ' is the p-transform of connection corresponding to ϵ.

To deal with the case when $i\colon X \to \overline{Y}$ is a general closed immersion, we we will use the following compatibility between the functor B and the functors $i_{\mathbb{A}_\phi}$ and $i_{\mathbb{A}}$. The lemma follows from the fact that B_X and B_Y induce equivalences of topoi, but can also be checked directly by checking the value of both sides on each object of $\mathbb{A}_\phi(X/S)$.

Lemma 6.31. *In the situation of Theorem 6.29, the map (coming from the commutative diagram of Proposition 6.26)*

$$B_Y^*(i'_{\mathbb{A}*}(E')) \to i_{\mathbb{A}_\phi *}(B_X^*(E'))$$

is an isomorphism. □

The general case of the theorem now follows easily. If E is a crystal of $\mathcal{O}_{X/S}$-modules on $\mathbb{A}_\phi(X/S)$, then $i_{\mathbb{A}_\phi *}(E)$ forms a crystal on $\mathcal{O}_{Y/S}$-modules on the small version of $\mathbb{A}_\phi(Y/S)$ whose value on Y is $E'_{\mathbb{A}^\phi_X(Y)}$, and the the special case we have already discussed shows that this sheaf carries a quasi-nilpotent connection. If $E = B^*_X(E')$ for some crystal E' of $\mathcal{O}_{X'/S}$-modules on $\mathbb{A}(X'/S)$, then

$$E_{\mathbb{A}^\phi_X(Y)} = i_{\mathbb{A}_\phi *}(E)_Y \cong B^*_Y(i_{\mathbb{A}} * (E'))_Y, ,$$

which we have shown is the F-transform of $i_{\mathbb{A}_\phi *}(E)_Y$. ☐

Proposition 6.32. *Let S be a formal ϕ-scheme and $X/\overline{S}$ a smooth morphism. There is a 2- commutative diagram of topoi:*

$$\begin{tikzcd}
(X/S)_{\mathbb{A}_\phi} \arrow[r, "B"] \arrow[d, "u_{X/S}"'] & (X'/S)_{\mathbb{A}} \arrow[d, "v'_{X/S}"] \\
X_{\acute{e}t} \arrow[r, "F_{X/S}"] & X'_{\acute{e}t}.
\end{tikzcd}$$

If E' is a crystal of $\mathcal{O}_{X'/S}$-modules on $\mathbb{A}(X'/S)$, there is a natural isomorphism

$$Rv'_{X/S*}(E') \to F_{X/S*}Ru_{X/S*}(B^*(E')).$$

Proof. The existence of the diagram follows immediately from the definitions. Since $F_{X/S*}$ is exact, we find an isomorphism:

$$Rv'_{X/S*} \circ RB_*(B^*(E')) \cong F_{X/S*} \circ Ru_{X/S*}(B^*(E'))$$

Since $B_{\mathbb{A}}$ is an equivalence of topoi, the functor B_* is exact, so $RB_*B^*(E') \cong B_*B^*(E') \cong E'$, and the result is proved. ☐

Our next result shows that the topos-theoretic formulation of the F-tranform is compatible with pullback and pushforward.

Corollary 6.33. *Let S be a ϕ-scheme for which $F_{\overline{S}}$ is flat, and let $f\colon X \to Y$ be a morphism of smooth $\overline{S}$-schemes.*

1. *If E' is a crystal of $\mathcal{O}_{X'/S}$-modules on $\mathbb{A}(X'/S)$, the following statements hold.*

   (a) *The natural maps*

   $$B^*_Y(R^q f'_{\mathbb{A}*}(E')) \to R^q(f_{\mathbb{A}_\phi *}(B^*_X(E'))$$

   are isomorphisms.

   (b) *Suppose that the restriction of $R^q f_{\mathbb{A}*}(E')$ to $\mathbb{A}_s(Y'/W)$ forms a crystal of $\mathcal{O}_{Y'/S}$-modules. Then if $\tilde{Y}/S$ is a formal ϕ-scheme lifting Y/S, the value of $R^q f_{\mathbb{A}_\phi *}(B^*_X(E'))$ on $\tilde{Y}$ is the F-transform of $R^q f'_{\mathbb{A}*}(E')_{Y'}$.*

2. *If E is a crystal of $\mathcal{O}_{Y/S}$-modules on $\Delta_\phi(Y/S)$, then the natural map*

$$f^*_{\Delta_\phi}(B_{X*}(E)) \to B_{Y*}(f'^*_{\Delta}(E))$$

is an isomorphism.

'

Proof. Theorem 6.24 tells us that the morphisms B_X and B_Y induce equivalences of topoi. It follows that B_{X*} and B_{Y*} are exact. Then if E is a crystal on $\Delta_\phi(X/S)$:

$$R^q f'_{\Delta*}(B_{X*}(E)) = R^q(f'_{\Delta} \circ B_X)_*(E') = R^q(B_Y \circ f_{\Delta_\phi *})(E) = B_{Y*} R^q f_{\Delta_\phi *}(E).$$

Now let $E = B^*_X(E')$. Then $B_{X*}(E) = E'$, and applying B_{Y*} to both sides of the previous equation yields statement (1), and statement (2) follows, thanks to Theorem 6.29 □

6.6 Applications

In this section we explain a few applications of our comparison theorems. These are not really new, but our results allow us to present formulations that are more explicit than has been possible previously. Throughout this section we assume that S is a formal ϕ-scheme and that $X/\overline{S}$ is a smooth morphism. As usual, we let $F_{X/S} \colon X \to X'$ denote the relative Frobenius morphism. Recall that we have canonical morphisms:

$$v_{X/S} \colon (X/S)_{\Delta} \to X_{\acute{e}t} \text{ and } u_{X/S} \colon (X/S)_{\mathbb{P}} \to X_{\acute{e}t}.$$

Compatibility of prismatic cohomology with base change, based on its computation by Čech-Alexander complexes, is discussed in [7, 4.18]. The use of p-de Rham complexes provides stronger results. This has been carried out by Y. Tian [27] who also deals with more general prisms. Here is our (slighlty different) version.

Theorem 6.34. *If E is a crystal of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$, the following statements hold.*

1. *$R^q v_{X/S*}(E) = 0$ if $q > \dim(X/\overline{S})$.*
2. *If E is locally free, then formation of $Rv_{X/S*}(E)$ commutes with base change $S \to S'$.*
3. *If $X/\overline{S}$ is proper of relative dimension n, if E is locally free of finite ran, and if $S = \operatorname{Spf} A$, then $R\Gamma(X/S_{\Delta}, E)$ is a perfect complex of p-adic A-modules, of amplitude in $[0, 2n]$.*

Proof. The first statement may be verified Zariski locally on X, so we may without loss of generality assume that there is a formal ϕ-scheme Y/S lifting $X/\overline{S}$. Then $Rv_{X/S*}(E)$ is calculated by the p-de Rham complex of (E_Y, ∇'), which has no terms in degrees greater than n. Statement (2) is proved in a similar manner, using the fact that formation of the p-de Rham complex is compatible with base change, and that its terms are locally free. (Note: if we were willing use a form of derived tensor product in the context of p-adic modules, then then local freeness assumption on E would be superfluous.) The last statement also follows; see [27] for details. $\square$

A key result of Bhatt-Scholze, [7, 5.2], compares prismatic and crystalline cohomology. It asserts the existence of a canonical quasi-isomorphism:

$$L\phi_S^* Rv_{X/S*}(\mathcal{O}_{X/S}) \to Ru_{X/S*}(\mathcal{O}_{X/S}), \tag{6.17}$$

where are the canonical projection morphisms. Strictly speaking, this result is only stated and proved in [7] when X and S are affine; presumably the general case follows by standard simplicial methods. Proposition 6.32 provides a generalization to the case of crystals.

Theorem 6.35. *Let E' be a crystal of $\mathcal{O}_{X/S}$-modules on $\Delta(X/S)$ and let E be the crystal of $\mathcal{O}_{X/S}$-modules on $\mathbb{P}(X/S)$ corresponding to the F-transform (4.9) of $\pi^*(E')$.*

1. *There is a canonical strict quasi-isomorphism*

$$Rv_{X'/S*}(\pi^*(E')) \to F_{X/S*}Ru_{X/S*}(E).$$

2. *If E' s locally free or if ϕ_S is p-completely flat, there is a canoncial strict quasi-isomorphism*

$$\phi_S^*(Rv_{X/S*}(E')) \to F_{X/S*}(Ru_{X/S*}(E)).$$

3. *In particular, there is a caonical strict quasi-isomorphism:*

$$\phi_S^*(Rv_{X/S*}(\mathcal{O}_{X/S})) \to F_{X/S*}(Ru_{X/S*}(\mathcal{O}_{X/S})).$$

Proof. Statement (1) here is just a restatement of (2) of Proposition 6.32. The remaining statements then follow from the base change results of Theorem 6.34. $\square$

Next we show that Frobenius induces an isogeny on prismatic cohomology. We can again include a version with coefficients in a prismatic crystal.

Proposition 6.36. *Let E' be a crystal of $O_{X'/S}$-modules on $\Delta(X'/S)$ and let $\phi_{X/S}^*(E')$ be its pullback to a crystal of $O_{X/S}$-modules on $\Delta(X/S)$. Then the natural map*

$$Rv_{X'/S*}(E') \to F_{X/S*}Rv_{X/S*}(\phi_{X/S}^*(E'))$$

is an isogeny. In particular, the Frobenius morphism

$$\phi_\Delta \colon H^{\cdot}((X'/S)_\Delta, \mathcal{O}_{X'/S}) \to H^{\cdot}((X/S)_\Delta, \mathcal{O}_{X/S})$$

is an isogeny.

Proof. We prove this under the assumption that X admits an embedding in a p-completely smooth formal ϕ-scheme Y/S. Let $E'_{Y'}$ denote the value of E' on the prismatic envelope of X' in Y'. Then the morphism in the statement is represented by the morphism of complexes:

$$(E'_{Y'} \otimes \Omega^{\cdot}_{Y'/S}, d') \to (\phi_{Y/S*}\phi^*_{Y/S}(E'_{Y'}), \Omega^{\cdot}_{Y/S}, d').$$

This is the morphism c in Theorem 4.7, the composite of the quasi-isomorphism a and the isogeny b. It follows that b becomes an isogeny in the derived category. The general case will follow by the usual simplicial methods. $\square$

The following result is stated in [7, 4.15] and [1, 3.2.1]. The treatments there only explicitly discuss the affine case, and the task of globalization (presumably using simplicial methods) is left to the reader. For quasi-projective schemes, our methods lead to a very explicit construction.

Theorem 6.37. *Let S be a formal ϕ-scheme, let $X/\overline{S}$ be a smooth and quasi-projective morphism, and let $\mathbb{L}_{X/S}$ denote the cotangent complex of X relative to S. Suppose that there is a closed immersion $X \to Y$, where Y/S is a p-completely smooth morphism of ϕ-schemes. Then there is a natural quasi-isomorphism;*

$$\mathbb{L}_{X/S} \to \tau^{\leq 0}\left(\mathcal{O}_{\overline{\mathbb{\Delta}}_X(Y)} \otimes \Omega^{\cdot}_{Y/S}[1]\right)$$

Proof. Strictly speaking, our formulation does not make sense without referring to an explicit complex representing $\mathbb{L}_{X/S}$. In fact, the closed immersion $X \to Y$, which is necessarily a regular immersion, provides us with such a representation: by [,], the cotangent complex of X/S can be identified with the complex:

$$\mathbb{L}_{X/Y/S} := I_{X/Y}/I^2_{X/Y} \xrightarrow{\overline{d}} \Omega^1_{Y/S} \otimes \mathcal{O}_X,$$

placed in degrees -1 and 0. Here $\overline{d}$ is the map induced by the Kahler differential $d\colon I_{X/Y} \to \Omega^1_{Y/S}$. It follows from the theory of cotangent complex that this construction is functorial in the derived sense. (In fact, it is elementary to check directly that if $g\colon Y \to Z$ is a morphism of smooth S-schemes such that $g \circ i$ is also a closed immersion, then the induced map of complexes $\mathbb{L}_{X/Z/S} \to \mathbb{L}_{X/Y/S}$ is a quasi-isomorphism.)

Recall from (2.1) that we have a morphism $\rho\colon I_{X/Y} \to \mathcal{O}_{\mathbb{\Delta}_X(Y)}$, with $p\rho(x) = \pi^\sharp_{\mathbb{\Delta}}(x)$ for all $x \in I_{X/Y}$. Thus ρ induces a morphism

$$\overline{\rho}\colon I_{X/Y}/I^2_{X/Y} \to \mathcal{O}_{\overline{\mathbb{\Delta}}_X(Y)}.$$

Furthermore,

$$pd'\rho(x) = d'\pi^\sharp_{\mathbb{\Delta}}(x) = pd\pi^\sharp_{\mathbb{\Delta}}(x) = p\pi^*_{\mathbb{\Delta}}(dx),$$

so $d'\rho(x) = \pi^*_{\Delta}(dx)$. We see that there is a commutative diagram of $\mathcal{O}_X$-linear maps:

$$
\begin{tikzcd}
I_{X/Y}/I^2_{X/Y} \arrow[r, "\overline{d}"] \arrow[d, "\overline{\rho}"'] & \Omega^1_{Y/S} \otimes \mathcal{O}_X \arrow[r, "\overline{d}"] \arrow[d, "\pi^*_{\Delta}"'] & \Omega^2_{Y/S} \otimes \mathcal{O}_X \arrow[d, "p\pi_{\Delta^*}"] \\
\mathcal{O}_{\overline{\Delta}_{X(Y)}} \arrow[r, "\overline{d}'"] & \mathcal{O}_{\overline{\Delta}_{X(Y)}} \otimes \Omega^1_{Y/S} \arrow[r, "\overline{d}'"] & \mathcal{O}_{\overline{\Delta}_{X(Y)}} \otimes \Omega^2_{Y/S}
\end{tikzcd}
$$

Since the rightmost vertical arrow is in fact zero, the diagram defines a morphism of complexes:

$$\mathbb{L}_{X/Y/S} \to \tau^{\leq 0}(\mathcal{O}_{\overline{\Delta}_{X(Y)}} \otimes \Omega^{\cdot}_{Y/S}[1]).$$

Suppose that $X \to Y'$ is another embedding of X into a p-completely smooth formal ϕ-scheme over S. Let $Z := Y \times Y'$ and let $X \to Z$ be the map induced by these two embeddings. We find a commutative diagram of complexes:

$$
\begin{tikzcd}
\mathbb{L}_{X/Y/S} \arrow[r] & \tau^{\leq 0}(\mathcal{O}_{\overline{\Delta}_{X(Y)}} \otimes \Omega^{\cdot}_{Y/S}[1]) \\
\mathbb{L}_{X/Z/S} \arrow[r] \arrow[u] \arrow[d] & \tau^{\leq 0}(\mathcal{O}_{\overline{\Delta}_{X(Z)}} \otimes \Omega^{\cdot}_{Z/S}[1]) \arrow[u] \arrow[d] \\
\mathbb{L}_{X/Y'/S} \arrow[r] & \tau^{\leq 0}(\mathcal{O}_{\overline{\Delta}_{X(Y')}} \otimes \Omega^{\cdot}_{Y'/S}[1)].
\end{tikzcd}
$$

The vertical maps on the right are quasi-isomorphisms by the prismatic Poincaré Lemma 3.11, and the maps on the left are quasi-isomorphisms as we have already explained. We can conclude that our construction gives a well-defined morphism in the derived category, as stated in the theorem. To show that it is a quasi-isomorphism, we may work locally, and in particular we may assume that Y is a lifting of X. In that case, the complex $\mathbb{L}_{X/Y/S}$ is $(p) \otimes \mathcal{O}_X$ in degree -1 and $\Omega^1_{X/S}$ in degree zero, with $\overline{d} = 0$, while the complex $\tau^{\leq 0}(\mathcal{O}_{\overline{\Delta}_{X(Y)}} \otimes \Omega^{\cdot}_{Y/S}[1])$ is $\mathcal{O}_X$ in degree -1 and $\Omega^1_{X/S}$ in degree zero, with $\overline{d}' = 0$. Furthermore, ρ sends the class of p to 1, and we see that the map of complexes is the identity map in this case. $\square$

7 Appendix: p-adic sheaves and modules

Let A be a p-adically separated and complete ring, not necessarily noetherian. We will need to work with A-modules which are also p-adically complete but not necessarily finitely generated. There are several possible strategies: "derived complete modules" as suggested in [7], topological modules, filtered modules, or p-adic inverse systems as in [9]. All these categories are additive, but only the first is abelian, which makes it a tempting choice, and is the strategy followed in [1]. However, this reference only studies derived p-complete modules which have bounded p-torsion, and as it turns out, these are necessarily p-adically separated and complete. Furthermore, we have been unable to prove the key exactness properties in Proposition 7.14 for derived p-complete modules in general. Consequently we will work instead with the category of p-adically separated and complete modules. This category can be profitably viewed as living either in the category of all modules, or in the category of inverse systems of torsion modules, each of which has its advantages and disadvantages. The following exposition, which is perhaps overly detailed and which includes many well-known results, summarizes our approach.

7.1 p-adic modules

Definition 7.1. *Let A be a p-adically separated and complete ring. We write $\mathbf{M}_A$ for the category of all A-modules, and $\mathbf{M}\cdot$ for the category of inverse systems $\{M_n \to M_{n-1} : n \in \mathbf{N}\}$ of A-modules such that $p^n M_n = 0$ for all n. A p-adic A-module is an object of $\mathbf{M}\cdot$ such that each map $M_n \otimes_A A_{n-1} \to M_{n-1}$ is an isomorphism.*

We have functors

$$\begin{aligned} \varprojlim : \mathbf{M}\cdot &\to \mathbf{M}_A : \quad M\cdot \mapsto \hat{M} \\ S\cdot : \mathbf{M}_A &\to \mathbf{M}\cdot : \quad M \mapsto \{M/p^n M \to M/p^{n-1}M\} \end{aligned} \tag{7.1}$$

Proposition 7.2. *With the definitions above, the following statements hold.*

1. *For every $M \in \mathbf{M}_A$, the object $S\cdot(M) \in \mathbf{M}\cdot$ is a p-adic A-module. For every $M\cdot$ in $\mathbf{M}\cdot$, the object $\varprojlim M\cdot$ of $\mathbf{M}_A$ is p-adically separated and complete.*

2. *If $M\cdot$ is a p-adic A-module and $\hat{M} := \varprojlim M\cdot$, then for each n, the natural map $\hat{M} \otimes_A A_n \to M_n$ is an isomorphism.*

3. *The functor $\varprojlim$ induces an equivalence from the category of p-adic A-modules to the category of p-adically separated and complete A-modules.*

Proof. The first part of statement (1) is obvious. For the second part, let $\hat{M} := \varprojlim M\cdot$, and for each n, let $F^n\hat{M}$ be the kernel of the map $\hat{M} \to M_n$. It follows from the construction that $\hat{M}$ is separated and complete for the F-adic topology. Since each $p^n M_n = 0$, this topology is weaker than the p-adic topology, and the following lemma shows s that $\hat{M}$ is aso p-adically separated and complete. This is proved in [21, A1] for any finitely generated ideal in place of (p); since the proof in this case is considerably simpler, we explain it again here.

Lemma 7.3. *Let M be an A-module, separated and complete with respect to an A-linear topology $F^{\cdot}$ such that $p^n M \subseteq F^n M$ for all n. Then M is also p-adically separated and complete.*

Proof. The separation is automatic, and for the completeness we check that every p-adically Cauchy sequence $(x.)$ in M has a p-adic limit. Since $(x.)$ is p-adically Cauchy, it suffices to find a subsequence which converges p-adically, so we may assume that $x_{n+1} - x_n \in p^n M$ for all n. Thus it suffices to prove that every series of the form $\sum p^n y_n$ converges p-adically. Since this series is also $F^{\cdot}$-adically Cauchy, it has an $F^{\cdot}$-adic limit y, and we claim that this is also the p-adic limit. Fix N, and consider the series $\sum\{p^{n-N} y_n : n \geq N\}$. This series is also $F^{\cdot}$-adically Cauchy and therefore has an $F^{\cdot}$-adic limit z_N, and it is clear that

$$p^N z_n = \sum_{n \geq N} p^n y^n = y - \sum_{n<N} p^n y^n.$$

Thus, the sequence of partial sums converges p-adically to y. $\square$

To prove (1), we repeat the argument of [9, 2.3.8] for the convenience of the reader. See also [2, 05GG] for a more general statement. Since the transition maps $M_n \to M_{n-1}$ are all surjective, so is each projection map $\hat{M} \to M_n$; let F^n denote its kernel. By construction, the module $\hat{M}$ is complete with respect to the F-adic topology. We know that $p^n \hat{M} \subseteq F^n \hat{M}$, and we must show that the reverse inclusion also holds. We have

$$\hat{M}/F^n \cong M_n \cong M_{n+1}/p^n M_{n+1} \cong \hat{M}/(F^{n+1} + p^n \hat{M}),$$

and it follows that that $F^n = p^n \hat{M} + F^{n+1}$. An element x of F^n can then be written as $p^n x_0' + x_1$ with $x_0 \in \hat{M}$ and $x_1 \in F^{n+1}$, and then we can write $x_1 = p^{n+1} x_1' + x_2$ with $x_2 \in F^{n+2}$. Continuing in this way, we find sequences $(x_0', x_1', x_2', \ldots)$ and $(x_1, x_2, \ldots)$, with $x_i' \in \hat{M}$, $x_i \in F^{n+i}$ and $x_i = p^{n+i} x_i' + x_{i+1}$ for all i. Let x' be the limit of the series $\sum p^i x_i'$, which converges $F^{\cdot}$-adically. Then $x = p^n x'$, since $\hat{M}$ is F-adically separated.

If M is p-adically separated and complete, then the natural map $M \to \varprojlim S.(M)$ is an isomorphism, by definition. On the other hand, if $M.$ is a p-adic A-module, statement tells us that $\varprojlim M.$ is p-adically complete and that the natural map of inverse systems $S.(\varprojlim M.) \to M.$ is an isomorphism. Statement (3) follows. $\square$

7.2 p-adic exactness

Example 7.4. The category of p-adically separated and complete A-modules is not abelian: a proper inclusion with dense image is a monomorphism and an epimorphism but not an isomorphism. For an example, let A be $\mathbf{Z}_p$, let M be the $\mathbf{Z}_p$-module consisting of the set of sequences $(a_0, a_1, \ldots)$ in $\mathbf{Z}_p$ which tend to zero p-adically, and let N be the set of all sequences in $\mathbf{Z}_p$. These module are p-adically separated and complete, as is easy to verify. Define a map $\alpha\colon M \to M$

by $\alpha(a_0, a_1, a_2 \ldots) := (a_0, pa_1, p^2a_2 \ldots)$ and a map $\beta\colon N \to M$ by the same formula. We have a commutative diagram with exact rows:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\alpha"] \arrow[d, "\gamma"'] & M \arrow[r] \arrow[d, "\mathrm{id}"] & M'' \arrow[r] \arrow[d, "\delta"] & 0 \\
0 \arrow[r] & N \arrow[r, "\beta"] & M \arrow[r] & N'' \arrow[r] & 0,
\end{tikzcd}$$

where γ is the inclusion. The image of β is the set of sequences $c\cdot$ such that each c_i is divisible by p^i, and is evidently closed in M. If $b\cdot \in N$ and $n > 0$, then

$$\begin{aligned}
\beta(b\cdot) &= (b_0, pb_1, p^2b_2, \cdots) \\
&= (b_0, pb_1, \ldots, p^{n-1}b_{n-1}, 0\cdots) + p^n(0, 0, \ldots, b_n, pb_{n+1} + \cdots)
\end{aligned}$$

which is of the form $\alpha(a\cdot) + p^n a'_\cdot$, with $a\cdot$ and $a'_\cdot \in M$. Thus $\alpha(M)$ is dense in $\beta(M)$, and in fact $\beta(M)$ is the closure of $\alpha(M)$ in M.

Note that M'' is the standard example of a module which is derived p-complete but not p-adically separated [2, 0G3F]. We also note that M'' can be written as an extension of two p-adically separated and complete modules, showing that the class of such modules does not form an exact subcategory of the category of all modules. To see this, we check first that the image of γ is closed in N. If $b\cdot \in N$ is in the closure of $\gamma(M)$, then for every $n > 0$, there exist $a\cdot \in M$ and $b'_\cdot \in N$ such that $b\cdot = \gamma(a\cdot) + p^n b'_\cdot$. Then there exists m_n such that $a_i \in p^n\mathbf{Z}_p$ for all $i \geq m_n$, which implies that the same holds for b_i. Thus $b\cdot$ also belongs to the image of γ. The snake lemma implies that $\mathrm{Ker}(\delta) \cong \mathrm{Cok}(\gamma)$. Since the image of γ is closed, these modules are p-adically separated, and since $\beta(N)$ is closed in N, the quotient N'' is also p-adically separated. Then

$$0 \to \mathrm{Ker}(\delta) \to M'' \to N'' \to 0$$

is the desired extension of separated and complete modules which is derived complete but not separated. We note further that $\mathrm{Ker}(\delta) = \cap_n p^n M''$ and that N''*quotient* is the p-adic completion of M'' and so by the exact sequence in [2, 0BKG], that $\mathrm{Ker}(\delta) = R^1 \lim M''[p^n]$. Note also that this module is p-torsion free.

Since the category of p-adic A-modules is not abelian, it is not safe to speak about exact sequences and the cohomology of complexes. We therefore introduce the following terminology.

Definition 7.5. *Let A be a p-adically separated and complete ring.*

1. *A morphism $M'_\cdot \to M\cdot$ of p-adic A-modules is a strict monomorphism if each $M'_n \to M_n$ is a monomorphism.*
2. *A sequence of p-adic A-modules $M'_\cdot \to M\cdot \to M''_\cdot$ is strictly short exact if each sequence $M'_n \to M_n \to M''_n$ is short exact.*

3. *A complex $M^{\cdot}_{\cdot}$ of p-adic A-modules is strictly acyclic if each $M^{\cdot}_n$ is acyclic.*
4. *A morphism $u_{\cdot}$ of complexes of p-adic A-modules is a strict quasi-isomorphism if each u_n is a quasi-isomorphism.*

Remark 7.6. The category $\mathbf{M}_{\cdot}$ is an abelian category, and a sequence in this category is exact if and only the corresponding sequences for each n are all exact. The category of p-adic A-modules is a full subcategory of $\mathbf{M}_{\cdot}$, and we claim that it is closed under extension. Indeed, if $0 \to M'_{\cdot} \to M_{\cdot} \to M''_{\cdot} \to 0$ is exact in $\mathbf{M}_{\cdot}$ and $M'_{\cdot}$ and $M''_{\cdot}$ are p-adic systems, then for each n we have a commutative diagram

$$\begin{tikzcd}
 & M'_n \otimes A_{n-1} \arrow[r] \arrow[d] & M_n \otimes A_{n-1} \arrow[r] \arrow[d] & M''_n \otimes A_{n-1} \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & M'_{n-1} \arrow[r] & M_{n-1} \arrow[r] & M''_{n-1} \arrow[r] & 0
\end{tikzcd}$$

whose rows are exact. The vertical arrows on the left and right are isomorphisms, and it follows that the central one is also. Thus our definition of "strict exact sequences" is consistent with Quillen's notion of an "exact category." We have two functors from the category of p-adic A-modules to an abelian category: the inclusion into the category $\mathbf{M}_{\cdot}$ and the inverse limit functor into the category $\mathbf{M}_{A}$. (or into the category of derived p-complete modules). By definition, a sequence $0 \to M'_{\cdot} \to M_{\cdot} \to M''_{\cdot} \to 0$ of p-adic A-modules is (strictly) exact if and only if it is so when viewed in $\mathbf{M}_{\cdot}$. The inverse limit of such a sequence is again exact, but converse does not hold.

Lemma 7.7. *Let M be a p-adically separated and complete A-module and let K be a submodule.*

1. *If K is closed in the p-adic topology of N, then it is p-adically separated and complete.*
2. *Let K^- be the closure of K in M. Then M/K^- is the p-adic completion of M/K.*

Proof. The topology of K induced by the p-adic topology of M could be weaker than the p-adic topology of K but it follows nonetheless that K is p-adically separated and complete, by Lemma 7.3. We have an inverse system of short exact sequences:

$$0 \to K/p^nM \cap K \to M/p^nM \to M/(p^nM + K) \to 0.$$

Since the transition maps on the left are surjective, we find an exact sequence

$$0 \to \varprojlim K/p^nM \cap K \to \varprojlim M/p^nM \to \varprojlim M/(p^nM + K) \to 0.$$

Since $K^- = \varprojlim K/p^nM \cap K$, the lemma is proved. $\square$

Proposition 7.8. *Let A be a p-adically separated, complete, and p-torsion free ring.*

1. *The category of p-adically separated and complete A-modules admits kernels, whose formation is compatible with the inclusion in the category $\mathbf{M}_A$.*
2. *Suppose that the kernel K of a homomorphism $u\colon M \to N$ of p-adically separated and complete A-modules agrees with the kernel of the corresponding homomorphism of $u.\colon M. \to N.$ in $\mathbf{M}.$. Then*
   1. *(a) The inclusion $K \to M$ is a strict monomorphism.*
   2. *(b) The image M'' of M in N is p-adically separated and complete and is closed in N, and the inclusion $M'' \to N$ is a strict monomorphism.*

   A homomorphism of p-adically separated and complete A-modules is said to be strict *if it satisfies these properties.*

Proof. If $u\colon M \to N$ is a homomorphism of p-adically separated and complete A-modules, let $K := \{x \in M : u(x) = 0\}$. This module is evidently closed in the p-adic topology of M, and so it is p-adically separated and complete by Lemma 7.7. As an example, if u is multiplication by p on $A := \mathbf{Z}_p$, we see that $K = \{0\}$, although u_n is not injective for $n > 0$.

Now suppose that $K_n = \mathrm{Ker}(u_n)$ for all n. Then necessarily each $K_n \to M_n$ is injective, so $K \to M$ is a strict monomorphism. For each n, we have a commutative diagram

$$\begin{tikzcd} & K \otimes_A A_n \arrow[r] \arrow[d] & M \otimes_A A_n \arrow[r] \arrow[d] & M'' \otimes_A A_n \arrow[r] \arrow[d] & 0 \\ 0 \arrow[r] & K_n \arrow[r] & M_n \arrow[r] & \mathrm{Im}(u_n) \arrow[r] & 0 \end{tikzcd}$$

with exact rows. Since the first two vertical arrows are isomorphisms, so is the third, and the left horizontal top arrow is injective. Since the transition maps for the system $K.$ are surjective, we can deduce that the map $M = \varprojlim M. \to \varprojlim M''.$ is surjective, and since $M''_n = \mathrm{Im}(u_n) \subseteq N_n$, that the map $\varprojlim M''. \to \varprojlim N. = N$ is injective. Thus $M'' = \hat{M}''$ is p-adically separated and complete. Moreover, since $M''_n \subseteq N_n$ for all n, the p-adic topology of M'' agrees with the induced topology from that of N, and the map $M'' \to N$ is a strict monomorphism. It follows also that M'' is closed in the p-adic topology of N. $\square$

Remark 7.9. We should remark that the category of p-adic A-modules also admits cokernels, whose formation is compatible with the inclusion into the category $\mathbf{M}.$. but not into the category $\mathbf{M}_A$. Namely, if $u.\colon M. \to N.$ is a homomorphism of p-adic A-modules, then the inverse system $\mathrm{Cok}(u.)$ is again a p-adic A-module, but its inverse limit need not be the cokernel of the map

$\hat{u} := \varprojlim u_\cdot$. Indeed, if $M''_n := \mathrm{Im}(u_n)$, then the inverse system $M''_\cdot$ has surjective transition maps, although it might not form a p-adic A-module. We thus get a surjective map $\varprojlim N_\cdot \to \varprojlim \mathrm{Cok}(u_\cdot)$, although the map $\varprojlim M_\cdot \to \varprojlim \mathrm{Im}(u_\cdot)$ might not be surjective. In fact it is easy to verify that $\varprojlim \mathrm{Im}(u_\cdot)$ is the closure of $\mathrm{Im}(\hat{u})$ in $\varprojlim N_\cdot$. Thus $\varprojlim \mathrm{Cok}(u_\cdot)$ is the quotient of $\varprojlim N_\cdot$ by the closure of $\mathrm{Im}(\hat{u})$.

Proposition 7.10. *Let A be a p-adically separated and complete ring.*

1. *The inverse limit of a a strictly exact sequence $0 \to M'_\cdot \to M_\cdot \to M''_\cdot$ (resp. $0 \to M'_\cdot \to M_\cdot \to M''_\cdot \to 0$) of p-adic A-modules is an exact sequence of A-modules.*

2. *The inverse limit of a strictly acyclic complex of p-adic A-modules is an acyclic complex of A-modules.*

3. *The inverse limit of a strict quasi-isomorphism of p-adic A-modules is a quasi-isomorphism of complexes of A-modules.*

Proof. If each sequence $0 \to M'_n \to M_n \to M''_n$ is exact, then so so is the limit sequence, since the inverse limit functor is left exact. Surjectivity is also preserved on the right, since the transition maps in the kernel sequence are all surjective. This proves statement (1).

For (2), suppose that $K^\cdot_\cdot$ is a complex of p-adic A-modules and that each complex $K^\cdot_n$ is acyclic For each $q \in \mathbf{Z}$, let $\overline{K}^q_n$ be the image of d^{q-1}_n, so that we have exact sequences:

$$0 \to \overline{K}^q_n \to K^q_n \to \overline{K}^{q+1}_n \to 0$$

for each n. Since we have a surjection $K^{q-1}_\cdot \to \overline{K}^q_\cdot$ and the transition maps of $K^{q-1\cdot}$ are surjective, the same is true of the transition maps of $\overline{K}^q_\cdot$, and hence the sequence

$$0 \to \varprojlim \overline{K}^q_\cdot \to \varprojlim K^q_\cdot \to \varprojlim \overline{K}^{q+1}_\cdot \to 0$$

is also exact. This holds for all q, and so the map $\varprojlim K^{q-1}_\cdot \to \varprojlim \overline{K}^q_\cdot$ is also surjective. Thus $\varprojlim \overline{K}^q_\cdot$ is the image of the map $\varprojlim d^{q-1}_\cdot$. Since we also know that it is the kernel of the map $\varprojlim d^q_\cdot$, we can conclude that $\varprojlim K^\cdot_\cdot$ is exact. Since the cone of a map of complexes of p-adic A-modules is again a complex of p-adic A-modules, statement (3) follows. □

We should also mention the following simple version of the derived Nakayama Lemma[2, 15.90.19].

Proposition 7.11. *Let $K^\cdot_\cdot$ be a complex of p-adic A-modules. Suppose that each K^q_n is flat over $\mathbf{Z}/p^n\mathbf{Z}$. If $H^m(K^\cdot_1)$ vanishes for some m, then $H^m(K^\cdot_n)$ vanishes for every n.*

Proof. Since $K_1^q \cong K_n^q/pK_n^q$ and K_n^q is flat, we have an exact sequence of complexes:

$$0 \to K_{n-1}^{\cdot} \to K_n^{\cdot} \to K_1^{\cdot} \to 0$$

Then the result follows by induction on n. □

7.3 p-complete flatness

Since the Artin-Rees lemma is not available in our context, the map from a module to its formal completion might not be flat, and as far as we know, even the map from a ring A to the p-adic completion of a localization of A might not be flat. The notion of "complete flatness" helps to overcome this difficulty. We add some details to the discussion in [7, Corollary 3.14].

Definition 7.12. *If I is an ideal in a ring A, then an A-module M is said to be I-completely flat if $\mathrm{Tor}_i^A(M,N)$ vanishes whenever $i > 0$ and N is an A-module with $IN = 0$. A module is I-completely faithfully flat if it is I-completely flat and in addition M/IM is faithfully flat as an A/I-module.*

In our case, the ideal I will be principally generated by p, and the notion simplifies. In particular, it shows that, if A is p-torsion free and p-adically separated and complete, then a p-adically separated and complete A-module M is p-completely flat if and only if the corresponding p-adic A-module $M.$ is flat as an inverse system of $A.$-modules.

Proposition 7.13. *Let A be a p-torsion free ring and M an A-module; write M_n for M/p^n if $n \in \mathbf{N}$. Then conditions (1)–(3) below are equivalent and imply condition (4). If M is p-adically separated, then all four conditions are equivalent.*

1. *M is p-completely flat (resp. faithfully flat).*
2. *M is p^n-completely flat (resp. faithfully flat) for all $n > 0$.*
3. *M is p-torsion free and M/pM is flat (resp. faithfully flat) over A/pA.*
4. *M_n is flat (resp. faithfully flat) over A_n for all $n > 0$.*

Proof. Since a flat A_n-module is faithfully flat if and only if its reduction modulo p is faithfully flat as an A_1-module, it suffices to prove the equivalence for flatness.

We prove that (1) implies (2) by induction on n. Suppose that M is p^n-completely flat and that $p^{n+1}N = 0$. Let $N' := \{x \in N : px = 0\}$ and let $N'' := N/N'$. Then $p^nN'' = p^nN' = 0$, and the exact sequence

$$\mathrm{Tor}_i^A(M,N') \to \mathrm{Tor}_i^A(M,N) \to \mathrm{Tor}_i^A(M,N'')$$

shows that $\mathrm{Tor}_i^A(M,N) = 0$ if $i > 0$, so M is p^{n+1}-completely flat.

If (2) holds, let $F. \to M$ be a flat resolution of M. Condition (2) implies that $\mathrm{Tor}_i^A(M,A_n) = 0$ for $i > 0$, so $F_n \to M_n$ is a flat resolution of the A_n-module M_n. If N is an A_n-module, the isomorphism

$$\mathrm{Tor}_i^{A_n}(M_n,N) \cong H_i(F_n^{\cdot} \otimes_{A_n} N) \cong H_i(F. \otimes_A N) \cong \mathrm{Tor}_i^A(M,N) \tag{7.2}$$

shows that $\mathrm{Tor}_i^{A_n}(M, N) \cong \mathrm{Tor}_i^A(M, N)$. Condition (2) implies that this vanishes if $i > 0$, and since N is an arbitrary A_n-module, it follows that M_n is A_n-flat. Thus (2) implies (4).

If (2) holds, then $\mathrm{Tor}_1^A(M, A/pA)$ vanishes, so M is p-torsion free, and since (4) also holds, we see that (2) implies (3).

To see that (3) implies (2), we again choose a flat resolution $F.$ of M. Condition (3) implies that M is p-torsion free and hence that $\mathrm{Tor}_i^A(M, A/pA) = 0$ for all $i > 0$, so $F. \otimes_A A/pA$ is a flat resolution of M/pM. In particular, equation 7.2 holds with $n = 1$, *i.e.*, $\mathrm{Tor}_i^A(M, N) \cong \mathrm{Tor}_i^{A/pA}(M/pM, N)$ for every A/pA-module N. Since M/pM is assumed to be flat over A/pA, this module vanishes for every $i > 0$, so M is p-completely flat. Thus conditions (1)–(3) are equivalent.

It remains only to show that (4) implies (3) if M is p-adically separated. The flatness of M_n implies that the sequence

$$0 \longrightarrow M/pM \xrightarrow{p^{n-1}} M_n \xrightarrow{p} M_n$$

is exact and hence that $\varprojlim M_n$ is p-torsion free. (See the proof of Lemma 7.15 below.) Thus any p-torsion element of M maps to zero in this limit and necessarily vanishes if M is p-adically separated. $\square$

Proposition 7.14. *Suppose that A is p-torsion free and p-adically separated and complete and that M is a p-completely flat A-module.*

1. *If N is a p-torsion free A-module, then $N \otimes_A M$ and $N \hat{\otimes}_A M$ are also p-torsion free.*

2. *If N is a p-completely flat A-module, then $N \hat{\otimes}_A M$ is also p-completely flat.*

3. *If $A \to A'$ is a homomorphism of p-torsion free p-adically separated and complete rings, then $A' \hat{\otimes}_A M$ is p-completely flat over A', and is faithfully so if M is p-completely faithfully flat over A.*

4. *The composition of p-completely flat (resp. faithfully flat) homomorphisms is again p-completely flat (resp. faithfully so). If $A \to A'$ is p-completely faithfully flat, then a p-adically separated and complete A-module N is p-completely flat (resp. faithfully so) over A if and only if $A' \hat{\otimes}_A N$ is so over A'.*

Proof. The short exact sequence $0 \longrightarrow N \xrightarrow{p} N \longrightarrow N/pN \longrightarrow 0$ gives rise to a long one:

$$\mathrm{Tor}_A^1(N, /pN, M) \longrightarrow N \otimes_A M \xrightarrow{p} N \otimes_A M \longrightarrow N/pN \otimes_A M \longrightarrow 0.$$

Since M is p-completely flat, the term on the left vanishes. This implies that $N \otimes M$ is p-torsion free, and the result for $N \hat{\otimes} M$ follows from Lemma 7.15 below.

If N is p-completely flat, then it is p-torsion free, and $N\hat{\otimes}M$ is p-torsion free by (1). Since N_1 and M_1 are flat over A_1, so is their tensor product. By (1) of Proposition 7.2, this is the same as $(N\hat{\otimes}M)_1$, so $N\hat{\otimes}M$ is p-completely flat by criterion (3) of Proposition 7.13.

To prove (3), first observe that statement (1) implies that $M' := A'\hat{\otimes}_A M$ is p-torsion free. Furthermore, $M'_1 \cong A'_1 \otimes_{A_1} M_1$, which is flat (resp. faithfully flat) over A'_1 because M_1 is flat (resp. faithfully flat) over A_1. Then criterion (3) of Proposition 7.13 implies that M' is p-completely flat (resp. faithfully flat) over A'.

Statement (4) follows immediately from criterion (4) of Proposition 7.13 and the analogous results for usual flatness. $\square$

Lemma 7.15. *The p-adic completion of a p-torsion free abelian group is again p-torsion free.*

Proof. If M is p-torsion free, then for every n we have a commutative diagram with exact rows:

$$\begin{array}{ccccccc}
0 & \longrightarrow & M/pM & \xrightarrow{p^n} & M/p^{n+1}M & \xrightarrow{p} & M/p^{n+1}M \\
 & & \Big\downarrow p & & \Big\downarrow \pi & & \Big\downarrow \pi \\
0 & \longrightarrow & M/pM & \xrightarrow{p^{n-1}} & M/p^nM & \xrightarrow{p} & M/p^nM
\end{array}$$

Since $\varprojlim$ is left exact, we find that the sequence

$$0 \longrightarrow \varprojlim(M/pM, p) \longrightarrow \varprojlim(M/p^nM, \pi) \xrightarrow{p} \varprojlim(M/p^nM, \pi)$$

is exact. The inverse limit on the left is zero, and the result follows. $\square$

We now investigate the relationship beween p-complete flatness and strict exactness. Note that if $N.$ is a p-adic A-module with inverse limit $\hat{N}$ and M is any A-module, then $\{N_n \otimes M \cong (\hat{N} \otimes M)_n\}$ forms a p-adic A-module whose inverse limit is $\hat{N}\hat{\otimes}M$. Furthermore $(\hat{N} \otimes M)_n \cong N_n \otimes M_n$ for every n, by statement (1) of Proposition 7.2.

Proposition 7.16. *Let $K^{\cdot}$ be a complex of p-adically separated and complete A-modules and let M be a p-completely flat A-module.*

1. *If $K^{\cdot}$ is strictly acyclic, then so is $K^{\cdot}\hat{\otimes}M$.*

2. *If M is p-completely faithfully flat and $K^{\cdot}\hat{\otimes}M$ is strictly acyclic, then so is $K^{\cdot}$.*

3. *If M is p-completely faithfully flat, then a homomorphism of complexes u of p-adically separated and complete modules is a strict quasi-isomorphism if and only if $u\hat{\otimes}\mathrm{id}M$ is.*

In particular, if Σ is a strictly short exact sequence of p-adically separated and complete A-modules, then $\Sigma\hat{\otimes}M$ is again strictly short exact.

Proof. If each $K^{\cdot}_n$ is acyclic, then, since each M_n is flat over A_n, so is each $K^{\cdot}_n \otimes_{A_n} M_n$. It follows from Proposition 7.10 that the inverse limit is again exact. If now M is p-completely faithfully flat, then each M_n is a faithfully flat A_n-module, and if each $K^{\cdot}_n \otimes_A M$ is acyclic, then so is each $K^{\cdot}_n$. The conclusion again follows from Proposition 7.10. Statement (3) follows by taking the mapping cone of u, which is again a complex of p-adically separated modules. □

Corollary 7.17. *Suppose that M is a p-completely flat A-module.*

1. *Let $N'^{\cdot} \to N^{\cdot}$ be a homomorphism of complexes of p-adically separated and complete A-modules. Each $N'_n \otimes_A M \to N_n \otimes_A M$ is a quasi isomorphism if and only if each each $N'^{\cdot}_n \to N^{\cdot}_n$ is. If this is true for all n, then $N'^{\cdot} \to N^{\cdot}$ and $N'^{\cdot}\hat{\otimes}_A M \to N^{\cdot}\hat{\otimes}_A M$ are quasi-isomorphisms.*

2. *If $N' \to N$ is a homomorphism of p-adically separated and complete A-modules, then $N' \to N$ is an isomorphism if and only if $N'\hat{\otimes}M \to N\hat{\otimes}M$ is an isomorphism.*

Proof. Statement (1) follows from Proposition 7.16 applied to the cone of $N'^{\cdot} \to N^{\cdot}$. Statement (2) follows from (1), or directly from Proposition 7.16. because $N' \to N$ is an isomorphism if and only if each $N'_n \to N_n$ is an isomorphism □

Corollary 7.18. *Let E_{tor} be the p-torsion submodule of a p-adically separated and complete A-module E and let E^-_{tor} be its closure in E. The quotient E/E^-_{tor} is p-torsion free and the inclusion $E^-_{tor} \to E$ is a strict monomorphism. Furthermore, if M is p-completely flat, the natural maps $E^-_{tor}\hat{\otimes}M \to (E\hat{\otimes}M)^-_{tor}$ and $(E/E^-_{tor})\hat{\otimes}M \to (E\hat{\otimes}M)/(E\hat{\otimes}M_{tor})^-$ are isomorphisms.*

Proof. Since E/E^-_{tor} is the completion of E/E_{tor} (see Lemma 7.7) and the latter is p-torsion free, Lemma 7.15 implies that E/E^-_{tor} is also p-torsion free. This implies that $p^nE \cap E^-_{tor} = p^nE^-_{tor}$ for all n and hence the strictness of the map $E^-_{tor} \to E$. Thus the sequence $0 \to E^-_{tor} \to E \to E/E^-_{tor} \to 0$ is strictly short exact, and remains so after forming the completed tensor product with M, by Proposition 7.16. This gives us the exactness of the top row of the following diagram:

$$\begin{tikzcd}
0 \arrow[r] & E^-_{tor}\hat{\otimes}M \arrow[r] \arrow[d] & E\hat{\otimes}M \arrow[r] \arrow[d] & (E/E^-_{tor})\hat{\otimes}M \arrow[r] \arrow[d, dashed] & 0 \\
0 \arrow[r] & (E\hat{\otimes}M)^-_{tor} \arrow[r] & E\hat{\otimes}M \arrow[r] & (E\hat{\otimes}M)/(E\hat{\otimes}M)^-_{tor} \arrow[r] & 0
\end{tikzcd}$$

It is clear that $E_{tor}\hat{\otimes}M$ maps to $(E\hat{\otimes}M)_{tor}$, and hence the same is true for the corresponding closures. This gives the existence of the dashed arrow (the "natural map") in the diagram. The module $E/E^{-}_{tor}\hat{\otimes}M$ is p-adically separated and complete, and it is p-torsion free, by Proposition 7.14. Thus the map $E\hat{\otimes}M \to E/E^{-}_{tor}\hat{\otimes}M$ factors through $(E\hat{\otimes}M)/(E\hat{\otimes}M)^{-}_{tor}$, and so the right vertical arrow is an isomorphism. It follows that the left vertical arrow is also an isomorphism. □

Proposition 7.19. *If $A \to B$ be a p-completely faithfully flat homomorphism of p-adically separated and complete p-torsion free rings, let $C^{\cdot}_B$ be the completed augmented Čech-Alexander complex*

$$C^{\cdot}_B := A \to B \to B\hat{\otimes}_A B \to B\hat{\otimes}_A B\hat{\otimes}_A B \cdots .$$

1. *If N is a p-adically separated and complete A-module, then the complex is strictly acyclic, $C^{\cdot}_B\hat{\otimes}N$, and $C^{\cdot}_B \otimes N_n$ is acyclic for every n.*
2. *If N is a p-adically separated and complete B-module endowed with formal descent data $\epsilon\colon B\hat{\otimes}_A B\hat{\otimes}N \to N\hat{\otimes}B\hat{\otimes}_A B$, let*

$$\begin{array}{rcl} N^{\epsilon} & := & \{x \in N : \epsilon(1\hat{\otimes}x) = x\hat{\otimes}1\} \\ (N_n)^{\epsilon} & := & \{x \in N_n : \epsilon_n(1\hat{\otimes}x) = x\hat{\otimes}1\}. \end{array}$$

   Then:

   (a) *$(N^{\epsilon})_n = (N_n)^{\epsilon}$ for all n, and we will denote both by N^{ϵ}_n. Thus $N^{\epsilon} \to N$ is a strict monomorphism.*

   (b) *The natural maps $B\hat{\otimes}_A N^{\epsilon} \to N$ and $B \otimes_A N^{\epsilon}_n \to N_n$ are isomorphisms for all n.*

Proof. Statement (1) of Proposition 7.2 implies that

$$(C_B\hat{\otimes}_A N)_n \cong C_{B_n} \otimes_{A_n} N_n$$

for every n. Since each $A_n \to B_n$ is faithfully flat, the acyclicty of each of these complexes is standard. Thus $C_B\hat{\otimes}_A N)$. forms a strictly acyclic complex of p-adic B-modules, and so Proposition 7.10 implies that the limit sequence is also acyclic. This proves (1).

For (2), we start with the fact that each $B\otimes_A (N_n)^{\epsilon} \to N_n$ is an isomorphism, by usual faithfully flat descent for the homomorphism $A_n \to B_n$. Since the maps $N_n \otimes_{B_n} B_{n-1} \to N_{n-1}$ are isomorphisms, the same is true for the maps $B \otimes_A (N_n)^{\epsilon} \otimes_{B_n} B_{n-1} \to B \otimes_A (N_{n-1})^{\epsilon}$. Again invoking faithfully flat descent, we conclude that the map $(N_n)^{\epsilon} \otimes_{A_n} A_{n-1} \to (N_{n-1})^{\epsilon}$ are isomorphisms. Thus $(N.)^{\epsilon}$ forms a p-adic A-module. Then (1) of Proposition 7.2 implies that each map $\varprojlim(N_n)^{\epsilon} \otimes_A A_n \to N^{\epsilon}_n$ is an isomorphism. The left exactness of $\varprojlim$ implies that the natural map $N^{\epsilon} \to \varprojlim(N_n)^{\epsilon}$ is an isomorphism, and (2a) follows. We have already seen that each map $B \otimes_A N^{\epsilon}_n \to N_n$, is an isomorphism, and hence so is the map

$$\varprojlim(B \otimes_A N^{\epsilon}_n) \to \varprojlim N_n = N.$$

Since $\varprojlim(B \otimes_A N^{\epsilon}_n) \cong B\hat{\otimes}_A N^{\epsilon}$, statement (2) follows. □

A morphism of formal schemes $S' \to S$ is said to be p-completely flat if for every open affine $U' \subseteq S'$ mapping to an open affine $U \subseteq S$, the corresponding map $\mathcal{O}_S(U) \to \mathcal{O}_{S'}(U')$ is p-completely flat. It follows from Proposition 7.14 that a p-completely flat map $A \to A'$ gives rise to a p-completely flat map $\operatorname{Spf} A' \to \operatorname{Spf} A$ and that the family of p-completely faithfully flat maps $S'' \to S' \to S$ forms a covering family for a site we denote by S_{pcf}. We note that the set of p-completely flat maps $U \to S$ with U affine forms a base for this topology.

7.4 p-complete quasi-coherence

We now briefly discuss how to sheafify these notions. The equivalence of the two conditions in the following definition is a consequence of Proposition 7.2.

Definition 7.20. *Let S be a p-torsion free p-adic formal scheme. A sheaf of $\mathcal{O}_S$-modules E on S_{pcf} is p-completely quasi-coherent if its value on each affine is p-adically separated and complete and the following equivalent conditions are satisfied.*

1. *For every p-completely flat S-morphism $U' \to U$ of affine p-adic formal schemes which are p-completely flat over S, the map*
$$\mathcal{O}_S(U')\hat{\otimes}_{\mathcal{O}_S(U)}E(U) \to E(U')$$
is an isomorphism.

2. *For every p-completely flat S-morphism $U' \to U$ of affine p-adic formal schemes which are p-completely flat over S, the map*
$$\mathcal{O}_{S_n}(U')\hat{\otimes}_{\mathcal{O}_{S_n}(U)}E_n(U) \to E_n(U')$$
is an isomorphism for every n.

In particular, the structure sheaf $\mathcal{O}_S$ is itself p-completely quasi-coherent, and if E is p-completely quasi-coherent, then each E_n is quasi-coherent as sheaf of $\mathcal{O}_{S_n}$-modules on the scheme S_n.

The following results follow from the discussions above by standard arguments, which we leave to the reader.

Proposition 7.21. *Let S be a p-torsion free p-adic formal scheme.*

1. *If E is a sheaf of $\mathcal{O}_S$-modules and there is a cover $\{U_i \to S\}$ of S_{pcf} such that each $E_{|U_i}$ is p-completely quasi-coherent, then E is p-completely quasi-coherent.*

2. *Suppose that $S = \operatorname{Spf} A$ and M is a p-adically separated and complete A-module. Define a presheaf $\tilde{M}$ on the family of affine p-completely flat maps $\operatorname{Spf} A' \to \operatorname{Spf} A$ by $\tilde{M}(\operatorname{Spf} A') := A'\hat{\otimes}_A M$. Then in fact $\tilde{M}$ is a sheaf, and the map $M \to \tilde{M}(S)$ is an isomorphism. Moreover, the functor $M \to \tilde{M}$ defines an equivalence from the category of p-adically separated and complete A-modules to the category of p-complete quasi-coherent sheaves of $\mathcal{O}_S$-modules.*

3. *If S is affine and E is a p-completely quasi-coherent sheaf of $\mathcal{O}_S$-modules on S_{pcf}, then $H^i(S_{pcf}, E)$ vanishes for $i > 0$.* □

Remark 7.22. It will sometimes be convenient for us to view p-completely quasi-coherent sheaves geometrically. We do this by adapting Grothendieck's $\mathbf{V}$ construction. Namely, if Y is a p-adic formal scheme and E is a p-completely quasi-coherent sheaf of $\mathcal{O}_Y$-modules, then for each $n \in \mathbf{N}$, the sheaf E_n is quasi-coherent on Y_n and we can form the scheme $\mathbf{V}E_n := \mathrm{Spec}(S^{\cdot}E_n)$, affine over Y_n. Since formation of $S^{\cdot}$ is compatible with base change, the system $S^{\cdot}E.$ forms a p-adic module, and $\varprojlim S^{\cdot}E.$ is p-adically separated and complete, by Proposition 7.2. Thus we can view the collection of Y-schemes $\{\mathbf{V}E_n : n \in \mathbf{N}\}$ as the family of reductions of a p-adic formal scheme $\mathbf{V}E$, affine over Y.

7.5 Very regular sequences

Various notions of "regular sequences" are used in the literature and in particular in [7]. We have found the following slightly stronger notion useful. (All the notions are equivalent in the noetherian case.)

Definition 7.23. *Let B be a ring and M a B-module. Recall that a sequence $(b_1, \ldots, b_r)$ is said to be M-regular if for all i, multiplication by b_i on $M/(b_1, \ldots, b_{i-1})M$ is injective, and in addition the quotient $M/(b_1, \ldots, b_r)$ is not zero. We say that $(b_1, \ldots, b_r)$ is very M-regular if each of its permutations, or equivalently, each of its subsequences, is M-regular.*

For the equivalence of the conditions in the definition, we refer to [2, 10.68.10].

Proposition 7.24. *Let Y be a p-torsion free p-adic formal scheme and let $X \to Y_1$ be a very regular closed immersion. Then $X \to Y$ is aso a very regular closed immersion.*

Proof. To check this, we work locally, where $Y = \mathrm{Spf}\, B$ and $(b_1, \ldots, b_r)$ is sequence in B lifting a sequence generating the ideal of X in Y_1 every permutation of which is B_1-regular. Our claim is that every permutation of $(p, b_1, \ldots, b_r)$ is B-regular. When $r = 1$, this follows from the following lemma.

Lemma 7.25. *Let B be a p-torsion free separated and complete ring, let b be an element of B such that (p, b) is B-regular. Then (b, p) is also B-regular, and furthermore B/bB is also p-torsion free, separated, and complete.*

Proof. We first show that b is a nonzero divisor in B. Namely, if bx vanishes in B, then it also vanishes in B/pB, and since multiplication by b on B/pB is injective, it follows that $x = px_1$ for some $x_1 \in B$. Then $pbx_1 = bpx_1 = bx = 0$, and since B is p-torsion free, this implies that $bx_1 = 0$. Repeating this process, we see that $x_1 = px_2$, that $x_2 = px_3$, and so on. Since B is p-adically separated, we conclude that $x = 0$. Next we check that $B' := B/bB$ is p-torsion free. Namely, if $px = by$, then since b is a nonzero divisor in B/pB, we can write $y = py'$ for some y'. Then $px = bpy'$, hence $x = by'$, so x also maps to zero in B'. It remains to check that B' is also p-adically separated and complete. (This is evident in the noetherian case, but requires an argument in general.) Note that for each n, the sequence

$$0 \longrightarrow B_n \xrightarrow{b} B_n \longrightarrow B'_n \longrightarrow 0$$

is exact, because B' is p-torsion free. Then we find a commutative diagram with exact rows:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{b} & B & \longrightarrow & B' & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \varprojlim B_n & \xrightarrow{b} & \varprojlim B_n & \longrightarrow & \varprojlim B'_n & \longrightarrow & 0.
\end{array}$$

The bottom row is exact because all the transition maps are surjective. Since the first two vertical arrows are isomorphisms, so is the third. $\square$

We proceed with the proof of proposition by induction, showing that every subsequence of $(p, b_1, \ldots, b_r)$ is B-regular. If such a subsequence contain p, then the statement is evident, because $(b_1, \ldots, b_r)$ is assumed to be very B/pB-regular. If the subsequence $(b'_1, \ldots, b'_{r'})$ is a proper subsequence of $(b_1, \ldots, b_r)$, then $(p, b'_1, \ldots, b'_{r'})$ is B-regular, because $(b_1, \ldots, b_r)$ was assumed to be very B/pB-regular. Then the induction hypothesis will apply since $r' < r$. So we need only check that $(b_1, \ldots, b_r)$ is B-regular. Let $b := b_1$ and $B' := B/bB$. ince $(b_1, \ldots, b_r)$ was assumed to be very B/pB-regular, it follows that $(b_2, \ldots, b_r)$ is very B'/pB'-regular. Then the induction hypothesis implies that $(b_2, \ldots, b_r)$ is B'-regular, and hence $(b_1, \ldots, b_r)$ is B-regular. $\square$

The following result is due to C. Huneke [13, 3.1]. We present an alternative proof.

Proposition 7.26. *Suppose an ideal I of B is generated by a very regular B-sequence. Then for all n, the map $S^n I \to I^n$ is an isomorphism.*

Proof. Suppose that $(b_1, \ldots, b_r)$ is a very regular B-sequence generating I. If $r = 1$, then I is free of rank one, generated by b_1, and the result is clear. We proceed by induction on r.

Our claim is that the canonical surjection $S^n I \to I^n$ is in fact an isomorphism. This is trivial if $n = 1$ and we proceed by induction on n. In the ensuing calculations, we use $\odot$ to indicate multiplication in the symmetric algebra $S^{\cdot} I$ and $\cdot$ for multiplication in the Rees algebra B_I. Let $b := b_r$, let $B' := B/bB$, and let I' be the image of I in B'. We have exact sequences

$$\begin{array}{ccccccccc}
0 & \longrightarrow & B & \xrightarrow{b} & I & \longrightarrow & I' & \longrightarrow & 0 \\
0 & \longrightarrow & B' & \xrightarrow{b} & I/bI & \longrightarrow & I' & \longrightarrow & 0,
\end{array} \tag{7.3}$$

the second of which is split. Namely, if J is the ideal of B generated by $(b_1, \ldots, b_{r-1})$, we claim that the composed map $J/bJ \to I/bI \to I'$ is an isomorphism. It is evidently surjective. The regularity of $(b_1, \ldots, b_r)$ implies that multiplication by b is injective on B/J. Say $x \in J$ maps to zero in $I' \subseteq B'$. Then $x = ba$ for some $a \in B$, and since b is a nonzero divisor in B/J, it follows that $a \in J$ as well, so $x \in bJ$.

Since the sequence $b.$ is regular, it is also quasi-regular [2, 10.69.2], meaning that the homomorphism from the polynomial algebra $B/I[x_1, \ldots, x_r]$ to $\oplus_n I^n/I^{n+1}$ sending x_i to the image of b_i in I/I^2 is an isomorphism. This implies in particular that multiplication by $b := b_r$ induces an injection $I^{n-1}/I^n \to I^n/I^{n+1}$ for all n. Hence if $x \in B$ and $bx \in I^n$, it follows by induction on n that $x \in I^{n-1}$. That is, $bB \cap I^n = bI^{n-1}$. Thus, if $[b]$ denotes the element $b \in I$ of B_I viewed in degree one, we have an exact sequence

$$0 \longrightarrow I^{n-1} \xrightarrow{\cdot[b]} I^n \longrightarrow I'^n \longrightarrow 0. \tag{7.4}$$

We claim that there is also an exact sequence:

$$S^{n-1}I \xrightarrow{\odot[b]} S^nI \longrightarrow S^nI' \longrightarrow 0. \tag{7.5}$$

Note first that, as a consequence of the split exact sequence (7.3), we have an exact sequence:

$$0 \to S^{n-1}I \otimes B' \xrightarrow{\odot b'} S^nI \otimes B' \longrightarrow S^nI' \longrightarrow 0$$

Observe also that, if $x_1, \ldots, x_n$ is a sequence in I, then

$$b([x_1] \odot \cdots \odot [x_n]) = [bx_1] \odot \cdots [x_n] = x_1[b] \odot \cdots [x_n]$$

Thus bS^nI is contained in the image of $\odot[b] \colon S^{n-1}I \to S^nI$. Now suppose that $x \in S^nI$ maps to zero in S^nI'. Then its image in $S^nI \otimes B'$ is in the image of $\odot[b']$ and hence can be further lifted to an element y of $S^{n-1}I$. Then $\odot[b]y$ and x have the same image in $S^nI \otimes B'$, and hence differ by an element z of bS^nI. As we have just seen, such an element also lies in the image of $\odot[b]$, and the exactness of (7.5) follows.

We find a commutative diagram with exact rows:

$$\begin{tikzcd}
S^{n-1}I \arrow[r, "\odot[b_r]"] \arrow[d] & S^nI \arrow[r] \arrow[d] & S^nI' \arrow[r] \arrow[d] & 0 \\
I^{n-1} \arrow[r, "\cdot b_r"] & I^n \arrow[r] & I'^n \arrow[r] & 0
\end{tikzcd}$$

The induction hypothesis on r implies that the right vertical map is injective, and the induction hypothesis on n that the left vertical map is injective. it follows that central vertical map is also injective, completing the proof. Let us note furthermore that, ince $(b_1, \ldots, b_r)$ is very regular, the map $\cdot b_r$ left is injective, and hence so is $\odot[b_r]$. $\square$

8 Appendix: Groupoids, stratifications, and crystals

Here we review the formalism of groupoid actions, stratifications, and crystals, in the context of fibered categories. If Y is an object in a category, by a "point of Y" we mean, of course, a morphism $T \to Y$ for some (often unnamed) object T, and we write $Y(T)$ for the class of such morphisms.

8.1 Groupoid objects

Definition 8.1. *Let* $\mathbf{C}$ *be a fixed category which admits fibered products. Then a category object* $\mathcal{C}$ *in* $\mathbf{C}$ *consists of the following data:*

- *An object Y (whose points are the "objects" of C).*
- *An object A (whose points are the "arrows" of C).*
- *Morphisms: $s, t\colon A \to Y$ ("source" and "target"). We view A as a left object via t and a right object via s.*
- *A morphism $c\colon A \times_Y A \to A$ (composition). Here the fiber product is taken with the morphism s on the left factor and the morphism t on the right factor. Thus if $a_1, a_2 \in A$, their composition $a_1 a_2 := c(a_1, a_2)$ is defined if $s(a_1) = t(a_2)$.*
- *A morphism $\iota\colon Y \to A$ ("identity section") such that $t \circ \iota = s \circ \iota = \mathrm{id}_Y$,*

We require that c satisfy the associative law and that ι be the identity for c; we do not write the relevant diagrams explicitly.

If T is an object of $\mathbf{C}$, then $Y(T)$ and $A(T)$ respectively form the set of objects and arrows of a category $\mathcal{C}(T)$, and a morphism $T' \to T$ induces a functor $\mathcal{C}(T) \to \mathcal{C}(T')$. Thus the category object $\mathcal{C}$ defines a presheaf of categories on the category $\mathbf{C}$.

A category object $\mathcal{C}$ in $\mathbf{C}$ is a *groupoid object* if, for every object T of $\mathbf{C}$, every element of $A(T)$ is an isomorphism. This is the case if and only if there exists an automorphism τ of A such that $\tau^2 = \mathrm{id}_Z$, $t(\tau) = s$, $s(\tau) = t$, and the diagrams

$$\begin{tikzcd} A \arrow[r, "{(\mathrm{id},\tau)}"] \arrow[dr, "\mathrm{id}_A"'] & A \times_Y A \arrow[d, "c"] \\ & A \end{tikzcd} \qquad \begin{tikzcd} A \arrow[r, "{(\tau,\mathrm{id})}"] \arrow[dr, "\mathrm{id}_A"'] & A \times_Y A \arrow[d, "c"] \\ & A \end{tikzcd}$$

commute. If $\mathcal{G}$ is a groupoid, we may sometimes follow the usual convention for groups and abusively write $\mathcal{G}$ instead of A for the object of arrows.

Example 8.2 (Indiscrete groupoids)**.** Suppose that Y is an object of $\mathbf{C}$ such that $Y(1) := Y \times Y$ is representable. We have maps

$$\begin{array}{rcl} s\colon Y(1) \to Y & : & (y_1, y_2) \mapsto y_2 \\ t\colon Y(1) \to Y & : & (y_1, y_2) \mapsto y_1 \\ \iota\colon Y \to Y(1) & : & y \mapsto (y, y) \\ c\colon Y(1) \times_Y Y(1) \to Y(1) & : & ((y_1, y_2), (y_2, y_3)) \mapsto (y_1, y_3) \\ \tau\colon Y(1) \to Y(1) & : & (y_1, y_2) \mapsto (y_2, y_1). \end{array}$$

This defines the *indiscrete groupoid* $\mathcal{G}_Y$ on the object Y: given any pair of points (y_1, y_2) there is a unique morphism from y_2 to y_1. For a variation of this theme, suppose that $f\colon Y \to S$ is a morphism in $\mathbf{C}$, and use the same formulas as above to define a composition law on $Y \times_S Y$. This groupoid, which denote by $\mathcal{G}_{Y/S}$, corresponds to the equivalence relation defined by the morphism f: given a pair of points (y_1, y_2) such that $f(y_1) = f(y_2)$, there is a unique morphism from y_2 to y_1.

8.2 Groupoid actions and stratifications

Before discussing groupoid actions on fibered categories, it may be wise to discuss such actions in the context of sets. In fact we may as well consider the action of a category $\mathbf{C}$. If $\mathbf{C}$ is a monoid, there is only one object, and an action of $\mathbf{C}$ on a set V is given by associating an endomorphism of V to each morphism of $\mathbf{C}$. If $\mathbf{C}$ has more than one object, then we replace V by a family of sets indexed by the class of objects of $\mathbf{C}$, and if a is a morphism from s to t, then a left (resp. right) action of $\mathbf{C}$ on V will map $V_s \to V_t$ (resp., V_t to V_s).

Now let $\mathbf{C}$ be a category and let $\mathbf{F}\colon \mathbf{V} \to \mathbf{C}$ be a functor. Recall that if Y is an object of $\mathbf{C}$, then $\mathbf{V}_Y$ is the category whose objects are those objects V of $\mathbf{V}$ such that $\mathbf{F}(V) = Y$ and whose morphisms are the morphisms v of $\mathbf{V}$ such that such that $\mathbf{F}(v) = \mathrm{id}_Y$. If $y\colon Y' \to Y$ is a morphism in $\mathbf{C}$, then a *y-morphism of* $\mathbf{V}$ is a morphism v in $\mathbf{V}$ such that $\mathbf{F}(v) = y$. We shall call the class of such morphisms the *fiber of* $\mathbf{V}$ *over* y, and denote it by $\mathbf{V}(y)$. If V is an object of $\mathbf{V}$ and $\mathbf{F}(V)$ is the target of y, then the *fiber of* V *over* y, denoted $V(y)$, is the subclass of $\mathbf{V}(y)$ whose target is V A y-morphism $\pi\colon V_y \to V$ is said to be (strongly) *Cartesian* if, for every pair (v, z) with $v\colon V'' \to V$ and $z\colon \mathbf{F}(V'') \to \mathbf{F}(V_y)$ such that $\mathbf{F}(v) = y \circ z$, there is a unique morphism $w\colon V'' \to V_y$ such that $\mathbf{F}(w) = z$ and $\pi \circ w = v$. Equivalently, for every point z of $\mathbf{F}(V_y)$, the map

$$w \mapsto \pi \circ w : V_y(z) \to V(y \circ z)$$

is a bijection. A Cartesian morphism is unique up to unique isomorphism. The functor $\mathbf{F}$ forms a *category fibered over* $\mathbf{C}$ if for every object V over Y and every morphism $y\colon Y' \to Y$, there is a Cartesian morphism $\pi_y\colon V_y \to V$ with $\mathbf{F}(V_y) = Y'$. If $v\colon V'' \to V$ and $z\colon \mathbf{F}(V'') \to Y'$ with $\mathbf{F}(v) = y \circ z$, we denote the corresponding morphism $V'' \to V_y$ by (v, z). The composition of Cartesian morphisms is Cartesian. Thus, if $Y = \mathbf{F}(V)$ and $z\colon Y'' \to Y'$ and $y\colon Y' \to Y$

are morphisms, in the diagram:

$$\begin{tikzcd}
(V_y)_z \arrow[r, "(\pi_y \circ \pi_z, z)"] \arrow[d, "\pi_z"'] & V_{y \circ z} \arrow[d, "\pi_{y \circ z}"] \\
V_y \arrow[r, "\pi_y"'] & V
\end{tikzcd}$$

the morphism $(\pi_y \circ \pi_z, z)$ is an isomorphism.

Example 8.3. If $\mathbf{C}$ is any category, the identity functor $\mathbf{C} \to \mathbf{C}$ makes $\mathbf{C}$ fibered over itself: every morphism is Cartesian in this case. For a less trivial but equally familiar example, suppose that $\mathbf{C}$ is a category with fibered products. Then let $\mathbf{AC}$ denote the category of morphisms of $\mathbf{C}$ and let $\mathbf{F}\colon \mathbf{AC} \to \mathbf{C}$ be the functor which takes a morphism to its target. If $y\colon Y' \to Y$ is a morphism in $\mathbf{C}$ and V an object of $\mathbf{AC}_Y$, then V is a morphism $X \to Y$, the projection map $V'\colon X \times_Y Y' \to Y'$ is an object of $\mathbf{AC}_{Y'}$, and the projection morphism $X \times_Y Y' \to X$ defines a Cartesian y-morphism $V' \to V$.

Definition 8.4. *Let $\mathcal{C} := (Y, A, s, t, c, \iota)$ be a category object in a category $\mathbf{C}$, and let $p_1, p_2\colon A \times_Y A \to A$ be the two projection mappings. Let $\mathbf{F}\colon \mathbf{V} \to \mathbf{C}$ be a category fibered over $\mathbf{C}$, and let V be an object of $\mathbf{V}_Y$.*

1. *A right action of $\mathcal{C}$ on V is a pair (r, π_t), where $\pi_t\colon V_t \to V$ is a Cartesian t-morphism and $r\colon V_t \to V$ is an s-morphism such that the following diagrams commute:*

$$\begin{tikzcd}
(V_t)_{p_1} \arrow[r, "(\pi_t \circ \pi_{p_1}, p_1)"] \arrow[d, "(r, p_2)"'] & V_{t \circ p_1} \arrow[r, "(\pi_{t \circ p_1}, c)"] & V_t \arrow[dl, "r"] \\
V_t \arrow[r, "r"] & V &
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[r, "(\mathrm{id}, \iota)"] \arrow[d, "\mathrm{id}"'] & V_t \arrow[dl, "r"] \\
V &
\end{tikzcd}$$

   A morphism of right actions of $\mathcal{C}$ is a Y-morphism $f\colon V \to V'$ such that $f \circ r = f' \circ f_t$.

2. *A $\mathcal{C}$-stratification of V is a triple (ϵ, π_t, π_s), where $\pi_t\colon V_t \to V$ and $\pi_s\colon V_s \to V$ are Cartesian morphisms and $\epsilon\colon V_t \to V_s$ is an A-isomorphism such that the following diagrams commute (the "cocycle conditions"):*

$$\begin{tikzcd}
(V_t)_{p_1} \arrow[r, "(\epsilon, p_1)"] \arrow[d, "(\epsilon, c)"'] & (V_s)_{p_1} \arrow[d, "\cong"'] \\
(V_s)_{p_2} & (V_t)_{p_2} \arrow[l, "(\epsilon, p_2)"']
\end{tikzcd}
\qquad
\begin{tikzcd}
V \arrow[d, "(\mathrm{id}, \iota)"] \arrow[dr, "(\mathrm{id}, \iota)"] & \\
V_t \arrow[r, "\epsilon"] & V_s
\end{tikzcd}$$

A morphism of $\mathcal{C}$-stratifications is a Y-morphism $f\colon V \to V'$ such that $f_s \circ \epsilon = \epsilon' \circ f_t$.

To unpack this definition a bit, let us note that if r is a right $\mathcal{C}$-action on V, then for every point a of A, we have a commutative diagram

$$\begin{tikzcd}
V_t(a) \arrow[d, "\cong"'] \arrow[dr, "r(a)"] & \\
V(t(a)) \arrow[r, "v \mapsto va"'] & V(s(a))
\end{tikzcd}$$

Furthermore, the commutative of the diagrams in the definition of r say that if (a_1, a_2) is a point of $A \times_Y A$ (so that $s(a_1) = t(a_2)$, and if $v \in V(t(a_1))$, then $(va_1)a_2 = v(a_1 a_2)$, and if $v \in V(y)$ then $vi(y) = v$.

Remark 8.5. If $\pi_s\colon V_s \to V$ and $\pi_t\colon V_t \to V$ are Cartesian morphisms over s and t respectively and $\epsilon\colon V_t \to V_s$ is an A-morphism satisfying the cocycle conditions, then $\pi_s \circ \epsilon\colon V_t \to V$ is a right $\mathcal{C}$-action. Conversely, if (r, π_s) is a right $\mathcal{C}$-action on V, there is a unique A-morphism $\epsilon_r\colon V_t \to V_s$ such that $\pi_s \circ \epsilon = r$. This morphism satisfies the cocycle conditions, and it is necessarily an isomorphism if $\mathcal{C}$ is a groupoid. Indeed, in that case we have a τ-isomorphism

$$\tau_V\colon V_s \to V_t := (\pi_s, \tau) \tag{8.1}$$

and a commutative diagram

$$\begin{tikzcd}
V_t \arrow[r, "\epsilon_r"] & V_s \arrow[dr, "\tau_V"] & \\
V_t \arrow[u, "\mathrm{id}"] & V_s \arrow[l, "\tau_V"'] & V_t \arrow[l, "\epsilon_r"']
\end{tikzcd}$$

Thus $\tau_V \circ \epsilon_r \circ \tau_V$ is a left inverse to ϵ_r, and, it follows that it is also a right inverse, since τ_V is an isomorphism. Thus we find that, if $\mathcal{C}$ is a groupoid, there is an equivalence between right actions of $\mathcal{C}$ and $\mathcal{C}$-stratifications.

Let us also remark that if (r, π_t) is a right action of $\mathcal{C}$ then the corresponding morphism $\epsilon\colon V_t \to V_s$ is an isomorphism if and only if $r\colon V_t \to V$ is a Cartesian s-morphism; in particular, this is always the case if $\mathcal{C}$ is a groupoid. Indeed, if ϵ is an isomorphism, it is Cartesian, and since π_s is Cartesian it follows that $r = \pi_s \circ \epsilon$ is also Cartesian. Conversely, if r is Cartesian, there is a unique A-morphism $\epsilon'\colon V_s \to V_t$ such that $r \circ \epsilon' = \pi_s$, and ϵ' is an isomorphism because π_s is also Cartesian. Then $\pi_s \circ \epsilon \circ \epsilon' = r \circ \epsilon' = \pi_s$, hence $\epsilon \circ \epsilon' = \mathrm{id}$, and since ϵ' is an isomorphism, so is ϵ.

Remark 8.6. A *left action of* $\mathcal{C}$ on an object V over Y is a pair (ℓ, π_s), where $\pi_s\colon V_s \to V$ is a Cartesian morphism and $\ell\colon V_s \to V$ is a t-morphism, satisfying the analogs of the conditions in Definition 8.4. If $\mathcal{C}$ is a groupoid and (r, π_t) is a right action of $\mathcal{C}$ on V, then $r \circ \tau_V : V_s \to V_t \to V$ is a left action.

Remark 8.7. If V and W are Y objects each of which is endowed with a right action of $\mathcal{G}$, then the fiber product $V \times_Y W$ of V and Y in $\mathbf{V}_Y$, if it exists, has a natural right action as well, defined by $r((v,w),a) := (r(v,a), r(w,a))$. (Here we are using the pointilist notation: v and w are morphism $X \to V, X \to W$ and a is a morphism $\mathbf{F}(X) \to A$, with $t \circ a = \mathbf{F}(v) = \mathbf{F}(w)$.)

Example 8.8. Suppose that $\mathbf{F}$ is the functor "target" from the arrow category $\mathbf{AC}$ of $\mathbf{C}$ to $\mathbf{C}$ described in Example 8.3, and let V be the identity morphism of Y. The projections

$$\begin{array}{ccc} Y \times_Y A \cong A & \xrightarrow{s} & Y \\ A \times_Y A \cong A & \xrightarrow{t} & Y \end{array} \tag{8.2}$$

define right and left actions of $\mathcal{C}$ on Y, respectively. These are the "tautological" actions of $\mathcal{C}$ on the points of Y.

Next, take $V \to Y$ to be $s\colon A \to Y$ (resp. $t\colon A \to Y$). We write these as A^s and A^t, respectively. The composition law defines a right (resp. left) action of $\mathcal{C}$ on A^s (resp A^t):

$$\begin{array}{ccc} (A^s)_t = A^s \times_Y A & \xrightarrow{c} & A^s \\ (A^t)_s = A \times_Y A^t & \xrightarrow{c} & A^t \end{array} \tag{8.3}$$

These are the right (resp. left) "regular" representations of $\mathcal{C}$ on itself. If $\mathcal{C} = \mathcal{G}$ is a groupoid , then, as we saw in Remark 8.6, we also have a right action of $\mathcal{G}$ on A^t:

$$c^t := A^t \times_Y A \xrightarrow{\mathrm{id} \times \tau} A^t \times_Y A \xrightarrow{c} A^t \tag{8.4}$$

We note that these actions are compatible with the tautological actions of $\mathcal{G}$ on itself. In particular, the diagram;

$$\begin{tikzcd} A^t \times_Y A \arrow[r, "c^t"] \arrow[d, "t \times \mathrm{id}"'] & A \arrow[d, "t"] \\ Y \times_Y A \arrow[r, "s"] & Y \end{tikzcd} \tag{8.5}$$

commutes.

A group can aso act by conjugation on itself. The analog for groupoids is a bit more complex. Assume that the category $\mathbf{C}$ has products, and let $Y(1) := Y \times Y$ and $A(1) := A \times A$. Then $\mathcal{G}(1) := (A(1), t \times t, s \times s, c \times c, \iota \times \iota)$ defines a groupoid over $Y(1)$. Furthermore, the pair (t, s) define a morphism $A \to Y(1)$, allowing us to view A as an object over $Y(1)$. Then we have a right action of $\mathcal{G}(1)$ on A:

$$A \times_{Y(1)} A(1) \to A : (a, (g_1, g_2)) \mapsto g_1^{-1} a g_2. \tag{8.6}$$

This makes sense, because for $(a,(g_1,g_2)) \in A \times_{Y(1)} A(1)$, we have $t(a) = t(g_1), s(a) = t(g_2)$, so $s(a^{-1}) = t(g_1)$, and hence the product $g_1^{-1} a g_2$ is defined.

Example 8.9. Let $\mathcal{G}_Y$ be the indiscrete groupoid with groupoid law $Y(1)$ as in Example 8.2. Then the tautological action 8.2 is given by the map

$$r \colon Y \times_Y Y(1) \longrightarrow Y : (y_1,(y_1,y_2)) \mapsto y_2. \tag{8.7}$$

The corresponding stratification is the map

$$\epsilon \colon Y \times_Y Y(1) \longrightarrow Y(1) \times_Y Y : (y_1,(y_1,y_2)) \mapsto ((y_1,y_2),y_2).$$

Let $Y(1)^t$ denote $Y(1)$ viewed as a Y-object via t, the left projection, and let $Y(1)^s$ denote $Y(1)$ viewed as a Y-object via s, the right projection. The morphism (8.1) for $V = Y(1)^t$ is given by

$$\begin{array}{rcl} \tau_{Y(1)^t} \colon (Y(1)^t)_s & \to & (Y(1)^t)_t \\ Y(1) \times_Y Y(1)^t & \to & Y(1)^t \times_Y Y(1) \\ ((y_1,y_2),(y_2,y_3)) & \mapsto & ((y_2,y_3),(y_2,y_3)) \end{array}$$

The regular representations (8.3) and (8.4) are given by:

$$\begin{array}{rcl} Y(1)^s \times_Y Y(1) \longrightarrow Y(1)^s & : & ((y_1,y_2),(y_2,y_3)) \mapsto (y_1,y_3) \\ Y(1) \times_Y Y(1)^t \longrightarrow Y(1)^t & : & ((y_1,y_2),(y_2,y_3)) \mapsto (y_1,y_3) \\ Y(1)^t \times_Y Y(1) \longrightarrow Y(1)^t & : & ((y_1,y_2),(y_1,y_3)) \mapsto (y_3,y_2). \end{array} \tag{8.8}$$

We used the following trivial-looking comparison in the proof of Shiho's theorem explained at the end of §5.2.

Proposition 8.10. *Let Y be an object of a category $\mathbf{C}$ with products and fibered products, let $\mathcal{G}_Y$ (resp. $\mathcal{G}_{Y(1)}$) be the indiscrete groupoid on Y (resp., on $Y(1)$), and let $\mathcal{G}_Y(1)$ be the groupoid over $Y(1)$ formed from $\mathcal{G}_Y$ as in the discussion of conjugation. Then there is an isomorphism $\alpha \colon \mathcal{G}_{Y(1)} \to \mathcal{G}_Y(1)$ of groupoids over $Y(1)$. Furthermore, α takes the tautological action (8.7) of $\mathcal{G}_{Y(1)}$ on $Y(1)$ to the conjugation action (8.6) of $\mathcal{G}_Y(1)$ on $Y(1)$.*

Proof. We write this out using points. Define α in the diagram below by the formula:

$$\alpha(y_1,y_2,y_3,y_4) := (y_1 y_3, y_2, y_4).$$

$$\begin{tikzcd} Y(1) \times Y(1) \arrow[r, "\alpha"] \arrow[d, shift right, "p_1"'] \arrow[d, shift left, "p_2"] & Y(1) \times Y(1) \arrow[d, shift right, "t \times t"'] \arrow[d, shift left, "s \times s"] \\ Y(1) \arrow[r, "\mathrm{id}"] & Y(1), \end{tikzcd}$$

For the sake of clarity, we check that the diagram commutes. Thus, if $y := (y_1,y_2,y_3,y_4) \in Y(1) \times Y(1)$, we have

$$(t \times t) \circ \alpha(y) = (t \times t)(y_1,y_3,y_2,y_4) = (y_1,y_2) = p_1(y)$$

$$(s \times s) \circ \alpha(y) = (s \times s)(y_1, y_3, y_2, y_4) = (y_3, y_4) = p_2(y).$$

Suppose y and z are elements of $Y(1) \times Y(1)$, viewed as arrows of $\mathcal{G}_{Y(1)}$. Then the composition $y \circ z$ is defined if $t(z) = s(y)$, *i.e.*, if $(y_3, y_4) = (z_1, z_2)$, in which case the composition is (y_1, y_2, z_3, z_4). On the other hand, if y' and z' are element of $Y(1) \times Y(1)$ viewed as arrows of $\mathcal{G}_Y(1)$, then their composition is defined if $(y'_2, y'_4) = (z'_1, z'_3)$, in which case the composition is (y'_1, y'_3, z'_2, z'_4). It follows that α is compatibile with composition.

Finally, let us check the compatibility with the actions. The tautological action is given by

$$(y_1, y_2) \cdot (y_1, y_2, y_3, y_4) = (y_3, y_4).$$

The conjugation action is given by $h \cdot (g_1, g_2) = g_1^{-1} h g_2$. Thus if $h = (y_1, y_2)$ and $(g_1, g_2) = (y'_1, y'_2, y'_3, y'_4)$, we have

$$(y_1, y_2) \cdot (y'_1, y'_2, y'_3, y'_4) = (y'_2, y'_1)(y_1, y_2)(y'_3, y'_4) = (y'_2, y'_4).$$

Thus, if $y' = \alpha(y)$, these agree. □

Remark 8.11. Suppose again that $\mathcal{C}$ is a category datum in $\mathbf{C}$ and that V is a Y-object of $\mathbf{V}$ endowed with a right action (π_t, r) of $\mathcal{C}$. Then we can form a category datum $\mathcal{V}$ in $\mathbf{V}$, which we call the *transporter category of* V, as follows. We let V be the object of objects and V_t the object of arrows, with $\pi_t \colon V_t \to V$ the "target" morphism and $r \colon V_t \to V$ the "source" morphism. The identity morphism $\iota_{\mathcal{C}} \colon Y \to A$ induces a morphism $\iota_{\mathcal{V}} \colon V \to V_t$ such that $\pi_t \iota_{\mathcal{V}} = r \iota_{\mathcal{V}} = \mathrm{id}_V$, which will serve as the identity section. In order to define the composition law in $\mathcal{V}$, we need a convenient description of the fiber product $V_t \times_V V_t$ (computed using r on the left and π_t on the right). Let $\pi_{tc} \colon V_{tc} \to V$ be a Cartesian morphism covering the map $t \circ p_1 = t \circ c \colon A \times_Y A \to Y$. Since π_t is Cartesian, there is a unique p_1 morphism $f_1 \colon V_{tc} \to V_t$ such that $\pi_t f_1 = \pi_{tc}$. Then $\mathbf{F}(r \circ f_1) = s \circ p_1 = t \circ p_2$, so there is a unique p_2-morphism $f_2 \colon V_{tc} \to V_t$ such that $\pi_t \circ f_2 = r \circ f_1$. Thus the pair (f_1, f_2) defines a V_{tc}-valued point of the $V_t \times_V V_t$.

Claim 8.12. *The map $(f_1, f_2) \colon V_{tc} \to V_t \times_V V_t$ defined above is an isomorphism*

To check this, let $g_1, g_2 \colon W \to V_t$, with $\pi_t \circ g_2 = r \circ g_1$. Then $t \circ \mathbf{F}(g_2) = s \circ \mathbf{F}(g_1)$, so we find a morphism $h \colon \mathbf{F}(W) \to A \times_Y A$ with $p_i \circ h = \mathbf{F}(g_i)$. In particular, $t \circ p_1 \circ h = t \circ \mathbf{F}(g_1) = \mathbf{F}(\pi_t \circ g_1)$, so there is a unique h-morphism $g \colon W \to V_{tc}$ such that $\pi_{tc} \circ g = \pi_t \circ g_1$. We claim that each $f_i \circ g = g_i$. By construction, $\mathbf{F}(f_i \circ g) = p_i \circ h = \mathbf{F}(g_i)$, so it is enough to check that $\pi_t \circ f_i \circ g = \pi_t \circ g_i$. For $i = 1$, we have $\pi_t \circ f_1 \circ g = \pi_{tc} \circ g$ by definition of f_1 and $\pi_t \circ g_1 = \pi_{tc} \circ g$ by definition of g, and it follows that $f_1 \circ g = g_1$. For $i = 2$, we have $\pi_t \circ g_2 = r \circ g_1 = r \circ f_1 \circ g = \pi_t \circ f_2 \circ g$, as required.

The map $c \colon A \times_Y A \to A$ then defines a map $V_t \times_V V_t \cong V_{tc} \to V_t$, which will serve as the composition law for $\mathcal{V}$.

8.3 Stratifications and crystals

Now we can describe the relations among the notions of actions, stratifications, and crystals.

Definition 8.13. *Let $\mathbf{F}\colon \mathbf{V} \to \mathbf{C}$ be a fibered category. A crystal in $\mathbf{F}$ is a Cartesian section of $\mathbf{F}$. Explicitly, this means a rule which assigns to each $T \in \mathbf{C}$ an object V_T of $\mathbf{V}$ and to each morphism $g\colon T' \to T$ of $\mathbf{C}$ a Cartesian morphism $\epsilon_g\colon V_{T'} \to V_T$ satisfying the following cocycle conditions:*

1. $\epsilon_{\mathrm{id}} = \mathrm{id}$.
2. *if $g\colon T' \to T$ and $h\colon T'' \to T'$ are morphisms in $\mathbf{C}$, then $\epsilon_{h\circ g} = \epsilon_h \circ \epsilon_g$.*

Recall that an object Y of $\mathbf{C}$ is *semifinal* if every object of $\mathbf{C}$ admits at least one morphism to Y. Suppose Y is such an object of $\mathbf{C}$, that $Y(1) := Y \times Y$ is representable, and that V is an object of $\mathbf{V}_Y$ equipped with a right action of the groupoid $Y(1)$ described in Example 8.9, or, equivalently, a $Y(1)$-stratification $\epsilon\colon V_t \to V_s$. Then we can define a crystal in $\mathbf{C}$ as follows. For each object T of $\mathbf{C}$, choose some morphism $y_T\colon T \to Y$ and let $\pi_T\colon V_T \to V$ be a Cartesian y_T-morphism in $\mathbf{V}$. If $f\colon T' \to T$ is a morphism in $\mathbf{C}$, then $y_{T'}$ and $f \circ y_T$ are two morphisms $T' \to Y$, and there is a morphism $g\colon T' \to Y(1)$ such that $t \circ g = f \circ y_T$ and $s \circ g = y_{T'}$. Then the $Y(1)$-isomorphism ϵ induces a T'-isomorphism:

$$\epsilon_g\colon V_{f\circ y_T} \cong (V_t)_g \to (V_s)_g \cong V_{y_{T'}}, \quad \textit{i.e.}$$

$$\epsilon(f)\colon (V_T)_f \to V_{T'}.$$

The cocycle condition guarantees.... This construction can be run backwards. We summarize these points of view as follows.

Proposition 8.14. *Let $\mathbf{F}\colon \mathbf{V} \to \mathbf{C}$ be a fibered category, where $\mathbf{C}$ is a category with fibered products. Suppose that Y is a semi-final object of $\mathbf{C}$, and let $\mathcal{G}_Y$ be the indiscrete groupoid on Y described in Example 8.2. Then the following notions are equivalent:*

1. *An object of $\mathbf{V}_Y$ equipped with a right action of $\mathcal{G}_Y$.*
2. *An object of $\mathbf{V}_Y$ equipped with a $\mathcal{G}_Y$-stratification.*
3. *A crystal in $\mathbf{F}$ (8.13).*

8.4 Differential $\mathcal{G}$-operators

We can also formulate a notion of differential operators and their linearization in the general context of groupoid actions. As motivation, we review the approach to the classical definitions explained in [6]. Suppose that A is an R-algebra and that E and F are two A-modules. An R-linear homomorphism $\theta\colon E \to F$ gives rise to (and is equivalent to) an A-linear homomorphism $A \otimes_R E \to F$; using the A-module structure on E, we can write this as an A-linear homomorphism

$$(A \otimes_R A) \otimes_A E \to E.$$

We say that θ is a *differential operator of order at most m* if this homomorphism factors through a map $(A\otimes_R A)/J^{m+1}\otimes_A E$, where J is the ideal of the diagonal. The composition law of the indiscrete groupoid corresponding to $\mathrm{Spec}(A/R)$ allows one to show that the composition of a differential operator of order m and a differential operator of order n is a differential operator of order $m+n$. In the crystalline context, a *hyper PD-differential operator* from E to F is by definition an A-linear homomorphism $D_J(A\otimes_R A)\otimes_A E\to F$, where $D_J(A\otimes_R A)$ is the divided power envelope of J. (In this case such an operator is not necessarily determined by its R-linear restriction to E.) We hope this discussion helps motivate the definition which follows.

We have discussed the notion of an action of a groupoid in category $\mathbf{C}$ on an object in a category $\mathbf{V}$ which is fibered over $\mathbf{C}$. It seems that to discuss differential operators in this context, one would need $\mathbf{V}$ to be cofibered as well as fibered. Rather that working in this general context, we will restrict to the case in which $\mathbf{V}$ is the category of arrows in $\mathbf{C}$, which will allow a simplification of the notations. Before stating the definitions, let us recast the motivating discussion above, but in a geometric and more general context.

Let $\mathbf{C}$ be a category with fibered products, let Y/S be a morphism in $\mathbf{C}$, and $\pi_V\colon V\to Y$ and $\pi_W\colon W\to Y$ be objects of $\mathbf{C}_Y$. Composing with the morphism $Y\to S$, we may view W and Y as objects of $\mathbf{C}_S$, and we suppose that we are given an S-morphism $f\colon V\to W$. To measure the failure of f to be a Y-morphism, we consider the morphism

$$\tilde f =: V \xrightarrow{(\pi_V,f)} Y\times_S W\cong Y(1)\times_Y W,$$

which on points takes v to $(\pi_V(v),\pi_W(f(v)),f(v))$. The morphism f is a Y-morphism if and only if $\tilde f$ factors through the map $W\to Y(1)\times_Y W$ induced by the diagonal. If $A\to Y(1)$ is a groupoid law over Y, we can hope that the difference between π_V and $\pi_W\circ f$ is mediated by A. We can also crudely "force" f to be a Y-morphism by forming

$$\mathcal{L}(f) := Y(1)\times_Y V\cong Y\times_S V\xrightarrow{\mathrm{id}_Y\times f)} Y\times_S W\cong Y(1)\times_Y W,$$

and hope to replace $Y(1)$ by A. This suggests the following definition.

Definition 8.15. *Let $\mathbf{C}$ be a category with products and fibered products, let $\mathcal{G}:=(Y,A,t,s,\iota,c)$ be a groupoid object in $\mathbf{C}$. If $\pi_V\colon V\to Y$ be an object of $\mathbf{C}_Y$, let*

$$\mathcal{L}_{\mathcal{G}}(V):=A\times_Y V \text{ and } \mathcal{R}_{\mathcal{G}}(V):=V\times_Y A,$$

where $A\times_Y V$ is viewed as an object over Y via the morphism t and $V\times_Y A$ is viewed as an object over Y via the morphism s. If $\pi_W\colon W\to Y$ is another object of $\mathbf{C}_Y$, then a differential $\mathcal{G}$-operator from V to W is a Y-morphism

$$D\colon V\to\mathcal{L}_{\mathcal{G}}(W),$$

If D is such an operator, then $\mathcal{L}_{\mathcal{G}}(D)\colon\mathcal{L}_{\mathcal{G}}(V)\to\mathcal{L}_{\mathcal{G}}(W)$ is defined by the *following diagram:*

$$\begin{array}{ccccc}
\mathcal{L}_{\mathcal{G}}(V) & = & A \times_Y V & \xrightarrow{\mathrm{id}_A \times D} & A \times_Y (A \times_Y W) \\
\downarrow{\scriptstyle \mathcal{L}_{\mathcal{G}}(D)} & & & & \downarrow{\scriptstyle \cong} \\
\mathcal{L}_{\mathcal{G}}(W) & = & A \times_Y W & \xleftarrow{c \times \mathrm{id}_W} & (A \times_Y A) \times_Y W
\end{array}$$

[6] (Note: If V and W are endowed with additional structure, *e.g.*, as group or module objects, then D should preserve such structure.)

Let us note that the following diagram commutes:

$$\begin{array}{ccc}
\mathcal{L}_{\mathcal{G}}(V) & \xrightarrow{\mathcal{L}_{\mathcal{G}}(D)} & \mathcal{L}_{\mathcal{G}}(W) \\
\downarrow & & \downarrow \\
Y(1) \times_Y V & \xrightarrow{\mathcal{L}(D)} & Y(1) \times_Y W
\end{array} \tag{8.9}$$

Since $\mathcal{L}_{\mathcal{G}}(V)$ is viewed as an object over Y using the morphism t, but the fiber product is formed using s, the left action (8.3) of $\mathcal{G}$ on $\mathcal{G}_t$ induces a left action $\ell_{\mathcal{G}}$ on $\mathcal{L}_{\mathcal{G}}(V)$. We can and shall view this as a right action $r_{\mathcal{G}}$ using the twist τ. Furthermore, if D is a differential $\mathcal{G}$-operator from V to W, the map $\mathcal{L}_{\mathcal{G}}(D)$ is compatible with the actions $\ell_{\mathcal{G}}$ and $r_{\mathcal{G}}$ of $\mathcal{G}$ on $\mathcal{L}_{\mathcal{G}}(V)$ and $\mathcal{L}_{\mathcal{G}}(W)$.

Suppose now that V itself is endowed with a right action r_V of $\mathcal{G}$, with corresponding stratification ϵ_V:

$$r_V \colon \mathcal{R}_{\mathcal{G}}(V) \to V; \qquad \epsilon_V \colon \mathcal{R}_G(V) \to \mathcal{L}_G(V).$$

Note that the projection mapping $\pi\colon \mathcal{L}_{\mathcal{G}}(V) \to V$ is in general not compatible with the right actions of $\mathcal{G}$. However, diagram (8.5) shows that it *is* compatible if $V = Y$ with its canonical action. Moreover, $\mathcal{R}_{\mathcal{G}}(V)$ inherits a right action $r_{\epsilon,\mathcal{G}}$ of $\mathcal{G}$, deduced from r_V and the right action (8.4) of $\mathcal{G}$ on A^t, and the projection $\mathcal{R}_{\mathcal{G}}(V) \to V$ *is* compatible with this right action. Moreover, the morphism ϵ_V takes $r_{\epsilon,\mathcal{G}}$ to $r_{\mathcal{G}}$. We will explain and prove a generalization of this in Proposition 8.16.

Thanks to the action r_V, a differential $\mathcal{G}$-operator D from W to W' induces a differential $\mathcal{G}$-operator $\epsilon(D)$ from $V \times_Y W$ to $V \times_Y W'$, given by

$$\begin{array}{ccc}
V \times_Y W & \xrightarrow{\mathrm{id}_V \times D} & V \times_Y (A \times_Y W') \cong (V \times_Y A) \times_Y W' \\
\downarrow{\scriptstyle \epsilon(D)} & & \downarrow{\scriptstyle \epsilon_V \times \mathrm{id}} \\
\mathcal{L}_{\mathcal{G}}(V \times_Y W') & = & A \times_Y (V \times_Y W') \cong (A \times_Y V) \times_Y W'
\end{array}$$

[6]danger here because of category.

Also, if W is any Y-object, we find an isomorphism:

$$\beta_\epsilon \colon V \times_Y \mathcal{L}_{\mathcal{G}}(W) \to \mathcal{L}_{\mathcal{G}}(V \times_Y W) \tag{8.10}$$

defined by the diagram:

$$\begin{tikzcd}
V \times_Y \mathcal{L}_{\mathcal{G}}(W) = V \times_Y (A \times_Y W) \cong (V \times_Y A) \times_Y W \arrow[d, "\beta_\epsilon"'] \arrow[d, "\epsilon \times \mathrm{id}_W"] \\
\mathcal{L}_{\mathcal{G}}(V \times_Y W) = A \times_Y (V \times_Y W) \cong (A \times_Y V) \times_Y W
\end{tikzcd}$$

The following result is an analog (generalization) of [6, 6.15].

Proposition 8.16. *Let $t, s \colon \mathcal{G} \to Y$ be a groupoid over Y, as described above, and let $V \to Y$ be a Y-object endowed with a right action of $\mathcal{G}$, and let $W \to Y$ be any Y-object.*

- *The map $\beta_\epsilon \colon V \times_Y \mathcal{L}_{\mathcal{G}}(W) \to \mathcal{L}_{\mathcal{G}}(V \times_Y W)$ is an isomorphism, compatible with the right actions of $\mathcal{G}$.*
- *If D is a differential $\mathcal{G}$-operator from W to W', the following diagram commutes:*

$$\begin{tikzcd}
V \times_Y \mathcal{L}_{\mathcal{G}}(W) \arrow[r, "\beta_\epsilon"] \arrow[d, "\mathrm{id} \times \mathcal{L}_{\mathcal{G}}(D)"] & \mathcal{L}_{\mathcal{G}}(V \times_Y W) \arrow[d, "\mathcal{L}_{\mathcal{G}}(\epsilon(D))"] \\
V \times \mathcal{L}_{\mathcal{G}}(W') \arrow[r, "\beta_\epsilon"] & \mathcal{L}_{\mathcal{G}}(V \times_Y W')
\end{tikzcd}$$

Proof. The fact that β_ϵ is an isomorphism is evident from its definition. To check its compatibility with the actions of $\mathcal{G}$, we calculate with points. Suppose (v, a, w) is a point of $V \times_Y \mathcal{L}_G(W)$, so $\pi_V(v) = t(a)$ and $s(a) = \pi_W(w)$. If $b \in A$ with $t(b) = \pi_W(v)$, then

$$\begin{aligned}
\beta_\epsilon((v, a, w)b) &= \beta_\epsilon(vb, b^{-1}a, w) \\
&= (\epsilon(vb, b^{-1}a), w) \\
&= (b^{-1}a, va, w)
\end{aligned}$$

$$\begin{aligned}
(\beta_\epsilon(v, a, w))b &:= (\epsilon(a, v), w), b) \\
&= (a, va, w)b \\
&= (b^{-1}a, va, w)
\end{aligned}$$

This proves (1). Statement (2) is straightforward and we omit its proof. $\square$

We shall also need the following compatibility. The proof is immediate and omitted.

Proposition 8.17. *With the notations above, let* $\iota_V : V \to \mathcal{L}_{\mathcal{G}}(V) := (\iota, \mathrm{id})$ *and let* $\pi_V : \mathcal{L}_{\mathcal{G}}(V) \to V$ *be the projection.*

1. $\pi_V \circ \iota_V = \mathrm{id}_V$.

2. *If* D *is a differential operator from* W *to* W'*, then*

$$\pi_{W'} \circ \mathcal{L}_{\mathcal{G}}(D) \circ \iota_W = \pi_{W'} \circ D. \quad \square$$

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