Logarithmic Geometry

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Outline

Introduction

The Language of Log Geometry

The Category of Log Schemes

The Geometry of Log Schemes

Applications

Conclusion
Emphasis

- What it’s for
- How it works
- What it looks like
History

Founders:
Deligne, Faltings, Fontaine–Illusie, Kazuya Kato, Chikara Nakayama, many others

Log geometry in this form was invented discovered assembled in the 80’s by Fontaine and Illusie with hope of studying $p$-adic Galois representations associated to varieties with bad reduction. Carried out by Kato, Tsuji, Faltings, and others. (The $C_{st}$ conjecture.)

I’ll emphasize geometric analogs—currently very active—today. Related to toric and tropical geometry
Motivating problem 1: Compactification

Consider

\[ S^* \xrightarrow{j} S \xleftarrow{i} Z \]

\( j \) an open immersion, \( i \) its complementary closed immersion. For example: \( S^* \) a moduli space of “smooth” objects, inside some space \( S \) of “stable” objects, \( Z \) the “degenerate” locus.

Log structure is “magic powder” which when added to \( S \) “remembers \( S^* \).”
Motivating problem 2: Degeneration

Study families, i.e., morphisms

\[
\begin{array}{cccc}
X^* & \rightarrow & X & \leftarrow & Y \\
\downarrow f^* & & \downarrow f & & \downarrow g \\
S^* & \rightarrow & S & \leftarrow & Z \\
j & & i & & \\
\end{array}
\]

Here \( f^* \) is smooth but \( f \) and \( g \) are only log smooth (magic powder).
The log structure allows \( f \) and even \( g \) to somehow “remember” \( f^* \).
Benefits

- Log smooth maps can be understood locally, (but are still much more complicated than classically smooth maps).
- Degenerations can be studied locally on the singular locus $Z$.
- Log geometry has natural cohomology theories:
  - Betti
  - De Rham
  - Crystalline
  - Etale
Roots and ingredients

- Toroidal embeddings and toric geometry
- Regular singular points of ODE’s, log poles and differentials
- Degenerations of Hodge structures

Remark: A key difference between local toric geometry and local log geometry:

- toric geometry based on study of cones and monoids.
- log geometry based on study of morphisms of cones and monoids.
Some applications

- Compactifying moduli spaces: K3’s, abelian varieties, curves, covering spaces
- Moduli and degenerations of Hodge structures
- Crystalline and étale cohomology in the presence of bad reduction—$C_{st}$ conjecture
- Work of Gabber and others on resolution of singularities (uniformization)
- Work of Gross and Siebert on mirror symmetry
What is Log Geometry?

What is geometry? How do we do geometry?
Locally ringed spaces: Algebra + Geometry

- Space: Topological space $X$ (or topos): $X = (X, \{U \subseteq X\})$
- Ring: $(R, +, \cdot, 1_R)$ (usually commutative)
- Monoid: $(M, \cdot, 1_M)$ (usually commutative and cancellative)

Definition
A locally ringed space is a pair $(X, \mathcal{O}_X)$, where

- $X$ is a topological space (or topos)
- $\mathcal{O}_X : \{\mathcal{O}_X(U) : U \subseteq X\}$ a sheaf of rings on $X$

such that for each $x \in X$, the stalk $\mathcal{O}_{X,x}$ is a local ring.
Example

\( X \) a complex manifold:
For each open \( U \subseteq X \), \( O_X(U) \) is the ring of analytic functions \( U \rightarrow \mathbb{C} \).
\( O_{X,x} \) is the set of germs of functions at \( x \), 
\( m_{X,x} := \{ f : f(x) = 0 \} \) is its unique maximal ideal.
Example: Compactification log structures

$X$ scheme or analytic space, $Y$ closed algebraic or analytic subset, $X^\ast = X \setminus Y$

\[
\begin{array}{ccc}
X^\ast & \xrightarrow{j} & X \\
& i & \leftarrow Y
\end{array}
\]

Instead of the sheaf of ideals:

\[
I_Y := \{ a \in O_X : i^*(a) = 0 \} \subseteq O_X
\]

consider the sheaf of multiplicative submonoids:

\[
M_{X^\ast/X} := \{ a \in O_X : j^*(a) \in O_{X^\ast}^\ast \} \subseteq O_X.
\]

Log structure:

\[
\alpha_{X^\ast/X} : M_{X^\ast/X} \rightarrow O_X \text{ (the inclusion mapping)}
\]
Notes

- This is generally useless unless codim \((Y, X) = 1\).
- \(\mathcal{M}_{X^*/X}\) is a sheaf of faces of \(\mathcal{O}_X\), i.e., a sheaf \(\mathcal{F}\) of submonoids such that \(fg \in \mathcal{F}\) implies \(f, g \in \mathcal{F}\).
- There is an exact sequence:

\[
0 \to \mathcal{O}^*_X \to \mathcal{M}_{X^*/X} \to \Gamma_Y(Div_X^-) \to 0.
\]
Definition of log structures

Let \((X, \mathcal{O}_X)\) be a locally ringed space (e.g. a scheme or analytic space).
A prelog structure on \(X\) is a morphism of sheaves of (commutative) monoids

\[ \alpha_X: \mathcal{M}_X \to \mathcal{O}_X. \]

It is a log structure if

\[ \alpha^{-1}(\mathcal{O}_X^*) \to \mathcal{O}_X^* \]

is an isomorphism. (In this case \(\mathcal{M}_X^* \cong \mathcal{O}_X^*\).)

Examples:

- \(\mathcal{M}_{X/X} = \mathcal{O}_X^*\), the trivial log structure
- \(\mathcal{M}_{\emptyset/X} = \mathcal{O}_X\), the empty log structure.
Logarithmic spaces

A log space is a pair \((X, \alpha_X)\), and a morphism of log spaces is a triple \((f, f^\#, f^\flat)\):

\[
f : X \to Y, f^\# : f^{-1}(\mathcal{O}_Y) \to \mathcal{O}_X, f^\flat : f^{-1}(\mathcal{M}_Y) \to \mathcal{M}_X
\]

Just write \(X\) for \((X, \alpha_X)\) when possible.

If \(X\) is a log space, let \(\underline{X}\) be \(X\) with the trivial log structure. There is a canonical map of log spaces:

\[
\underline{X} \to X : (X, \mathcal{M}_X \to \mathcal{O}_X) \to (X, \mathcal{O}_X^* \to \mathcal{O}_X)
\]

\[
(id : X \to X, id : \mathcal{O}_X \to \mathcal{O}_X, inc : \mathcal{O}_X^* \to \mathcal{M}_X)
\]

Variant: Idealized log structures

Add \(\mathcal{K}_X \subseteq \mathcal{M}_X\), sheaf of ideals, such that \(\alpha_X : (\mathcal{M}_X, \mathcal{K}_X) \to (\mathcal{O}_X, 0)\).
Example: torus embeddings and toric varieties

Example

The log line: $A^1$, with the compactification log structure from:

$$
\begin{array}{ccc}
G_m & \xrightarrow{j} & A^1 \\
& & \leftarrow^i 0 \\
on points: & \mathbb{C}^* & \longrightarrow_{i} \mathbb{C} \quad \leftarrow 0.
\end{array}
$$

Generalization

$$(G_m)^r \subseteq A_Q$$

Here $(G_m)^r$ is a commutative group scheme: a (noncompact) torus,

$A_Q$ will be a monoid scheme, coming from a toric monoid $Q$, with $Q^{gp} \cong \mathbb{Z}^r$. 
Notation Let $Q$ be a cancellative commutative monoid.

$Q^* :=$ the largest group contained in $Q$.  
$Q^{gp} :=$ the smallest group containing $Q$.  
$\overline{Q} := Q / Q^*$.  

$\text{Spec } Q$ is the set of prime ideals of $Q$, i.e, the complements of the faces of $Q$.  

N.B. A face of $Q$ is a submonoid $F$ which contains $a$ and $b$ whenever it contains $a + b$. 

Terminology: We say $Q$ is:

- **integral** if $Q$ is cancellative
- **fine** if $Q$ is integral and finitely generated
- **saturated** if $Q$ is integral and $nx \in Q$ implies $x \in Q$, for $x \in Q^{gp}$, $n \in \mathbb{N}$
- **toric** if $Q$ is fine and saturated and $Q^{gp}$ is torsion free
- **sharp** if $Q^* = 0$. 
Generalization: toric varieties

Assume $Q$ is toric (so $Q^{gp} \cong \mathbb{Z}^r$ for some $r$). Let

$$A^*_Q := \text{Spec } \mathbb{C}[Q^{gp}],$$

a group scheme (torus). Thus

$$A^*_Q(\mathbb{C}) = \{ Q^{gp} \to \mathbb{C}^* \} \cong (\mathbb{C}^*)^r, \quad \mathcal{O}_{A^*_Q}(A^*_Q) = \mathbb{C}[Q^{gp}]$$

$$A_Q := \text{Spec } \mathbb{C}[Q],$$

a monoid scheme. Thus

$$A_Q(\mathbb{C}) = \{ Q \to \mathbb{C} \}, \quad \mathcal{O}_{A_Q}(A_Q) = \mathbb{C}[Q]$$

$$A_Q := \text{the log scheme given by the open immersion } j : A^*_Q \to A_Q.$$ 

Have $\Gamma(\mathcal{M}) \cong \mathbb{C}^* \oplus Q$. 
Examples

- If $Q = \mathbb{N}^r$, $A_Q(\mathbb{C}) = \mathbb{C}^r$, $A_Q^*(\mathbb{C}) = (\mathbb{C}^*)^r$.

- If $Q$ is the submonoid of $\mathbb{Z}^4$ spanned by
  \{(1, 1, 0, 0), (0, 0, 1, 1), (1, 0, 1, 0), (0, 1, 0, 1)\}, then

  $$A_Q(\mathbb{C}) = \{(z_1, z_2, z_3, z_4) \in \mathbb{C}^4 : z_1z_2 = z_3z_4\}.$$

  $$A_Q^* \cong (\mathbb{C}^*)^3.$$
Pictures

Pictures of $Q$:
$	ext{Spec } Q$ is a finite topological space. Its points correspond to the orbits of the action of $\mathbb{A}_Q^*$ on $\mathbb{A}_Q$, and to the faces of the cone $C_Q$ spanned by $Q$.

Pictures of a log scheme $X$
Embellish picture of $X$ by attaching $\text{Spec } \mathcal{M}_{X,x}$ to $X$ at $x$. 
Example: The log line \((Q = \mathbb{N}, C_Q = \mathbb{R}_{\geq})\)

\[
\text{Spec}(\mathbb{N})
\]

\[
\text{Spec}(\mathbb{N} \rightarrow \mathbb{C}[\mathbb{N}])
\]
Example: The log plane \((Q = \mathbb{N} \oplus \mathbb{N}, \ C_Q = \mathbb{R}_{\geq} \times \mathbb{R}_{\geq})\)
Log points

The standard (hollow) log point

\[ t := \text{Spec } \mathbb{C}. \text{ (One point space). } \mathcal{O}_t = \mathbb{C} \text{ (constants)} \]

Add log structure:

\[ \alpha : \mathcal{M}_t := \mathbb{C}^* \oplus \mathbb{N} \rightarrow \mathbb{C} \quad (u, n) \mapsto u0^n = \begin{cases} u & \text{if } q = 0 \\ 0 & \text{otherwise} \end{cases} \]

We usually write \( P \) for a log point.

Generalizations

- Replace \( \mathbb{C} \) by any field.
- Replace \( \mathbb{N} \) by any sharp monoid \( Q \).
- Add ideal to \( Q \).
Example: log disks

$V$ a discrete valuation ring, e.g, $\mathbb{C}\{t\}$ (germs of holomorphic functions)

$K := \text{frac}(V), m_V := \text{max}(V), k_V := V/m_V,$

$\pi \in m_V$ uniformizer, $V' := V \setminus \{0\} \cong V^* \oplus \mathbb{N}$

$T := \text{Spec } V = \{\tau, t\}, \tau := T^* := \text{Spec } K, t := \text{Spec } k.$

Log structures on $T$: $\Gamma(\alpha_T): \Gamma(T, M_T) \to \Gamma(T, \mathcal{O}_T)$:

- trivial: $\alpha_{T/T} = V^* \to V$ (inclusion): $T_{\text{triv}}$
- standard: $\alpha_{T^*/T} = V' \to V$ (inclusion): $T_{\text{std}}$
- hollow: $\alpha_{\text{hol}} = V' \to V$ (inclusion on $V^*$, 0 on $m_V$): $T_{\text{hol}}$
- split$_m$ $\alpha_m = V^* \oplus \mathbb{N} \to V$ (inc, $1 \mapsto \pi^m$): $T_{\text{spl}_m}$

Note: $T_{\text{spl}_1} \cong T_{\text{std}}$ and $T_{\text{spl}_m} \to T_{\text{hol}}$ as $m \to \infty$
Inducing log structures

Pullback and pushforward

Given a map of locally ringed spaces $f: X \to Y$, we can:

*Pushforward* a log structure on $X$ to $Y$: $f_*(\mathcal{M}_X) \to \mathcal{O}_Y$.

*Pullback* a log structure on $Y$ to $X$: $f^*(\mathcal{M}_Y) \to \mathcal{O}_X$.

A morphism of log spaces is *strict* if $f^*(\mathcal{M}_Y) \to \mathcal{M}_X$ is an isomorphism.

A *chart* for a log space is strict map $X \to \mathbb{A}^Q$ for some $Q$.

A log space (or structure) is *coherent* if locally on $X$ it admits a chart.

Generalization: *relatively coherent* log structures.
Example: Log disks and log points

Let $T$ be a log disk, $t$ its origin. Then the log structure on $T$ induces a log structure on $t$:

<table>
<thead>
<tr>
<th>Log structure on $T$</th>
<th>Induced structure on $t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Trivial</td>
<td>Trivial</td>
</tr>
<tr>
<td>Standard</td>
<td>Standard</td>
</tr>
<tr>
<td>Hollow</td>
<td>Standard</td>
</tr>
<tr>
<td>Split</td>
<td>Standard</td>
</tr>
</tbody>
</table>
Fiber products

The category of coherent log schemes has fiber products. \( \mathcal{M}_{X \times_Z Y} \to \mathcal{O}_{X \times_Z Y} \) is the log structure associated to

\[
p^{-1}_X \mathcal{M}_X \oplus p^{-1}_Z \mathcal{M}_Z \overset{p^{-1}_Y \mathcal{M}_Y}{\to} \mathcal{O}_{X \times_Z Y}.
\]

Danger: \( \mathcal{M}_{X \times_Z Y} \) may not be integral or saturated. Fixing this can “damage” the underlying space \( X \times_Z Y \).
Properties of monoid homomorphisms

A morphism $\theta: P \to Q$ of integral monoids is

- **strict** if $\bar{\theta}: \bar{P} \to \bar{Q}$ is an isomorphism
- **local** if $\theta^{-1}(Q^*) = P^*$
- **vertical** if $Q/P := \text{Im}(Q \to \text{Cok}(\theta^{gp}))$ is a group.
- **exact** if $P = (\theta^{gp})^{-1}(Q) \subseteq P^{gp}$

A morphism of log schemes $f: X \to Y$ has $P$ if for every $x \in X$, the map $f^\flat: M_{Y,f(x)} \to M_{X,x}$ has $P$. 

Examples of monoid homomorphisms

Examples:

- $\mathbb{N} \to \mathbb{N} \oplus \mathbb{N} : n \mapsto (n, n)$
  $\mathbb{C}^2 \to \mathbb{C} : (z_1, z_2) \mapsto z_1 z_2$
  Local, exact, and vertical.

- $\mathbb{N} \oplus \mathbb{N} \to \mathbb{N} \oplus \mathbb{N} : (m, n) \mapsto (m, m + n)$
  $\mathbb{C}^2 \to \mathbb{C}^2 : (z_1, z_2) \mapsto (z_1, z_1 z_2)$ (blowup)
  Local, not exact, vertical

- $\mathbb{N} \to Q := \langle q_1, q_2, q_3, q_4 \rangle/(q_1 + q_2 = q_3 + q_4) : n \mapsto n q_4$
  Local, exact, not vertical.
Differentials

Let $f : X \to Y$ be a morphism of log schemes, 
Universal derivation:

$$(d, \delta) : (\mathcal{O}_X, \mathcal{M}_X) \to \Omega^1_{X/Y} \quad \text{(some write } \omega^1_{X/Y})$$

$$d\alpha(m) = \alpha(m)\delta(m) \quad \text{so } \delta(m) = d \log m \quad \text{(sic)}$$

Geometric construction:
(gives relation to deformation theory)
Infinitesimal neighborhoods of diagonal $X \to X \times_Y X$ made strict:
$X \to \mathcal{P}^N_{X/Y}, \Omega^1_{X/Y} = J/J^2$. 
If \( \alpha_X = \alpha_{X^*/X} \) where \( Z := X \setminus X^* \) is a DNC relative to \( Y \),

\[
\Omega^1_{X/Y} = \Omega^1_{\log Z}
\]

In coordinates \((t_1, \ldots t_n)\), \( Z \) defined by \( t_1 \cdots t_r = 0 \).
\( \Omega^1_{X/Y} \) has basis: \((dt_1/t_1, \ldots dt_r/t_r, dt_{r+1} \ldots dt_n)\).
Logarithmic de Rham complex

\[ 0 \to \mathcal{O}_X \to \Omega^1_{X/Y} \to \Omega^2_{X/Y} \cdots \]

Logarithmic connections:

\[ \nabla : E \to \Omega^1_{X/Y} \otimes E \]

satisfying Liebnitz rule + integrability condition: \( \nabla^2 = 0 \).

Generalized de Rham complex

\[ 0 \to E \to E \otimes \Omega^1_{X/Y} \to E \otimes \Omega^2_{X/Y} \cdots \]
Smooth morphisms

The definition of smoothness of a morphism \( f : X \to Y \) follows Grothendieck’s geometric idea: “formal fibration”: Consider diagrams:

\[
\begin{array}{ccc}
T & \xrightarrow{g} & X \\
\downarrow{i} & & \downarrow{f} \\
T' & \xrightarrow{h} & Y
\end{array}
\]

Here \( i \) is a strict nilpotent immersion. Then \( f : X \to Y \) is

- **smooth** if \( g' \) always exists, locally on \( T \),
- **unramified** if \( g' \) is always unique,
- **étale** if \( g' \) always exists and is unique.
Examples: monoid schemes and tori

Let \( \theta : P \to Q \) be a morphism of toric monoids. \( R \) a base ring. Then the following are equivalent:

- \( A_\theta : A_Q \to A_P \) is smooth
- \( A^*_\theta : A^*_Q \to A^*_P \) is smooth
- \( R \otimes \text{Ker}(\theta^{gp}) = R \otimes \text{Cok}(\theta^{gp})_{\text{tors}} = 0 \)

Similarly for étale and unramified maps.
In general, smooth (resp. unramified, étale) maps look locally like these examples.
The space $X_{\log}$ (Kato–Nakayama)

$X/\mathbb{C}$: (relatively) fine log scheme of finite type,

$X_{an}$: its associated log analytic space.

$X_{\log}$: topological space, defined as follows:

Underlying set: the set of pairs $(x, \sigma)$, where $x \in X_{an}$ and

$\mathcal{O}^*_{X,x} \xrightarrow{X^\#} \mathbb{C}^* \xrightarrow{u} u/u|u|$

commutes. Hence:

$X_{\log} \xrightarrow{\tau} X_{an} \xrightarrow{\alpha} X$
Each $m \in \tau^{-1}M_X$ defines a function $\text{arg}(m) : X_{\log} \to S^1$. $X_{\log}$ is given the weakest topology so that $\tau : X_{\log} \to X_{an}$ and all $\text{arg}(m)$ are continuous.

Get $\tau^{-1}M_X^{gp} \xrightarrow{\text{arg}} S^1$ extending $\text{arg}$ on $\tau^{-1}O_X^*$.

Define *sheaf of logarithms of sections of* $\tau^{-1}M_X^{gp}$:

$$
\begin{array}{ccc}
\mathcal{L}_X & \xrightarrow{\text{exp}} & S^1 \\
\downarrow & & \downarrow \\
\mathbb{R}(1) & \xrightarrow{\text{exp}} & S^1
\end{array}
$$
Get “exponential” sequence:

\[
0 \to \mathbb{Z}(1) \to \tau^{-1}\mathcal{O}_X \to \tau^{-1}\mathcal{O}_X^* \to 0
\]

\[
0 \to \mathbb{Z}(1) \to \mathcal{L}_X \to \tau^{-1}\mathcal{M}^{gp}_X \to 0
\]

Here: \( \tau^{-1}\mathcal{O}_X \to \mathcal{L}_X : a \mapsto (\exp a, \text{Im}(a)) \in \tau^{-1}\mathcal{M}^{gp}_X \times \mathbb{R}(1) \).

Construct universal sheaf of \( \tau^{-1}\mathcal{O}_X \)-algebras \( \mathcal{O}_X^{log} \) containing \( \mathcal{L}_X \)
Compactification of open immersions

The map $\tau$ is an isomorphism over the set $X^*$ where $M = 0$, so we get a diagram

$$
\begin{array}{ccc}
X_{an} & \xrightarrow{j} & X_{an} \\
\downarrow \tau & & \downarrow \tau \\
X_{log} & \xrightarrow{j_{log}} & X_{log}
\end{array}
$$

The map $\tau$ is proper, and for $x \in X$, $\tau^{-1}(x)$ is a torsor under $T_x := \text{Hom}(M_{x}^{gp}, S^1)$ (a finite sum of compact tori). We think of $\tau$ as a relative compactification of $j$. 
Example: monoid schemes

\[ X = A_Q := \text{Spec}(Q \to \mathbb{C}[Q]), \text{ with } Q \text{ toric.} \]

\[ X_{\log} = A_Q^{\log} = R_Q \times T_Q \xrightarrow{\tau} X = \overline{A}_Q \]

where

\[ A_Q(\mathbb{C}) = \{ z : Q \to (\mathbb{C}, \cdot) \} \text{ (algebraic set)} \]

\[ R_Q := \{ r : Q \to (\mathbb{R}_\geq, \cdot) \} \text{ (semialgebraic set)} \]

\[ T_Q := \{ \zeta : Q \to (\mathbb{S}^1, \cdot) \} \text{ (compact torus)} \]

\[ \tau : R_Q \times T_Q \to A_Q(\mathbb{C}) \text{ is multiplication: } z = r\zeta. \]

So \( A_Q^{\log} \) means polar coordinates for \( \overline{A}_Q \).
Example: log line, log point

If $X = \mathbb{A}_N$, then $X_{\text{log}} = \mathbb{R}_{\geq} \times S^1$. 

![Diagram](image-url)
or

(Real blowup)

If $X = P = x_\mathbb{N}$, $X_{\log} = S^1$. 

\[
\begin{array}{c}
\text{circle} \\ \rightarrow
\end{array}
\quad \rightarrow \\
\text{circle}
\]
Example: $\mathcal{O}_P^{log}$

$$\Gamma(P_{log}, \mathcal{O}_P^{log}) = \Gamma(S^1_{log}, \mathcal{O}_P^{log}) = \mathbb{C}.$$ 

Pull back to universal cover $\exp : \mathbb{R}(1) \to S^1$

$$\Gamma(\mathbb{R}(1), \exp^* \mathcal{O}_P^{log}) = \mathbb{C}[\theta],$$
generated by $\theta$ (that is, $\log(0))$.

Then $\pi_1(P_{log}) = Aut(\mathbb{R}(1)/S^1) = \mathbb{Z}(1)$ acts, as the unique automorphism such that $\rho_\gamma(\theta) = \theta + \gamma$. In fact, if $N = d/d\theta$, 

$$\rho_\gamma = e^{\gamma N}.$$
Application—Compactification

Theme: $j_{\text{log}}$ compactifies $X^* \rightarrow X$ by adding a boundary.

Theorem
If $X/\mathbb{C}$ is (relatively) smooth, $j_{\text{log}} : X^*_\text{an} \rightarrow X_{\text{log}}$ is locally aspheric. In fact, $(X_{\text{log}}, X_{\text{log}} \setminus X^*_\text{an})$ is a manifold with boundary.

Proof.
Reduce to the case $X = A_Q$. Reduce to $(R_Q, R^*_Q)$. Use the moment map, a homeomorphism:

$$(R_Q, R^*_Q) \cong (C_Q, C^o_Q) : r \mapsto \sum_{a \in A} r(a) a$$

where $A$ is a finite set of generators of $Q$ and $C_Q$ is the real cone spanned by $Q$. 

Example: The log line
Cohomology of log compactifications

Let $X/\mathbb{C}$ be (relatively) smooth, and $X^*$ the open set where the log structure is trivial.

Theorem

\[
\begin{align*}
H^*(X_{\log}, \mathbb{Z}) & \cong H^*(X^*, \mathbb{Z}) \\
& \cong H^*(X_{an}, \mathbb{Z})
\end{align*}
\]
Log de Rham cohomology

Three de Rham complexes:

- $\Omega^\cdot_{X/\mathbb{C}}$ (log DR complex on $X$)
- $\Omega^{log^\cdot}_{X/\mathbb{C}}$ (log DR complex on $X_{log}$)
- $\Omega^{\cdot}_{X^*/\mathbb{C}}$ (ordinary DR complex on $X^*$)

Theorem:
There is a commutative diagram of isomorphisms:

$$
\begin{align*}
H_{DR}(X) & \longrightarrow H_{DR}(X_{log}) \longrightarrow H_{DR}(X^*) \\
\downarrow & \downarrow \\
H_B(X_{log}, \mathbb{C}) & \longrightarrow H_B(X_{an}^*, \mathbb{C})
\end{align*}
$$
$X/S$ (relatively) smooth map of log schemes.

**Theorem (Riemann-Hilbert)**

Let $X/\mathbb{C}$ be (relatively) smooth. Then there is an equivalence of categories:

$$MIC_{nil}(X/\mathbb{C}) \equiv L_{un}(X_{\log})$$

$$(E, \nabla) \mapsto \text{Ker}(\tau^{-1}E \otimes \mathcal{O}^\log_X \overset{\nabla}{\longrightarrow} \tau^{-1}E \otimes \Omega^1_{X,\log})$$
Example: $X := P$ (Standard log point)

$\Omega^1_{P/C} \cong N \otimes \mathbb{C} \cong \mathbb{C}$, so

$MIC(P/C) \equiv \{(E, N) : \text{vector space with endomorphism}\}$

$P_{log} = S^1$, so $L(P_{log})$ is cat of reps of $\pi_1(P_{log}) \cong \mathbb{Z}(1)$. Thus:

$L(P_{log}) \equiv \{(V, \rho) : \text{vector space with automorphism}\}$

Conclusion:

$\{(E, N) : N \text{ is nilpotent}\} \equiv \{(V, \rho) : \rho \text{ is unipotent}\}$

Use $O^\log_P = \mathbb{C}[\theta]$:

$$(V, \rho) = \text{Ker} \left( \tau^* E \otimes \mathbb{C}[\theta] \to \tau^* E \otimes \mathbb{C}[\theta] \right)$$

$N \mapsto e^{2\pi i N}$
Application: Degenerations

Theme: replacing $f$ by $f_{\log}$ smooth out singularities of mappings.

**Theorem (Nakayama-Ogus)**

Let $f : X \to S$ be a (relatively) smooth exact morphism. Then $f_{\log} : X_{\log} \to S_{\log}$ is a topological submersion, whose fibers are orientable topological manifolds with boundary. The boundary corresponds to the set where $f_{\log}$ is not vertical.
Example

Semistable reduction $\mathbb{C} \times \mathbb{C} \rightarrow \mathbb{C} : (x_1, x_2) \mapsto x_1 x_2$
This is $A_\theta$, where $\theta : \mathbb{N} \rightarrow \mathbb{N} \oplus \mathbb{N} : n \mapsto (n, n)$
Topology changes: (We just draw $\mathbb{R} \times \mathbb{R} \rightarrow \mathbb{R}$):

![Diagram of topology changes]
Log picture: $R_Q \times T_Q$

Just draw $R_Q \to R_\mathbb{N}: \mathbb{R}_\geq \times \mathbb{R}_\geq \to \mathbb{R}_\geq : (x_1, x_2) \mapsto x_1 x_2$

Topology unchanged, and in fact is homeomorphic to projection mapping. Proof: (Key is exactness of $f$, integrality of $C_\theta$.)
Consequences

**Theorem**

\( f : X \to S \) (relatively) smooth, proper, and exact,

1. \( f_{\log} : X_{\log} \to S_{\log} \) is a fiber bundle, and

2. \( R^q f_{\log*}(\mathbb{Z}) \) is locally constant on \( S_{\log} \).
Monodromy

In the above situation, $R^q f_*(Z)$ defines a representation of $\pi_1(S_{\log})$. We can study it locally, using $X_{\log} \to X \times S_{\log}$. (Vanishing cycles)
Restrict to $D \subseteq S$, $D$ a log disk. Even better: to $P \subseteq D$, $P$ a log point.

Theorem
Let $X \to P$ be (relatively) smooth, saturated, and exact.

- The action of $\pi_1(P_{\log})$ on $R^q f_*(Z)$ is unipotent.
- Generalized Picard-Lefschetz formula for graded version of action in terms of linear data coming from: $\overline{M}_P \to \overline{M}_X$.

Proof uses a log construction of the Steenbrink complex

$$\Psi^\cdot : = \mathcal{O}_P^{\log} \to \mathcal{O}_P^{\log} \otimes \Omega^1_{X/P} \otimes \cdots$$
Example: Dwork families

Degree 3: Family of cubic curves in $P^3 : X \rightarrow S$:

$$t(X_0^3 + X_1^3 + X_2^3) - 3X_0X_1X_2 = 0$$

At $t = 0$, get union of three complex lines: At $t = \infty$, get smooth elliptic curve.

$X_{log} \rightarrow S_{log}$ is a fibration. How can this be?
Fibers of $X \to S$
Fibers of $X_{\log} \rightarrow S_{\log}$
Dehn twist
Degree 4:

\[ t(X_0^4 + X_1^4 + X_2^4 + X_3^4) - 4X_0X_1X_2X_3 = 0 \]

At \( t - 0 \), get a (complex) tetrahedron. At \( t = \infty \), get a K3 surface. Need to use \textit{relatively} coherent log structure for verticality. Still get a fibration!
Degree 5:

\[ t(X_0^5 + X_1^5 + X_2^5 + X_3^5 + X_4^5) - 5X_0X_1X_2X_3X_4 = 0 \]

Famous Calabi-Yau family from mirror symmetry. Also used in proof of Sato-Tate

**Nostalgia**

\( t = 5/3 \) was subject of my first colloquium at Berkeley more than thirty years ago.
Conclusion

- Log geometry provides a uniform geometric perspective to treat compactification and degeneration problems in topology and in algebraic and arithmetic geometry.
- Log geometry incorporates many classical tools and techniques.
- Log geometry is not a revolution.
- Log geometry presents new problems and perspectives, both in fundamentals and in applications.
Log:
It’s better than bad, it’s good.