ERRATUM TO "NOTES ON CRYSTALLINE COHOMOLOGY"

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Assertion (B2.1) of Appendix B to [BO] is incorrect as stated: a necessary condition for its conclusion to hold is that the transition maps $D^q_n \to D^q_{n-1}$ be surjective for all q and $n \geq 1$. However, [BO] only uses the weaker version (B2.1) below, which takes place in the derived category and holds for any $D \in K^-(\mathbb{N}, A_\bullet)$. Therefore, this incorrect statement has no consequence on the validity of the rest of [BO] and the text can be corrected by the following modifications.

1) Replace paragraph (B2.1) by the following text:

   "(B2.1) There exists in $D^-(\mathbb{N}, A_\bullet)$ an isomorphism $F \xrightarrow{\sim} D$ where F is such that each F^q_n is a projective A_n-module and each map $F^q_n \to F^q_{n-1}$ is surjective."

2) Replace from page B.3, line -2 to page B.4, line -9 by the following text:

   "Proof. To prove statement (B2.1), we first observe that we may assume that the transition maps $D^q_n \to D^q_{n-1}$ are surjective for all q and all $n \geq 1$. Indeed, for any $A_\bullet$-module E, the flasque resolution $\Gamma(E) \to \Gamma(E)/E \to 0 \to \ldots$ defined in (B.1.6) is a length 1 resolution of E by $A_\bullet$-modules with surjective transition maps. As it is functorial in E, we can apply it to each term D^q of the complex $D \in K^-(\mathbb{N}, A_\bullet)$, and we obtain in this way a double complex of $A_\bullet$-modules with surjective transition maps. The associated total complex D' belongs to $D^-(\mathbb{N}, A_\bullet)$, its terms have surjective transition maps and the natural morphism $D \to D'$ is a quasi-isomorphism. Therefore it is sufficient to prove (B2.1) for D'. In fact we shall prove the following more precise statement, which clearly suffices.

   (B2.1a) Assume that $D \in K^-(\mathbb{N}, A_\bullet)$ is such that, for all q, the inverse system D^q has surjective transition maps. Then there exists an $F \in K^-(\mathbb{N}, A_\bullet)$ and a surjective quasi-isomorphism $F \to D$ such that each F^q_n is a projective A_n-module and each map $F^q_n \to F^q_{n-1}$ is surjective.

   To prove (B2.1a), we begin by observing that if $K^\bullet_{n-1}$ is an acyclic complex of projective A_{n-1}-modules and is bounded above, then there exists an acyclic complex $K^\bullet_n$ of projective A_n-modules, still bounded above, and a surjective map $K^\bullet_n \to K^\bullet_{n-1}$. In fact, one sees easily by descending induction on the degree q that one can write each K^q_{n-1} as a direct sum $P^{q-1}_{n-1} \oplus P^q_{n-1}$, where the boundary map is given by the formula: $d^q(p^{q-1}, p^q) = (p^q, 0)$. Now write P^q_{n-1} as a quotient of a free A^q_n-module P^q_n and take $K^q_n = P^{q-1}_n \oplus P^q_n$ with similarly defined boundary maps. (As a matter

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of fact, it is even true that $K^\bullet_{n-1}$ lifts to A_n, as Houzel shows [SGA 5, Exp. XV], but we shall not need this result.)

We now prove (B2.1a) by constructing the complexes $\{F^\bullet_n : n \in \mathbb{N}\}$ inductively. Given $F^\bullet_{n-1} \to D^\bullet_{n-1}$, it is standard to find a $P^\bullet_n \in K^-(A_n)$ consisting of projective A_n-modules and a surjective quasi-isomorphism $P^\bullet_n \to D^\bullet_n$, and then a morphism $P^\bullet_n \to F^\bullet_{n-1}$ covering the given $D^\bullet_n \to D^\bullet_{n-1}$. We shall add an acyclic complex to $P^\bullet_n$ to construct $F^\bullet_n$ so that $F^\bullet_n \to F^\bullet_{n-1}$ is surjective. Let $K^\bullet_{n-1}$ be the mapping cone of the identity endomorphism of $F^\bullet_{n-1}$; by the previous paragraph, we can find a surjective map $K^\bullet_n \to K^\bullet_{n-1}$, where $K^\bullet_n \in K^-(A_n)$ is acyclic and has projective terms. Then $K^\bullet_n$ is also split: there exists a family of projective A_n-modules $\{E^q : q \in \mathbb{Z}\}$ such that $K^q_n \cong E^{q-1} \oplus E^q$ for all q, where the boundary map is as above. It follows that, for any complex $G^\bullet \in K(A_n)$, there is a canonical isomorphism

$$\prod_q \mathrm{Hom}_{A_n}(E^q, G^q) \xrightarrow{\sim} \mathrm{Hom}_{K(A_n)}(K^\bullet_n, G^\bullet).$$

Namely, if $h^q \colon E^q \to G^q$ for all q,

$$\{(d^{q-1}_G \circ h^{q-1}, h^q) \colon E^{q-1} \oplus E^q \to G^q : q \in \mathbb{Z}\}$$

defines a morphism of complexes $K^\bullet_n \to G^\bullet$. Since each E^q is projective, it follows that $\mathrm{Hom}_{K(A_n)}(K^\bullet_n,\ )$ is an exact functor. Now set $F^\bullet_n = P^\bullet_n \oplus K^\bullet_n[-1]$ and take the obvious surjection $F^\bullet_n \to F^\bullet_{n-1}$ extending $P^\bullet_n \to F^\bullet_{n-1}$. To define the morphism $F^\bullet_n \to D^\bullet_n$, take the given morphism on the summand $P^\bullet_n$. Since D^q_n maps surjectively to D^q_{n-1}, we can find on the second summand a morphism of complexes $K^\bullet_n[-1] \to D^\bullet_n$ lifting the morphism $K^\bullet_n[-1] \to F^\bullet_{n-1} \to D^\bullet_{n-1}$. This completes the construction."

References

[BO] P. Berthelot, A. Ogus, *Notes on Crystalline Cohomology*, Mathematical Notes **21** (1978), Princeton University Press.

[SGA 5] A. Grothendieck, *Cohomologie ℓ-adique et fonctions L*, avec la collaboration de I. Bucur, C. Houzel, L. Illusie, J.-P. Jouanolou et J.-P. Serre, Lecture Notes in Math. **589** (1977), Springer-Verlag.

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