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Frobenius and the Hodge Spectral Sequence

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Let X/k be a smooth proper scheme over a perfect field of characteristic p and let n be a natural number. A fundamental theorem of Barry Mazur relates the Hodge numbers of X/k to the action of Frobenius on the crystalline cohomology $H^n_{cris}(X/W)$ of X over the Witt ring W of k, which can be viewed as a linear map $\Phi\colon F^*_W H^n_{cris}(X/W) \to H^n_{cris}(X/W)$. If $H^n_{cris}(X/W)$ is torsion free, then since W is a discrete valuation ring, the source and target of Φ admit (unrelated) bases with respect to which the matrix of Φ is diagonal. For $i, j \in \mathbf{Z}$ with $i+j=n$, the (i, j)*th Hodge number* $h^{i,j}(\Phi)$ of Φ is defined to be the number of diagonal terms in this matrix whose p-adic ordinal is i. Mazur's theorem [1, 8.26] asserts that if the crystalline cohomology is torsion free and the Hodge spectral sequence of X/k degenerates at E_1, then these "Frobenius" Hodge numbers coincide with the "geometric" Hodge numbers:

$$h^{i,j}(\Phi) = h^{i,j}(X/k) := \dim H^j(X, \Omega^i_{X/k}). \tag{0.0.1}$$

In fact, Mazur's result is more precise. When the crystalline cohomology is torsion free, the De Rham cohomology $H_{DR}(X/k)$ of X/k can be identified with the reduction modulo p of the crystalline cohomology $H_{cris}(X/W)$, and using this identification, Mazur defines "abstract" Hodge and conjugate filtrations M_Φ and N_Φ on the De Rham cohomology $H_{DR}(X/k)$ in terms of the Frobenius action on $H_{cris}(X/W)$. He then proves that (under the above hypotheses), these abstract filtrations M_Φ and N_Φ coincide with the "geometric" Hodge and conjugate filtrations F_{Hdg} and F_{con} on $H_{DR}(X/k)$. An important consequence of his result is Katz's conjecture, which asserts that the Newton polygon [5] of Φ lies on or above the Hodge polygon (formed from the geometric Hodge numbers of X/k).

The main technical goal of this paper is to investigate what one can say about the Frobenius Hodge numbers of X when its Hodge spectral sequence does not degenerate. If X/k lifts to W and has dimension $n<p$, then degeneracy is automatic [3]. When the weight n is large compared to p,

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however, "pathologies" such as nondegeneration of the Hodge spectral sequence seem typical, and in view of increasing interest in motives of high weight, it seems important to develop techniques to deal with these phenomena. It has long been known [1, 8.36] that the Hodge polygon of Φ always lies over the polygon formed by the geometric Hodge numbers $h^{i,j}(X/k)$. This result is not very useful, however, because when the Hodge spectral sequence is not degenerate, $\sum_{i+j=n} h^{i,j}(X/k)$ is larger than the nth Betti number, and the aforementioned inequality of polygons contains very little information. It seems natural to ask if one can replace the numbers $h^{i,j}(X/k)$ by the "reduced" Hodge and/or conjugate numbers:

$$h^{ij}_{\infty}(X/k) := \dim \mathrm{Gr}^i_{F_{Hdg}} H^{i+j}_{DR}(X/k)$$

$$\bar{h}^{ij}_{\infty}(X/k) := \dim \mathrm{Gr}^i_{F_{con}} H^{i+j}_{DR}(X/k).$$

When the Hodge spectral sequence degenerates, it follows from the Cartier isomorphism that $h^{i,j}_{\infty} = \bar{h}^{j,i}_{\infty}$, but since this fails in general, it is not *a priori* clear which set of numbers to use.

It turns out that, with suitable hypotheses, we can prove that in fact the Hodge polygon of Φ lies *over* the reduced Hodge polygon and *under* the reduced conjugate polygon. When the Hodge spectral sequence degenerates, the Hodge and conjugate polygons coincide, and it follows that all three polygons are in fact equal. Thus our result generalizes the original result of Mazur. As we shall see in a subsequent article [6], even when this is no longer the case, it is sometimes possible to use additional information provided by a closer examination of the spectral sequences and/or duality to determine the Hodge numbers of Φ exactly.

Our technical results about Hodge polygons depend on a new framework in which to place the proof and statement of Mazur's fundamental theorem. The key is to study the Hodge and conjugate filtrations on crystalline cohomology instead of De Rham cohomology. These filtrations, which we denote again by F_{Hdg} and F_{con}, are (very nearly) *p-good* (1.1), and a simple abstract construction attaches to any W-module H with a p-good filtration F:

- a W-module with an *abstract p-good conjugate filtration* $(\bar{H}, \bar{F})$
- an *abstract F-span* Φ
- *abstract Hodge and conjugate spectral sequences* E_{Hdg} and E_{con}.

These spectral sequences degenerate if and only if (H, F) satisfies the p-adic Griffiths transversality condition of [7, 2.1.2], and this is always true of the Hodge and conjugate filtrations M_{Φ} and N_{Φ} associated to Φ. Furthermore,

$F^iH \subseteq M^i_\Phi H$ and $\bar{F}^i\bar{H} \subseteq N^i_\Phi H$ for all i, and it follows that, with suitable indexing,

$$\sum_{i=0}^{k} h^i_\infty(Hdg) \geqslant \sum_{i=0}^{k} h^i(\Phi) \geqslant \sum_{i=n-k}^{n} h^i_\infty(con),$$

for all k. The result on Hodge polygons is an immediate consequence of these inequalities. As it turns out, equality for all k is even *equivalent* to degeneration of the Hodge and conjugate spectral sequences (cf. Corollary 2.5).

In order to apply these simple constructions to crystalline cohomology, we first generalize them to the context of complexes. It is then possible to state a rather attractive derived category version of Mazur's theorem (Theorem 4.2.) Passing to cohomology, we easily deduce the corresponding estimates for crystalline cohomology. There still remains the difficulty of comparing the p-adic versions of the Hodge and conjugate filtrations with their better-known mod p incarnations. It is here that some hypotheses seem to be needed, and we attempt to find some reasonably useful ones which suffice for the applications we have in mind and which guarantee that the crystalline filtrations induce the standard filtrations on De Rham cohomology. If there is no torsion, these hypotheses are satisfied, for example, in the highest weight at which nondegeneration occurs (cf. Theorem 4.5). With yet additional hypotheses, it is even true that the geometric Hodge and conjugate spectral sequences coincide with the abstract Hodge and conjugate spectral sequences E_{Hdg} and E_{con}, in a suitable range.

All these results apply to cohomology with coefficients in an F-crystal, and in a subsequent article [6] we shall analyze (as our main motivating example) the F-crystals introduced by Scholl in his study of p-adic properties of modular forms of higher weight. It was in fact Scholl who suggested that I apply the machinery of [7] to his crystals, and the difficulties I encountered in attempting to carry out this calculation led eventually to the ideas developed here. I am grateful for his suggestion, and to Ofer Gabber for a marvelous conversation in Bures. Thanks also go to Matthew Emerton, who read a preliminary version of the manuscript and found many errors, and to the referee, who found many more.

1. FILTRATIONS AND SPANS

Let $\mathscr{A}$ be an abelian category, for example, the category of sheaves of modules in some ringed topos. If A is an object in $\mathscr{A}$ and m is a natural number, m_A will denote the endomorphism of A induced by multiplication by m, and A is said to be *m-torsion free* if m_A is injective. To focus ideas and simplify the terminology, we shall fix once and for all a prime number p, and we shall suppose that m_A is an isomorphism for every object A of

$\mathscr{A}$ and every m relatively prime to p. Then if A and A' are objects of $\mathscr{A}$, $\mathrm{Hom}(A', A)$ is a module over the localization of $\mathbf{Z}$ at p, and $\mathbf{Q} \otimes \mathrm{Hom}(A', A)$ can be identified with the localization of $\mathrm{Hom}(A', A)$ by p.

Let K be a p-torsion free object of $\mathscr{A}$ and let $j_A: A \to K$ be a subobject of K. For $i \in \mathbf{N}$, $p^i j_A: A \to K$ is again a subobject of K; frequently we shall just write A instead of j_A and $p^i A$ instead of $p^i j_A$. It will be convenient to extend this notation to negative values of i as well. Thus, if $k \geqslant 0$, $p^{-k} j_A$ is an element of $\mathbf{Q} \otimes \mathrm{Hom}(A, K)$, which we usually just denote by $p^{-k}A$ and shall call a *virtual subobject of* K. Of course, one can always find an injection $u: K \to K'$ such that $u \circ j_A$ is divisible by p^k, so that K, A, and $p^{-k}A$ can be thought of as subobjects of K'. If A is a virtual subobject of K and i and j are integers, then $p^i(p^j A) = p^{i+j}A$. If K is p-torsion free, then by a *virtual filtration of* K we mean a family F of virtual subobjects of K such that $F^iK \subseteq F^jK$ for $i \geqslant j$. The virtual filtrations we shall consider here will usually have the property that there is an injection $K \to K'$ such that $F^jK \subseteq K'$ for all j. Throught this paper, we will simply say "filtration" to mean a virtual filtration.

If (K, F) and (K', F') are (virtually) filtered torsion free objects of $\mathscr{A}$, an element Φ of $\mathbf{Q} \otimes \mathrm{Hom}(K', K)$ is said to *preserve the filtrations* if for every i, Φ induces a morphism $F'^iK' \to F^iK$. We shall say that Φ is a *filtered quasi-isomorphism* if it preserves the filtrations and in addition each map $F'^iK' \to F^iK$ is an isomorphism. If (K, F) is a (virtually) filtered object and d is an integer, $F(d)$ denotes the virtual filtration of K defined by $F(d)^i K := F^{i+d}K$.

If I is an ideal in a ring A and M is an A-module, the notion of an *I-good* filtration of M is classical and widely useful. We shall consider especially the following situation:

Definition 1.1. Let F be a (virtual) filtration on a p-torsion free object K of $\mathscr{A}$ and let a and b be integers with $a \leqslant b$.

1. F is *saturated* if $pF^iK \subseteq F^{i+1}K$ for all $i \in \mathbf{Z}$
2. If F is saturated, $\bar{F}^iK := p^iF^{-i}K$ for all $i \in \mathbf{Z}$.
3. F is *good with level in* $[a, b]$ if it is saturated and in addition $F^aK = F^iK$ for $i \leqslant a$ and $pF^iK = F^{i+1}K$ for $i \geqslant b$.

For example, if K is a p-torsion free object of K, then the canonical p-adic filtration F_p defined by

$$F^i_p K := \begin{cases} p^iK & \text{if} \quad i \geqslant 0 \\ K & \text{if} \quad i \leqslant 0 \end{cases} \tag{1.1.2}$$

is good with level in $[0, 0]$, and in fact it is the unique such filtration.

If (K, F) is saturated,

$$\bar{F}^{i+1}K = p^{i+1}F^{-i-1}K = p^i(pF^{-i-1})\,K \subseteq p^iF^{-i}K = \bar{F}^iK,$$

so that $\bar{F}$ is again a (virtual) filtration of K, called the (abstract) *conjugate* of F. Moreover,

$$p\bar{F}^iK = p^{i+1}F^{-i}K \subseteq p^{i+1}F^{-i-1}K = \bar{F}^{i+1}K,$$

so $(K, \bar{F})$ is again saturated.

LEMMA 1.2. *If F is a saturated filtration of a p-torsion free object K of $\mathscr{A}$, the following are equivalent*:

1. *For all i, $F^iK \cap pF^{i-2}K = pF^{i-1}K$.*
2. *For all i and j with $j < i$, $F^iK \cap pF^jK = pF^{i-1}K$.*
3. *For all i, j, and k with $j \leqslant i-k$ and $k \geqslant 0$, $F^iK \cap p^kF^jK = p^kF^{i-k}K$.*
4. *For all i, multiplication by p induces an injection*

$$F^{i-1}K/F^iK \to F^iK/F^{i+1}K.$$

If they are satisfied, F is said to be G-transversal to p.

Proof. For ease of exposition, we just give the proof for a category of modules, so that we can work with elements. The right side of each of the purported equalities in (1)–(3) is contained in the left because the filtration is saturated. Suppose that (1) holds and that $x \in F^iK \cap pF^jK$. If $j = i-1$ then (2) is trivial, and if $j = i-2$ it is exactly the statement (1). If $j < i-2$, then $x := py \in F^{j+2}K \cap pF^jK$, so by (1) with $i = j+2$, in fact $y \in F^{j+1}K$. It follows that (2) holds in general by induction on $i-j$. Moreover (2) implies (3) by induction on k. Indeed, (3) is trivial if $k=0$ and is exactly (2) when $k=1$. If $x = p^ky \in F^iK$ with $y \in F^jK$ and $k>1$, let $x' := p^{k-1}y$. Then $px' \in F^iK \cap pF^jK = pF^{i-1}K$, so that in fact $x' \in F^{i-1}K \cap p^{k-1}F^jK$, and by induction on k we can conclude that $y \in F^{i-k}K$. The equivalence of (1) and (4) is obvious. ∎

Remark 1.3. For a discussion of G-transversal filtrations, and in particular the relationship with Griffiths transversality, see [7]. We should point out that the definition of G-transversality given there is slightly different—the current one is better adapted to filtrations which are not necessarily exhaustive. (For exhaustive filtrations, the two definitions are equivalent, as the previous lemma shows.) If $\mathscr{A}$ has infinite direct sums, then one can form a graded object $\mathrm{Gr}_F K$ which is in a natural way an $\mathbf{F}_p[X]$ module, and (4) implies that F is G-transversal to p if and only if $\mathrm{Gr}_F K$ is torsion free as a module over this ring.

Let $\mathscr{A}F_{sat}$ denote the category of torsion-free objects of A endowed with a saturated filtration, with morphisms the morphisms preserving the filtrations. We denote by $\mathscr{A}F_a^b$ the full subcategory corresponding to the objects with a good filtration with level in $[a, b]$, and by $\mathscr{A}F_g$ the category of objects with a good (virtual) filtration of any level.

PROPOSITION 1.4. *Let F be a (virtual) filtration of a p-torsion free object K of $\mathscr{A}$.*

1. *If F is saturated, $\bar{F}$ is saturated, and $\bar{\bar{F}} = F$.*
2. *If (K, F) is good with level in $[a, b]$, then $(K, \bar{F})$ is good with level in $[-b, -a]$.*
3. *If $d \in \mathbf{Z}$, then multiplication by p^d induces an isomorphism*

$$(K, \bar{F}(-d)) \to (p^d K, \overline{F(d)}).$$

4. *If $0 \to (K', F') \to (K, F) \to (K'', F'') \to 0$ is a strict short exact sequence in $\mathscr{A}F_{sat}$ then the induced exact sequence*

$$0 \to (K', \bar{F}') \to (K, \bar{F}) \to (K'', \bar{F}'') \to 0$$

is again strict and exact.

5. *If (K, F) is G-transversal to p, then so is $(K, \bar{F})$.*
6. *Let $h: (K', F') \to (K, F)$ be a morphism of filtered p-torsion free objects of $\mathscr{A}$, and suppose that F' is the filtration of K' induced by F: $F'^i K' := h^{-1}(F^i K)$ for all i. Then if F is saturated (resp. G-transveral to p), the same is true for F', and $\bar{F}'$ is the filtration of K' induced by $\bar{F}$.*

Proof. We have already seen that if F is a saturated virtual filtration of K, then the same is true of $\bar{F}$. Moreover, the definition says that

$$\bar{\bar{F}}^i \bar{K} = p^i \bar{F}^{-i} K = p^i (p^{-i} F^i K) = F^i K,$$

proving (1).

Suppose that (K, F) has level in $[a, b]$. Then for $i \geqslant -a$,

$$p\bar{F}^i K = p(p^i F^{-i} K) = p(p^i F^{-i-1} K) = p^{i+1} F^{-i-1} K = \bar{F}^{i+1} K,$$

and for $i \leqslant -b$,

$$\bar{F}^{i-1} K = p^{i-1} F^{-i+1} K = p^{i-1} p F^{-i} K = p^i F^{-i} K = \bar{F}^{-i} K.$$

Thus $(K, \bar{F})$ has level in $[-b, -a]$, proving (2).

To prove (3), let $G := F(d)$. Multiplication by p^d induces an isomorphism $K \to p^d K$, and the image of $\bar{F}^i K$ is

$$p^d(\bar{F}^i K) = p^d(p^i F^{-i} K) = p^{d+i} F^{d-i-d} K = p^{d+i} G^{-i-d} K = \bar{G}^{i+d} K = \bar{G}(d)^i\, K.$$

That is, p^d_K induces a filtered isomorphism $(K, \bar{F}) \to (p^d K, \bar{G}(d))$, and (3) follows by shifting.

If

$$0 \to (K', F') \to (K, F) \to (K'', F'') \to 0$$

is strict and exact, then for all j the sequence

$$0 \to F'^j K' \to F^j K \to F''^j K'' \to 0$$

is exact. It follow that for any i, the sequence

$$0 \to p^i F'^j K' \to p^i F^j K \to p^i F''^j K'' \to 0$$

is again exact, so (4) holds.

Although (5) will be a consequence of Proposition (1.9) below, let us give a direct proof here. Suppose that (K, F) is G-transversal to p, and note that if we write $j := 2 - i$, then

$$\begin{aligned} \bar{F}^i K \cap p\bar{F}^{i-2} K &= p^i F^{-i} K \cap p p^{i-2} F^{2-i} K \\ &= p^i F^{-i} K \cap p^{i-1} F^{2-i} K \\ &= p^{i-1}(pF^{-i} K \cap F^{2-i} K) \\ &= p^{1-j}(pF^{j-2} K \cap F^j K) \\ &= p^{1-j} p F^{j-1} K \\ &= p\bar{F}^{1-j} K \\ &= p\bar{F}^{i-1} K. \end{aligned}$$

The verification of the last statement is immediate. ∎

Remark 1.5. Fontaine, Ekedahl, and Kato have worked with *F-gauges* (I suggest the terminology *Fontaine-gauges*) instead of filtered objects, and it would certainly have been possible, and perhaps even preferable, to do so here, with only a change in packaging. Recall that an F-gauge in $\mathscr{A}$ is a sequence of objects A^i and maps $F\colon A^i \to A^{i-1}$ and $V\colon A^i \to A^{i+1}$ such that FV and VF are multiplication by p. If the objects A^i are p-torsion free, then they define a p-saturated filtration on $\bigcup A^i$, and indeed, the category of p-torsion free F-gauges is equivalent to the category of p-torsion free saturated and exhaustively filtered objects. The operation of taking the conjugate of a p-saturated filtration translates, in the category of F-gauges, to the operation of replacing A by $A^i := A^{-i}$, with F and V reversed. We have chosen to use the language of filtered objects primarily because of the analogy with Hodge theory.

The operation of taking the conjugate of a good filtration is also related to Mazur's notion of a gauge, although the latter bears little resemblance to a Fontaine-gauge. Recall from [7] that a 1-*gauge* is a function $\varepsilon\colon \mathbf{Z} \to \mathbf{Z}$ such that $0 \leqslant \varepsilon(i) - \varepsilon(i+1) \leqslant 1$ for all i, and that if (K, F) is a filtered object in an abelian category with direct limits and ε is a gauge,

$$F^{\varepsilon}K := \sum_i p^{\varepsilon(i)} F^i K.$$

If F is good with level in $[a, b]$, then in fact

$$F^{\varepsilon}K = \sum_{i=a}^{b} p^{\varepsilon(i)} F^i K.$$

The following lemma shows the relationship between forming the conjugate of a filtration and the conjugate of a gauge, and will enable us to adapt the results of [7] to our present language.

LEMMA 1.6. *Let F be a good filtration on a p-torsion free object K of $\mathscr{A}$ and let $\varepsilon\colon \mathbf{Z} \to \mathbf{Z}$ be a* 1-*gauge. Then $\bar{\varepsilon}\colon \mathbf{Z} \to \mathbf{Z}$ defined by $\bar{\varepsilon}(i) := \varepsilon(-i) - i$ is again a* 1-*gauge, and $F^{\bar{\varepsilon}}K = \bar{F}^{\varepsilon}K$.*

Proof. By definition,

$$\begin{aligned} F^{\bar{\varepsilon}}K &= \sum_i p^{\bar{\varepsilon}(i)} F^i K \\ &= \sum_i p^{\varepsilon(-i) - i} F^i K \\ &= \sum_i p^{\varepsilon(i) + i} F^{-i} K \\ &= \sum_i p^{\varepsilon(i)} \bar{F}^i K \\ &= \bar{F}^{\varepsilon} K. \quad \blacksquare \end{aligned}$$

Mazur [5] defines a *span in $\mathscr{A}$* to be a morphism $\Phi\colon H' \to H$ of torsion-free objects of $\mathscr{A}$. In fact it will be convenient to generalize slightly, and to call any element Φ of $\mathbf{Q} \otimes \mathrm{Hom}(H', H)$ a span; if $\Phi \in \mathrm{Hom}(H', H)$, we say that Φ is *effective*. A span is *nondegenerate* if it is an isogeny, i.e., if there exists a $\Phi^{-1} \in \mathbf{Q} \otimes \mathrm{Hom}(H, H')$ such that $\Phi^{-1} \circ \Phi = \mathrm{id}_{H'}$ and $\Phi \circ \Phi^{-1} = \mathrm{id}_H$. If $\Phi\colon H' \to H$ is a span and r is an integer, its rth *Tate twist* $\Phi(r)$ is the span $p^{-r}\Phi\colon H' \to H$. A nondegenerate span Φ has *level in* $[a, b]$ if $\Phi(a)$ and $\Phi^{-1}(-b) = (\Phi(b))^{-1}$ are effective. Associated to a nondegenerate span are an abstract Hodge filtration M_Φ on H' and an abstract conjugate filtration N_Φ on H. Conversely, we can associate to any good filtration F on an object K an abstract span Φ_F and a spectral sequence $E_{Hdg}(K, F)$:

DEFINITION 1.7. 1. Let $\Phi\colon H' \to H$ be an effective span in $\mathscr{A}$. Then M_Φ is the pull-back by Φ of the standard p-adic filtration F_p on H, and N_Φ is the image of the conjugate $\bar{M}_\Phi$ of M_ϕ via Φ. That is,

$$M^i_\Phi H' := \begin{cases} \Phi^{-1}(p^i H) & \text{if } \; i \geqslant 0 \\ H' & \text{if } \; i \leqslant 0 \end{cases}$$

$$N^i_\Phi H := \begin{cases} p^i \Phi(M^{-i} H') & \text{if } \; i \leqslant 0 \\ p^i \Phi(H') & \text{if } \; i \geqslant 0. \end{cases}$$

2. Let F be a good filtration on K, with level in $[a, b]$. Then $H'_F := F^a K$, $H_F := \bar{F}^{-b} K$, and Φ_F is the span $\Phi_F\colon H' \to H$ such that

$$\begin{array}{ccc} H' & \xrightarrow{\Phi_F} & H \\ {\scriptstyle =}\downarrow & & \downarrow{\scriptstyle =} \\ F^a K & \xrightarrow{\Phi_F} & \bar{F}^{-b} K \\ & \searrow & \downarrow \\ & & K \end{array}$$

commutes, in $\mathbf{Q} \otimes \mathrm{Hom}(H', K)$.

3. Let (K, F) be a p-torsion free object of $\mathscr{A}$ endowed with a saturated filtration. Then $Q(K, F)$ is the filtered complex in degrees -1 and 0

$$Q(K, F) := (K, F(-1)) \xrightarrow{d} (K, F),$$

where d is the map induced by multiplication by p. The associated spectral sequence $E_{Hdg}(K, F)$ is the *abstract Hodge spectral sequence* of (K, F), and $E_{con}(K, F) := E_{Hdg}(K, \bar{F})$ is the *abstract conjugate spectral sequence* of (K, F).

If $\Phi\colon H' \to H$ is an effective span and $r \geqslant 0$, then $\Phi' := \Phi(-r)$ is again effective, and $M_{\Phi'} = M_\Phi(-r)$ and $N_{\Phi'} = N_\Phi(r)$. Then if Φ is any span, we can choose $r \geqslant 0$ so that $\Phi' := \Phi(-r)$ is effective and define $M_\Phi := M_{\Phi'}(r)$

and $N_\Phi := N_{\Phi'}(-r)$. The resulting filtrations M_Φ and N_Φ are independent of the choice of r and are respectively called the *Frobenius Hodge filtration* and the *Frobenius conjugate filtration* associated to Φ.

For any span Φ, the filtrations M_Φ and N_Φ are saturated and G-transversal to p, e.g., by (1.4). Furthermore, M_Φ is always exhaustive and $pN^i_\Phi = N^{i+1}_\Phi$ for $i \gg 0$. If Φ is nondegenerate of level in $[a, b]$, then M_Φ is good with level in $[a, b]$ and N_Φ is good with level in $[-b, -a]$. If (K, F) has level in $[a, b]$, then $H_F := \bar{F}^{-b}K := p^{-b}F^bK$, and since $b-a \geqslant 0$ and F is saturated,

$$p^{b-a}H'_F = p^{b-a}F^aK \subseteq F^bK = p^b\bar{F}^{-b}K = p^bH_F.$$

Thus $p^{-a}H'_F \subseteq H_F$, and so $p^{-a}\Phi_F$ is effective; in particular Φ_F is effective if $a \geqslant 0$. Moreover, $\bar{F}$ has level in $[-b, -a]$, so $H'_{\bar{F}} := \bar{F}^{-b}K = H_F$ and $H_{\bar{F}} := \bar{\bar{F}}^aK = H_F$, and $\Phi_{\bar{F}} = \Phi_F^{-1}$. Thus $p^b\Phi_F^{-1} = p^b\Phi_{\bar{F}}$ is effective, and Φ_F is nondegenerate of level in $[a, b]$. One can easily check that if $\Phi: H' \to H$ is a nondegenerate effective span, then Φ_{M_Φ} is naturally isomorphic to Φ, and that the formation of the span associated to a filtration is functorial and compatible with twists and shifts.

Proposition 1.8. 1. *Let $\Phi: H' \to H$ be a nondegenerate span. Then Φ induces filtered quasi-isomorphisms*:

$$(H', M_\Phi) \cong (H, \bar{N}_\Phi)$$

$$(H', \bar{M}_\Phi) \cong (H, N_\Phi).$$

2. *Let (K, F) be a good filtration and let $\Phi := \Phi_F$ be the associated span. Then for all i,*

$$F^iK \subseteq M^i_\Phi H' \qquad \textit{and} \qquad \bar{F}^iK \subseteq N^i_\Phi H.$$

Proof. Because of the compatibility of the formation of the filtrations associated to a span with twisting and shifting, it suffices to prove these results in the effective case. It follows immediately from the definitions that Φ maps $\bar{M}^i_\Phi H'$ isomorphically to $N^i_\Phi H$, so that Φ induces a filtered quasi-isomorphism:

$$(H', \bar{M}_\Phi) \cong (H, N_\Phi).$$

Taking conjugates, we see that it also induces a filtered quasi-isomorphism:

$$(H', M_\Phi) \cong (H, \bar{N}_\Phi).$$

This proves (1).

Say F has level in $[0, b]$, so that Φ_F is effective. Then if $i \geqslant 0$,

$$(F^i K) = p^i p^{-i} F^i K = p^i \bar{F}^{-i} K \subseteq p^i \bar{F}^{-b} K = p^i H.$$

That is, $\Phi_F(F^i K) \subseteq p^i H$, so $F^i K \subseteq M^i_\Phi H'$. If $i \leqslant 0$, $F^i K = F^0 K = M^0_\Phi H' = M^i_\Phi H'$ by definition. Thus in any case $F^i K \subseteq M^i_\Phi H'$, and hence $\bar{F}^i K \subseteq \bar{M}^i_\Phi H' = N^i_\Phi H$. The general case follows by twisting. ∎

Proposition 1.9. *Let (K, F) be an object of $\mathscr{A}$ endowed with a good filtration and let $\Phi\colon H' \to H$ be the corresponding span. Then for each i, multiplication by p^i induces isomorphisms*:

$$E_1^{i,-i}(K, \bar{F}) \cong E_1^{-i,i}(K, F)$$
$$E_1^{i,-i-1}(K, \bar{F}) \cong E_1^{-i,i-1}(K, F).$$

Moreover, the following conditions are equivalent:

1. *For all i, $F^i K = M^i_\Phi H'$.*
2. *The filtration F of K is G-transversal to p.*
3. *For all i, $E_1^{i,-i-1}(K, F) = 0$.*
4. *The spectral sequence $E(K, F) := E_{Hdg}(K, F)$ degenerates at E_1.*
5. *Any of the statements corresponding to (1)–(4) with $(K, \bar{F})$ in place of (K, F) is true.*

If F is exhaustive, these conditions are also equivalent to the statement that the boundary map of the complex $Q_{Hdg}(K, F)$ is strictly compatible with the filtration F.

Proof. Deligne's definition of the spectral sequence of a filtered complex [2] (C, F) is:

$$E_r^{i,j}(C, F) = (F^i C^{i+j} \cap d^{-1}(F^{i+r} C^{i+j+1})) \\ /(F^{i+1} C^{i+j} \cap d^{-1}(F^{i+r} C^{i+j+1}) + dF^{i-r+1} C^{i+j-1}).$$

In particular,

$$E_1^{i,j}(C, F) = (F^i C^{i+j} \cap d^{-1}(F^{i+1} C^{i+j+1}))/(F^{i+1} C^{i+j} + dF^i C^{i+j-1}).$$

Thus if $i \geqslant 0$,

$$\begin{aligned} E_1^{i,-i}(K, \bar{F}) &= \bar{F}^i K/(\bar{F}^{i+1} K + p\bar{F}^{i-1} K) \\ &= p^i F^{-i} K/(p^{i+1} F^{-i-1} K + pp^{i-1} F^{1-i} K) \\ &= p^i (F^{-i} K/(pF^{-i-1} K + F^{1-i} K)) \\ &= p^i E_1^{-i,i}(K, F). \end{aligned}$$

Similarly,

$$\begin{aligned}E_1^{i,-i-1}(K,\bar F)&=(\bar F^{i-1}K\cap p^{-1}(\bar F^{i+1}K))/\bar F^iK\\&=(p^{i-1}F^{1-i}K\cap p^{-1}(p^{i+1}F^{-i-1}K))/p^iF^{-i}K\\&=p^i(p^{-1}F^{1-i}K\cap F^{-i-1}K)/F^{-i}K)\\&=p^iE_1^{-i,i-1}(K,F).\end{aligned}$$

If $i<0$ we can achieve the same result by reversing the roles of F and $\bar F$, proving the first statement of the proposition.

For the remainder of the proof we shall assume without loss of generality that F, and hence also M_Φ, has level in $[0,b]$. Since M_Φ is G-transversal to p, it is obvious that (1) implies (2). It follows from the formula above that

$$E_1^{i,-i-1}(K,F)=p^{-1}F^{1+i}K\cap F^{i-1}K/F^iK\cong F^{i+1}K\cap pF^{i-1}K/pF^iK.$$

Thus by the definition in Lemma (1.2), F is G-transversal to p if and only if $E_1^{i,-i-1}(K,F)=0$ for all i, so (2) and (3) are equivalent. Since $E_1^{i,j}(K,F)$ vanishes unless $i+j$ is 0 or 1, (3) implies that $E_1(K,F)=E_\infty(K,F)$, i.e., that the spectral sequence degenerates at E_1. On the other hand, since K is p-torsion free, $H^{-1}(Q,F)=0$, so if the spectral sequence degenerates at E_1, $E_1^{i,-i-1}(K,F)=0$, and hence (3) and (4) are also equivalent.

It follows from the definitions that for $i\geqslant 0$,

$$M_\Phi^iH'=\{x\in F^0K:\Phi_F(x)\in p^iH=p^{i-b}F^bK\}.$$

In other words, $M_\Phi^iH'=F^0K\cap p^{i-b}F^bK$. If $i\geqslant b$, $M_\Phi^iH'=p^{i-b}F^bK=F^iK$ since F has level in $[0,b]$. Suppose that F is G-transversal to p and $x\in M_\Phi^iH'$ with $i<b$. Then $y:=p^{b-i}x\in F^bK$, and by Lemma 1.2, $x\in F^iK$. Thus (3) and (2) imply (1). The equivalence of (4) and (5) follows from the fact that $E_1^{i,i-1}(K,F)\cong E_1^{-i,-i-1}(K,\bar F)$, or from 1.4.5.

Suppose that F is exhaustive. Then by Lemma 1.2, F is G-transversal to p if and only if $F^iK\cap pK=pF^{i-1}K$ for all i, i.e., if and only if the differential d of $Q_{Hdg}(K,F)$ is strictly compatible with the filtrations. The equivalence with the degeneration of the spectral sequence is Deligne's [2, 1.3.2]. (The statement assumes that the filtration is biregular, but the proof only uses that it is exhaustive.) ∎

Let $\mathscr{B}$ be another abelian category in which m_B is an isomorphism for every m relatively prime to p and every object B and let $T\colon\mathscr{A}\to\mathscr{B}$ be an additive functor. If A is an object of $\mathscr{A}$, let $T_f(A)$ denote the quotient of $T(A)$ by its p-torsion, and if (K,F) is a filtered object in $\mathscr{A}$, let $F^iT(K)$ denote the image of $T(F^iK)\to T(K)$.

PROPOSITION 1.10. *Let $T: \mathscr{A} \to \mathscr{B}$ be an additive functor, let (K, F) be an object of $\mathscr{A}F^b_a$, and let $K' := F^aK$ and $\bar{K} := \bar{F}^{-b}K$. Then:*

1. *The map $T(F^iK) \to F^iT(K')$ induces an isomorphism*

$$T_f(F^iK) \cong F^iT_f(K').$$

2. *The filtration F on $T_f(K')$ is good, with level in $[a, b]$.*

3. *For every i, the image of $T(\bar{F}^iK)$ in $T_f(\bar{K})$ is $p^iF^{-i}T_f(K)$. That is, the filtration on $T_f(K)$ induced by the filtration $\bar{F}$ of K can be identified with the conjugate of the filtration F of $T_f(K')$.*

4. *If $\Phi_F: H'_F \to H_F$ is the span associated to F, then $T_f(\Phi_F)$ can be identified with the span associated to $(T_f(K'), F)$.*

Proof. If $i = a + j$ with $j \geqslant 0$, then $p^jK' = p^jF^aK \subseteq F^iK$, and hence there are commutative diagrams

$$\begin{array}{ccc} F^iK & \xrightarrow{\alpha} & K' \\ & \searrow^{p^j} & \downarrow{\scriptstyle\beta} \\ & & F^iK \end{array} \qquad\qquad \begin{array}{ccc} T_f(F^iK) & \xrightarrow{\alpha^*} & T_f(K') \\ & \searrow^{p^j} & \downarrow{\scriptstyle\beta^*} \\ & & T_f(F^iK) \end{array}$$

Thus α^* is injective, and $T_f(F^iK)$ can be identified with its image $F^iT_f(K)$ in $T_f(K)$. Since the filtration on K is saturated, there are also commutative diagrams

$$\begin{array}{ccc} F^iK & \xrightarrow{p} & F^iK \\ \downarrow{\scriptstyle\gamma} & \nearrow_{\delta} & \\ F^{i+1}K & & \end{array} \qquad\qquad \begin{array}{ccc} T_f(F^iK) & \xrightarrow{p} & T_f(F^iK) \\ \downarrow{\scriptstyle\gamma^*} & \nearrow_{\delta^*} & \\ T_f(F^{i+1}K) & & \end{array}$$

Thus the filtration F of $T_f(K')$ is saturated. Furthermore, α and γ, hence α^* and γ^*, are isomorphisms for $i \geqslant b$, and it follows that $(T_f(K'), F)$ is good, with level in $[a, b]$.

If $i \geqslant 0$, there are commutative diagrams

$$\begin{array}{ccc} F^{-i}K & \xrightarrow{\cong} & \bar{F}^iK \\ & \searrow^{p^i} & \downarrow \\ & & F^{-i}K \end{array} \qquad\qquad \begin{array}{ccc} T_f(F^{-i}K) & \xrightarrow{\cong} & T_f(\bar{F}^iK) \\ & \searrow^{p^i} & \downarrow \\ & & T_f(F^{-i}K) \end{array}$$

This shows that the image of $T_f(\bar{F}^iK)$ in $T_f(F^{-i}K)$ can be identified with $p^iT_f(F^{-i}K)$. When $i \leqslant 0$, we use the same diagram with F and $\bar{F}$ interchanged, and (3) follows; (4) is an immediate consequence. ▮

COROLLARY 1.11. *Let $\Gamma\colon \mathscr{A} \to \mathscr{B}$ be a left exact functor of abelian categories as above. Then Γ induces a functor $\Gamma_g\colon \mathscr{A}F_g \to \mathscr{B}F_g$, and Γ_g commutes with formation of conjugates and spans.*

Proof. Since Γ is left exact, $\Gamma(K)$ is torsion free whenever K is, so the corollary follows immediately from Proposition (1.10). ▮

2. ESTIMATES

DEFINITION 2.1. Let F be a filtration of an object V of $\mathscr{A}$ of finite length. Then

$$h^i(V, F) := \ln(\mathrm{Gr}^i_F(V))$$

$$q_i(V, F) := \ln(V/F^i) = h^{i-1}(V, F) + h^{i-2}(V, F) + \cdots = \sum_{j<i} h^j(V, F)$$

$$l^i(V, F) := \ln(F^iV) = h^i(V, F) + h^{i+1}(V, F) + \cdots = \sum_{j \geqslant i} h^j(F, V)$$

$$l_i(V, F) := \ln(F^{1-i}V) = h^{1-i}(V, F) + h^{2-i}(V, F) + \cdots = \sum_{j<i} h^{-j}(V, F).$$

If $\Phi\colon H' \to H$ is a span in $\mathscr{A}$ and $H'_0 := H/pH$ has finite length, the *Hodge numbers of* Φ are the numbers $h^i(\Phi) := h^i(H'_0, M_\Phi)$, and $q_i(\Phi)$, $l^i(\Phi)$, and $l^i(\Phi)$ are defined similarly.

We have chosen the notation because q_i and l_i are increasing functions of i, while l^i is decreasing. Note that $l_i(V, F) = l^{1-i}(V, F)$, and that if F and F' are filtrations on V which are n-opposed [2], then for all i $F'^{n-i+1}V \cong V/F^iV$, so that $q_i(V, F) = l_{i-n}(V, F')$. On the other hand, if $F \subseteq F'$, there are surjections $V/F^i \to V/F'^i$ for all i, hence $q_i(V, F) \geqslant q_i(V, F')$, and $l_i(V, F) \leqslant l_i(V, F')$.

THEOREM 2.2. *Let (K, F) be an object of $\mathscr{A}F_g$, let $\Phi\colon H' \to H$ be the corresponding span, and let $E(K, F)$ be the corresponding Hodge spectral sequence* (1.7.3). *Then H'_0 and H_0 have the same length, and if this length is finite, then the lengths $e^{i,j}_r(K, F)$ of $E^{i,j}_r(K, F)$ and $E^{i,j}_r(K, \bar{F})$ of $E^{i,j}_r(K, \bar{F})$ are finite for all i, j and r. Furthermore, for any i,*

1. $h^i(H'_0, M_\Phi) = h^{-i}(H_0, N_\Phi)$ *and* $q_i(H'_0, M_\Phi) = l_i(H_0, N_\Phi)$.
2. $q_i(H'_0, F) \geqslant q_i(\Phi) \geqslant l_i(H_0, \bar{F})$.
3. $q_i(H_0, \bar{F}) \geqslant l_i(\Phi) \geqslant l_i(H'_0, F)$.

Proof. The filtrations M_Φ of H' and N_Φ of H are G-transversal to p and exhaustive, and $N_\Phi \cong \bar{M}_\Phi$. Thus by Proposition (1.9), the spectral sequences $E_{Hdg}(H', M_\Phi)$ and $E_{Hdg}(H, N_\Phi)$ degenerate at E_1, and multiplication by p^{-i} induces an isomorphism $\mathrm{Gr}^i_{M_\Phi}(H'_0) \to \mathrm{Gr}^{-i}_{N_\Phi}(H_0)$. It follows that $h^i(H'_0, M_\Phi) = h^{-i}(H_0, N_\Phi)$ and that $q_i(H'_0, M_\Phi) = l_i(H_0, N_\Phi)$ for all i. Moreover, M_Φ and N_Φ are good, so the filtrations they induce on H'_0 and H_0 are biregular. Thus $\ln(H'_0) = q_i(H'_0, M_\Phi)$ for $i \gg 0$ and $\ln(H_0) = l_i(H_0, N_\Phi)$ for $i \gg 0$, so H'_0 and H_0 have the same length. If this length is finite, then $F^iK/pF^iK = H'/pH' = H'_0$ has finite length for $i \leqslant a$. For $i \geqslant a$ there is an exact sequence

$$F^{i-1}K/pF^{i-1}K \xrightarrow{\alpha} F^iK/pF^iK \xrightarrow{\beta} F^{i-1}K/pF^{i-1}K,$$

where α is induced by multiplication by p and β by the inclusions. Thus it follows by induction that F^iK/pF^iK has finite length for all i, and hence so does its quotient $F^iK/F^{i+1}K$. Then the explicit description of the E_1 term of the spectral sequences (see the proof of (1.6)) shows that $E_1^{i,j}(K, F)$ and $E_1^{i,j}(K, \bar{F})$ have finite length for all i and j.

By Proposition (1.8), $F \subseteq M_\Phi$ and $\bar{F} \subseteq N_\Phi$, and the same remains true of the induced filtrations on H'_0 and H_0. Consequently $q_i(H'_0, F) \geqslant q_i(H'_0, M_\Phi) = q_i(\Phi)$ and $q_i(\Phi) = l_i(H_0, N_\Phi) \geqslant l_i(H_0, \bar{F})$. This proves (2), and (3) can be proved in the same way, or by subtracting (2) from the equality $\ln(H'_0) = \ln(H_0)$, or by interchanging F and $\bar{F}$. ∎

Remark 2.3. The Hodge polygon associated with a sequence of integers h^i is formed by concatenating the line segments of slope i over the intervals $[q_i, q_{i+1}]$. That is, q_i is the abscissa of the ith breakpoint of the Hodge polygon. Thus (2.2.2) says that the breakpoints of the Φ-Hodge polygon lie to the *left* of those of the F-Hodge polygon, so that the Φ-Hodge polygon lies *above* the F-Hodge polygon. If one creates a polygon using the numbers $l_i(H_0, \bar{F})$, which amounts to using the Hodge numbers of $(H_0, \bar{F})$ in reverse order, then (2.2.2) also implies that the corresponding polygon lies *above* the Φ-Hodge polygon.

Mazur's original result (in the form proved in [1]) stated that the Frobenius Hodge numbers and the geometric Hodge numbers agree when the Hodge spectral sequence degenerates. Our next result gives a similar result when one has degeneracy in a suitable range.

THEOREM 2.4. *Let (K, F) be an object of $\mathscr{A}F_g$, let $\Phi: H' \to H$ be the corresponding span, and suppose that H'_0 has finite length. Fix an integer k, and consider the following conditions:*

1. $e_1^{i,-i}(K, F) = e_\infty^{i,-i}(K, F)$ *for all* $i > k$.
2. $h^i(H', F) = h^{-i}(H, \bar{F})$ *for all* $i > k$.
3. $h^i(H', F) = h^i(\Phi) = h^{-i}(H, \bar{F})$ *for all* $i > k$.
4. $F^i H'_0 = M^i_\Phi H'_0$ *for all* $i > k$ *and* $\bar{F}^i H_0 = N^i_\Phi H_0$ *for all* $i \leqslant -k$
5. $h^i(H, \bar{F}) = h^{-i}(\Phi)$ *for all* $i < -k$.
6. $\bar{F}^i H_0 = N^i_\Phi H_0$ *for all* $i \leqslant -k$.
7. $\bar{F}^i K = N^i_\Phi H$ *for all* $i \leqslant -k$.
8. $E_1^{i,-i-1}(K, \bar{F}) = 0$ *for all* $i \leqslant -k$.
9. $E_1^{i,-i-1}(K, F) = 0$ *for all* $i \geqslant k$.

Then $(1) \Rightarrow (2) \Leftrightarrow (3) \Leftrightarrow (4) \Rightarrow (5) \Leftrightarrow (6) \Leftrightarrow (7) \Rightarrow (8) \Leftrightarrow (9)$.

Proof. For any j, $e_1^{j,-j}(K, \bar{F}) \geqslant e_\infty^{j,-j}(K, \bar{F}) = h^j(H_0, \bar{F})$. Hence statement (3) of Theorem (2.2) implies, with the obvious abbreviations, that for any i,

$$\sum_{j<i} e_1^{j,-j}(\bar{F}) \geqslant \sum_{j<i} h^j(\bar{F}) \geqslant \sum_{j<i} h^{-j}(\Phi) \geqslant \sum_{j<i} h^{-j}(F).$$

Replace i by $-i$ and j by $-j$ to get that for any i,

$$\sum_{j>i} e_1^{-j,j}(\bar{F}) \geqslant \sum_{j>i} h^j(\bar{F}) \geqslant \sum_{j>i} h^j(\Phi) \geqslant \sum_{j>i} h^j(F) = \sum_{j>i} e_\infty^{j,-j}(F). \qquad (2.4.3)$$

If (1) holds and $k \leqslant i$, then by Proposition (1.9),

$$\sum_{j>k} e_\infty^{j,-j}(F) = \sum_{j>k} e_1^{j,-j}(F) = \sum_{j>k} e_1^{-j,j}(\bar{F}).$$

Then all the inequalities in (2.4.3) are equalities, and it follows by induction that $h^i(F) = h^i(\Phi) = h^{-i}(\bar{F})$ for all $i > k$. This argument shows that in fact (1) implies (2) and that (2) and (3) are equivalent. The implications $(7) \Rightarrow (6) \Leftrightarrow (5) \Leftarrow (3) \Leftrightarrow (4)$ are obvious. Supposing that (6) holds, we prove that $\bar{F}^i K = N^i_\Phi H$ for all $i \leqslant -k$ by induction on i. If F has level $[a, b]$, then $\bar{F}^i K = H = N^i_\Phi H$ for any $i \leqslant -b$. Assume that $i \leqslant -k$ and that $\bar{F}^{i-1} K = N^{i-1}_\Phi H$. If $x \in N^i_\Phi H$, then by (6) there exist $y \in \bar{F}^i K$ and $z \in H$ such that $x = y + pz$. Then $pz \in N^i_\Phi H$, and since N_Φ is G-transversal to p, $z \in N^{i-1}_\Phi H = \bar{F}^{i-1} K$. Then $pz \in \bar{F}^i K$, and it follows that $x \in \bar{F}^i K$ also. This completes the proof that (5)–(7) are equivalent. Assume that (7) holds and

$x \in \bar{F}^{i-1}K \cap p^{-1}\bar{F}^{i+1}K$ with $i \leqslant -k$. Then $px \in \bar{F}^{i+1}K \subseteq N^{i+1}_{\Phi}H$, so $x \in N^{i}_{\Phi}H = \bar{F}^{i}K$. Thus

$$E_1^{i,-i-1}(K, \bar{F}) = (\bar{F}^{i-1}K \cap p^{-1}\bar{F}^{i+1}K)/\bar{F}^{i}K = 0,$$

so (7) ⇒ (8), which is equivalent to (9) by Proposition (1.9). ∎

COROLLARY 2.5. *Let F be an exhaustive good filtration on an object K of $\mathcal{A}$ such that K_0 has finite length. Then the following conditions are equivalent*:

1. *For all i, $h^i(\Phi) = h^i(H'_0, F)$.*
2. *For all i, $h^i(\Phi) = h^{-i}(H_0, \bar{F})$.*
3. *For all i, $F^iK = M^{i}_{\Phi}H'$.*
4. *The Hodge spectral sequence of (K, F) degenerates at E_1.*
5. *The conjugate spectral sequence of (K, F) degenerates at E_1.*

PROPOSITION 2.6. *Let F be an exhaustive good filtration of an object K of $\mathcal{A}$, and let $Q := Q(K, F)$ as in* (1.7.3). *Then for any k the following are equivalent*:

1. *For all $i > k$, the map $H^0(F^{i+1}Q) \to H^0(Q)$ is injective.*
2. *For all $i > k$, $H^{-1}(Q/F^{i+1}Q) = 0$*
3. *For all $i \geqslant k$, $F^iK = M_{\Phi}{}^{i}K$*

Proof. Since K is p-torsion free, $H^{-1}(Q) = 0$, and the equivalence of (1) and (2) follows from this and the long exact sequence of cohomology. The equivalence of (2) and (3) can be proved by induction, using the following lemma. ∎

LEMMA 2.7. *In the situation of* (2.6), *suppose that $F^{i+1}K = M^{i+1}_{\Phi}H'$. Then*

$$H^{-1}(Q/F^{i+1}Q) \cong M^{i}_{\Phi}H'/F^{i}K.$$

Proof. Note that since F is exhaustive, $K' = K$ and M_{Φ} can be regarded as an exhaustive filtration of K. It follows from the definitions that

$$H^{-1}(Q/F^{i+1}Q) \cong (K \cap p^{-1}F^{i+1}K)/F^{i}K.$$

If $F^{i+1}K = M^{i+1}_{\Phi}H'$, then since M_{Φ} is G-transversal to p,

$$K \cap p^{-1}F^{i+1}K = K \cap p^{-1}M^{i+1}_{\Phi}H' = M^{i}_{\Phi}H'. \quad ∎$$

3. COMPLEXES AND COHOMOLOGY

As in the previous sections, $\mathscr{A}$ will denote an abelian category and p a prime such that m_A is invertible for every object A of $\mathscr{A}$ and every integer m relatively prime to p. If K is a complex in $\mathscr{A}$ such that each K^q is p-torsion free, than a *virtual filtration* of K is a family of virtual filtrations of each K^q stable under the boundary maps. If F is a saturated virtual filtration of K, then its conjugate $\bar{F}$ is again a saturated virtual filtration of K. A filtration F is *good with level in* $[a, b]$ if each (K^q, F) is good with level in $[a, b]$. A good filtration of a complex induces a span of complexes, as in (1.7).

Let KF_{sat} (resp. KF_g) denote the category whose objects are complexes K endowed with a saturated (resp. good) virtual filtration F and whose morphisms are the filtered homotopy classes of maps. If (K, F) is an object of KF_{sat}, its translate $(K, F)[1]$ is again an object of KF_{sat}, and if u is a morphism in KF_{sat}, its mapping cone $C(u)$ is an object of KF_{sat}. Thus KF_{sat} has the structure of a triangulated category, and KF_g is a triangulated subcategory.

A morphism $u\colon (K', F) \to (K, F)$ of filtered complexes is said to be a *filtered quasi-isomorphism* if for every i and n the map

$$H^n(u)\colon H^n(F^iK') \to H^n(F^iK)$$

is an isomorphism. The set *Qis* of quasi-isomorphisms is a multiplicative system compatible with the triangulation [4, I, 4.2] and we define DF_{sat} and DF_g to be the triangulated categories obtained from KF_{sat} and KF_g by localizing by *Qis*.

Remark 3.1. It is not clear whether or not the functor from DF_{sat} to the filtered derived category DF is fully faithful. However, it follows from [4, I, 3.3] that DF_g is a full subcategory of DF_{sat}. Indeed, suppose that $s\colon (K', F) \to (K, F)$ is a filtered quasi-isomorphism, with (K, F) good with level in $[a, b]$ and (K', F) saturated. Define a new filtration G on K' by letting $G^iK' := F^aK'$ if $i \leqslant a$ and $G^iK' := p^{i-b}F^bK'$ if $i \geqslant b$. Since F is saturated, G is finer than F, so there is a morphism $f\colon (K', G) \to (K', F)$. Furthermore, (K', G) is good, and the arrow $(K', G) \to (K, F)$ is a filtered quasi-isomorphism.

Let us say that (K, F) has *quasi-level in* $[a, b]$ if the maps $F^aK \to F^iK$ are quasi-isomorphisms for $i \leqslant a$ and the maps $pF^iK \to F^{i+1}K$ are quasi-isomorphisms for $i \geqslant b$, and let KF_{qg} denote the full subcategory of KF_{sat} consisting of those objects which have quasi-level $[a, b]$ for some $[a, b]$.

Then the construction in (3.1) shows that an object of KF_{sat} is quasi-isomorphic to an object of KF_g if and only if it belongs to KF_{qg}, so that the derived categories DF_g and DF_{qg} are equivalent.

The operation

$$(K, F) \mapsto (K, F)^- := (K, \bar{F})$$

defines a functor from KF_g to itself, compatible with translation and formation of mapping cones. Furthermore, it follows from (1.4.4) that $(K, F)^-$ is filtered acyclic if (K, F) is, and consequently that conjugation takes quasi-isomorphisms to quasi-isomorphisms and localizes to a triangulated functor $DF_g \to DF_g$.

The following construction of derived functors although not the most general possible statement, will suffice for our purposes. It applies, for example, if $\mathscr{A}$ is the category of sheaves of W-modules on a topological space and Γ is the global section functor, since one can in that case take G to be Godement's sheaf of discontinuous sections functor.

PROPOSITION 3.2. *Let $\Gamma: \mathscr{A} \to \mathscr{B}$ be a left exact functor. Suppose there exists an exact functor $G: \mathscr{A} \to \mathscr{A}$ and an injective natural transformation $\varepsilon: \mathrm{id}_{\mathscr{A}} \to G$ such that $G(A)$ is acyclic for Γ for every object A of $\mathscr{A}$. Then the right derived functor $R^+\Gamma_g$ of $\Gamma_g: \mathscr{A}F_g \to \mathscr{B}F_g$ exists and fits into a commutative diagram (up to isomorphisms of functors):*

$$\begin{array}{ccc} D^+F_g(\mathscr{A}) & \longrightarrow & D^+F(\mathscr{A}) \\ \Big\downarrow{\scriptstyle \mathrm{R}^+\Gamma_g} & & \Big\downarrow{\scriptstyle \mathrm{R}^+\Gamma} \\ D^+F_g(\mathscr{B}) & \longrightarrow & D^+F(\mathscr{B}) \end{array}$$

Moreover, $R^+\Gamma_g$ commutes with formation of conjugates and spans.

Proof. Since Γ is left exact, $\Gamma(E)$ is torsion free if E is, and it follows from (1.11) that if $(K, F) \in \mathscr{A}F_g$, then $\Gamma(K, F) \in \mathscr{B}F_g$. Thus Γ induces a functor $KF_g(\mathscr{A}) \to KF_g(\mathscr{B})$. Since we do not know that $D^+F_g(\mathscr{A})$ is a full subcategory of $D^+F(\mathscr{A})$, the construction of $R^+\Gamma_g$ requires an additional argument. By [4, I, 5.1], it will suffice to show that every object (K, F) of $KF_g^+(\mathscr{A})$ is quasi-isomorphic to an object (K', F) in $KF_g(\mathscr{A})$ such that each F^iK' is acyclic for Γ. Since G is left exact, it induces a functor $\mathscr{A}F_g \to \mathscr{A}F_g$, and if $(K, F) \in KF_g^+(\mathscr{A})$, (K, F) is quasi-isomorphic to the filtered complex (K', F) obtained by taking the associated simple complex to the filtered double complex $G(K, F)$. It is clear that formation of conjugates and spans commutes with Γ and G and hence with $\mathrm{R}^+\Gamma_g$. ∎

EXAMPLE 3.3. Let K be any complex in $\mathscr{A}$ and let F be a filtration of K. Then the *décalé* [2] F_{dec} of F is defined by

$$F^i_{dec}K^q := \{\omega \in F^{i+q}K^q : d\omega \in F^{i+q+1}K^{q+1}\}.$$

Suppose that K is p-torsion free. Then F_{dec} is saturated (resp. G-transversal to p) if F is. Furthermore, if F is G-transversal to p and has level in $[a, b]$, then $F^i_{dec}K^q$ has level in $[a-q, b-q]$. Thus if K is bounded, then if F is G-transversal to p and good, the same is true of F_{dec}. For example, the *conjugate filtration* F_{con} of K is by definition the décalé of the canonical p-adic filtration (1.1.2):

$$F^i_{con}K^q := \begin{cases} \omega \in p^{i+q}K^q : d\omega \in p^{i+q+1}K^{q+1} & \text{if } \ i \geqslant -q \\ \omega \in K^q & \text{if } \ i < -q. \end{cases}$$

The filtration induced by F_{con} on the reduction K_0 of K modulo p is the "filtration canonique" [2, 1.4.6], associated with the second spectral sequence of hypercohomology. We shall call the conjugate of F_{con} the *standard filtration* of K; it is given by

$$F^i_{std}K^q := \bar{F}^i_{con}K^q = \begin{cases} \omega \in p^q K^q : d\omega \in p^{q+1}K^{q+1} & \text{if } \ i \leqslant q \\ \omega \in p^i K^q & \text{if } \ i > q. \end{cases}$$

The next result shows that the formation of the span associated to a filtration is compatible with passing to cohomology. Although the proof is an immediate consequence of (1.10) and the definitions, we state it as a theorem, because of its central role.

THEOREM 3.4. *Let (K, F) be an object of KF_g, of level in $[a, b]$, let $K' := F^aK$, $\bar{K} := \bar{F}^{-b}K$, and let $\Phi \colon K' \to \bar{K}$ be corresponding span. Then for any integer n, the map*

$$H^n_f(\Phi) : H^n_f(K') \to H^n_f(\bar{K})$$

is a nondegenerate span and coincides with the span

$$\Phi_n \colon H'_F \to \bar{H}_F$$

associated to the good filtration induced by F on $H := H^n_f(K)$. The filtrations F of H'_F and $\bar{F}$ of H_F are finer than the filtrations M_{Φ_n} and N_{Φ_n}, respectively. ▮

Remark 3.5. In the situation of (3.4), suppose that $H^i(K')$ and $H^i(\bar{K})$ are p-torsion free when $i=n$ and $n+1$. Then the natural maps

$$H'_0 \to H^n(K'_0) \qquad \text{and} \qquad \bar{H}_0 \to H^n(\bar{K}_0)$$

are isomorphisms, and it is easy to see that the filtration F (resp. $\bar{F}$) induced on H'_0 (resp. $\bar{H}'_0$) is *finer* than the filtration corresponding to the filtered complex (K_0, F) (resp. $(\bar{K}_0, \bar{F})$). Assuming that the cohomology modules have finite length, it follows that

$$q_i(H', F) \geqslant q_i(\Phi) \geqslant l_i(\bar{H}, \bar{F}),$$

and that

$$q_i(H', F) \geqslant q_i(H(K'_0), F) \qquad \text{and} \qquad l_i(H, F) \leqslant l_i(H(\bar{K}_0), \bar{F}).$$

Thus we cannot in general use the Hodge numbers derived from (K'_0, F) and $(\bar{K}_0, \bar{F})$ to bound the Hodge numbers of Φ. In practice it is these mod p Hodge numbers that are more amenable to calculation than the Hodge numbers of (K, F). This difficulty motivates the next result, which shows that, with some additional hypotheses, the two sets of numbers coincide.

PROPOSITION 3.6. *Let (K, F) be an object of KF_g, let $\Phi\colon K' \to \bar{K}$ be the corresponding span, and suppose that F is G-transversal to p and that n is an integer such that $H^{n+1}(K')$ is torsion free. Consider the following conditions*:

1. *For all i, the map $H^{n+1}(F^iK'_0) \to H^{n+1}(K'_0)$ is injective*
2. *For all i, the map $H^{n+1}(F^iK) \to H^{n+1}(K')$ is injective*
3. *For all i, $H^{n+1}(F^iK)$ is torsion free.*

Then $(1) \Rightarrow (2) \Leftrightarrow (3)$, *and* (2) *and* (3) *imply that the natural map*

$$H^n(K')_0 \to H^n(K'_0)$$

is an isomorphism and takes

$$F^iH^n(K)_0 := \operatorname{Im} H^n(F^iK) \to H^n(K')_0$$

isomorphically onto

$$F^iH^n(K_0) := \operatorname{Im} H^n(F^iK'_0) \to H^n(K'_0)$$

Proof. Lemma (4.4.4) of [7] shows that (1) implies (2). (The hypothesis of compatibility with direct limits is not needed here.) Since $H^{n+1}(K')$ is torsion free, (2) implies (3), and it follows from Proposition (1.10.1) that (3) implies (2). The universal coefficient theorem shows that if $H^{n+1}(K')$ is torsion free, $H^n(K')_0 \cong H^n(K'_0)$. The G-transversality of (K, F) to p implies the existence of an exact sequence

$$0 \to F^{i-1}K \to F^iK \to F^iK'_0 \to 0$$

and hence also

$$H^n(F^iK) \to H^n(F^iK'_0) \to H^{n+1}(F^{i-1}K) \to H^{n+1}(F^iK).$$

Then (2) implies that the map $H^n(F^iK) \to H^n(F^iK'_0)$ is surjective. ∎

COROLLARY 3.7. *Let* (K, F) *be an object of* KF_g *which is G-transversal to* p, *let* n *be an integer for which* (2) *and* (3) *of Proposition* (3.6) *above hold, and let* $\Phi\colon H' \to \bar{H}$ *be the span* $H^n(\Phi)$. *Then the filtrations of* H'_0 *and* $\bar{H}_0$ *induced by* (K'_0, F) *and* $(\bar{K}_0, \bar{F})$ *are respectively finer than the Frobenius Hodge and conjugate filtrations* M_Φ *and* N_Φ *on* H'_0 *and* $\bar{H}_0$. *If* H'_0 *has finite length,*

$$q_i(H(K'_0), F) \geqslant q_i(\Phi) \geqslant l_i(H(\bar{K}_0), \bar{F}) \quad \textit{and} \quad q_i(\bar{H}_0, \bar{F}) \geqslant l_i(\Phi) \geqslant l_i(H'_0, F).$$

Under some circumstances it is even possible to identify (a portion of) the spectral sequences of the filtered complexes (K'_0, F) and $(\bar{K}_0, \bar{F})$ with the Hodge and conjugate spectral sequences of the filtered object (H', F). For the application we have in mind, the following result will suffice.

DEFINITION 3.8. Let (K, F) be an object of KF_g and let n be an integer. Then (K, F) is *cohomologically concentrated in degree n* if $H^n(K')$ is torsion free and for all i, the maps $H^n(F^iK) \to H^n(K')$ are injective and $H^q(F^iK)$ vanishes for $q \neq n$.

PROPOSITION 3.9. *Let* $\Phi\colon K' \to \bar{K}$ *be the span associated to an object* (K, F) *of* KF_g *and let* $H' := H^n(K')$ *with the filtration* F *induced by the filtration* F *of* K. *Then if* (K, F) *is cohomologically concentrated in degree* n,

1. *If* (H', F) *is regarded as a filtered complex placed in degree* n, *then* (H', F) *is good, and there is an isomorphism in* DF_g

$$(K', F) \simeq (H', F).$$

2. *The filtered complex* $(\bar{K}, \bar{F})$ *is cohomologically concentrated in degree* n, *and there is an isomorphism in* DF_g

$$(\bar{K}, \bar{F}) \simeq (\bar{H}, \bar{F}).$$

3. *If (K, F) is G-transversal to p, then there is an isomorphism in the filtered derived category of $\mathscr{A}$*

$$(K'_0, F) \simeq Q(H', F).$$

Proof. The first two statements will follow from the following simple lemma.

LEMMA 3.10. *If (K, F) is an object of KF_g and n is an integer, let $(T_{\leqslant n}K, F)$ denote the filtered complex which is (K^i, F) in degrees $<n$, $\mathrm{Ker}(d^n, F)$ in degree n, and zero in degrees greater than n. Then $(T_{\leqslant n}K, F)$ is good, and the natural map*

$$(T_{\leqslant n}K, F) \to (K, F)$$

is a filtered quasi-isomorphism if $H^q(F^iK) = 0$ for $q > n$.

Proof. Let $(C, F) := (T_{\leqslant n}K, F)$. Then C is torsion free, and if $x \in F^iC$, $px \in F^{i+1}C$ so (C, F) is saturated. Evidently $F^iC = F^{i-1}C$ if $F^iK = F^{i-1}K$, and if $F^{i+1}K = pF^iK$ and $x \in F^{i+1}C^n$, then $x = py$ with $y \in F^iK^n$ and $dx = pdy = 0$. Since K is torsion free, $dy = 0$ and $x \in pF^iC$. Thus (C, F) is good. It is standard that $H^q(F^iC) \cong H^q(F^iK)$ if $q \leqslant n$ and is zero if $q > n$, and the lemma follows. ∎

It follows from the lemma that the natural map $(T_{\leqslant n}K', F) \to (K', F)$ is a filtered quasi-isomorphism because of the fact that $H^q(F^iK) = 0$ for $q > n$. Furthermore, because $H^n(F^iK) \cong F^iH'$, and $H^q(F^iT_{\leqslant n}K) \cong H^q(F^iK) = 0$ for $q < n$, the natural map $(T_{\leqslant n}K, F) \to (H', F)$ is a filtered quasi-isomorphism. This proves the first statement of the Proposition (3.9). Because K is p-torsion free, multiplication by p^i induces an isomorphism $F^{-i}K \to \bar{F}^i\bar{K}$, and hence its cohomology vanishes except in degree n and is torsion free. Since $\bar{F}$ is saturated, for any i there are commutative diagrams

$$\begin{array}{ccccc} \bar{F}^{i+1}K & \longrightarrow & \bar{F}^iK & & \\ & \searrow^{p} & \downarrow & \searrow^{p} & \\ & & \bar{F}^{i+1}K & \longrightarrow & \bar{F}^iK \end{array} \qquad \begin{array}{ccccc} H^n(\bar{F}^{i+1}K) & \longrightarrow & H^n(\bar{F}^iK) & & \\ & \searrow^{p} & \downarrow & \searrow^{p} & \\ & & H^n(\bar{F}^{i+1}K) & \longrightarrow & H^n(\bar{F}^iK) \end{array}$$

Since $H^n(\bar{F}^{i+1}K)$ is torsion free, the maps $H^n(\bar{F}^{i+1}K) \to H^n(\bar{F}^iK)$ are injective.

To prove (3.9.3), we need the following lemma:

LEMMA 3.11. *If (K, F) is a p-torsion free complex equipped with a good filtration F, let u be the morphism of filtered complexes*

$$u\colon (K, F(-1)) \to (K, F)$$

induced by multiplication by p, let $C(u)$ denote its mapping cone, and let $Q(K, F) := C(u)$. Then the natural morphism in DF_g

$$Q(K, F) \to (K'_0, F)$$

is a quasi-isomorphism if F is G-transversal to p.

Proof. Since (K, F) is good, the natural map $(K', F) \to (K, F)$ is a quasi-isomorphism, and hence $Q(K', F) \cong Q(K, F)$ in DF_g. Thus we may assume without loss of generality that $K = K'$. Because K is torsion free, u is a monomorphism. Let C denote its cokernel, with the filtration induced from F, i.e., $(C, F) = (K_0, F)$. Then the natural map $C(u) \to (C, F)$ is compatible with the filtrations, and is a filtered quasi-isomorphism if u is strictly compatible with the filtrations. By (1.2), this is the case if (and only) if F is G-transversal to p. ∎

If (K, F) is cohomologically concentrated in degree n, there is a diagram of distinguished triangles in the filtered derived category DF_g,

$$\begin{array}{ccccccccc}
(K, F(-1)) & \longrightarrow & (K, F) & \longrightarrow & Q(K, F) & \longrightarrow & (K, F(-1))[1] & \longrightarrow \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & \\
(H', F(-1)) & \longrightarrow & (H', F) & \longrightarrow & Q(H', F) & \longrightarrow & (H', F(-1))[1] & \longrightarrow
\end{array}$$

and by the 5-lemma, the arrow $Q(K, F) \to Q(H', F)$ is a filtered quasi-isomorphism. If F is G-transversal to p, $Q(K, F) \cong (K'_0, F)$. This proves (3.9.3). ∎

COROLLARY 3.12. *Let $\Phi\colon K' \to \bar{K}$ be the span associated to an object (K, F) of KF_g which is G-transversal to p and cohomologically concentrated in degree n. Then the spectral sequence of the filtered complex (K'_0, F) can be identified with the spectral sequence $E_{Hdg}(H', F)$, and the spectral sequence of the filtered complex $(\bar{K}_0, \bar{F})$ can be identified with the spectral sequence $E_{con}(H', F)$.*

Remark 3.13. If (K, F) is a complex satisfying the hypotheses of (3.6), then it is easy to see that the map $(T_{\leqslant n} K_0, F) \to (K_0, F)$ induces a map of spectral sequences which is an isomorphism on all terms $E_r^{i,j}$ with $i + j \leqslant n$ and $r \geqslant 1$. Moreover, for any k, the following are equivalent:

1. For all $i \geqslant k$, the map $H^n(F^iK_0) \to H^n(K_0)$ is injective.
2. For all $i \geqslant k$, $E_1^{i,n-i}(K_0, F) = E_\infty^{i,n-i}(K_0, F)$.

The following result is helpful in establishing the validity of the hypotheses of Proposition (3.9)

PROPOSITION 3.14. *Let Φ: $K' \to \bar{K}$ be the span corresponding to an object (K, F) of KF_g which is G-transversal to p and such that $H^m(K')$ and $H^m(\bar{K})$ are p-adically separated for all m, and let n be an integer.*

1. *If $H^{n+1}(\mathrm{Gr}_F^i K_0)$ vanishes for all i, then $H^{n+1}(F^iK)$ and $H^{n+1}(F^iK_0)$ vanish for all i. Furthermore, $(\bar{K}, \bar{F})$ satisfies the same hypotheses and conclusions.*
2. *Suppose that $H^{n-1}(\bar{K}_0)$, $H^{n-1}(K'_0)$, and $H^{n-2}(\mathrm{Gr}_F^i K'_0)$ vanish for all i. Then $H^{n-1}(K')$ vanishes, $H^n(K')$ is torsion free, and the maps $H^n(F^iK) \to H^n(K')$ are injective for all i. Furthermore, $(\bar{K}, \bar{F})$ satisfies the same hypotheses and conclusions.*

Proof. Without loss of generality we may assume that $K = K'$. Suppose that K has level in $[a, b]$ and that $H^{n+1}(\mathrm{Gr}_F^i K_0)$ vanishes for all i. Since $F^iK_0 = 0$ for $i > b$, it follows from the exact sequences

$$H^{n+1}(F^{i+1}K_0) \to H^{n+1}(F^iK_0) \to H^{n+1}(\mathrm{Gr}_F^i K_0)$$

and descending induction on i that $H^{n+1}(F^iK_0)$ vanishes for all i. Since (K, F) is G-transversal to p, there are exact sequences

$$H^{n+1}(F^{i-1}K) \xrightarrow{\alpha} H^{n+1}(F^iK) \to H^{n+1}(F^iK_0),$$

where α is induced by multiplication by p. Then in fact α is surjective, and applying this with $i = a$, we see that multiplication by p is a surjective endomorphism of $H^{n+1}(K)$. Since this object is p-adically separated, it must vanish, and it follows by induction and the surjectivity of α that the same is true of $H^{n+1}(F^iK)$ for every i. The isomorphisms $\mathrm{Gr}_F^i K_0 \cong \mathrm{Gr}_{\bar{F}}^{-i} \bar{K}_0$ show that $(\bar{K}, \bar{F})$ inherits the hypotheses from (K, F). This proves (1).

Suppose the hypotheses of (2) are satisfied. Then the exact sequence

$$H^{n-1}(\bar{K}) \xrightarrow{p} H^{n-1}(\bar{K}) \to H^{n-1}(\bar{K}_0) \to H^n(\bar{K}) \xrightarrow{p} H^n(\bar{K})$$

and the vanishing of $H^{n-1}(\bar{K}_0)$ show that $H^n(\bar{K})$ is torsion free and that $H^{n-1}(\bar{K})_0$ vanishes. Since $H^{n-1}(\bar{K})$ is p-adically separated, it follows that it vanishes. Furthermore, $F^bK \cong \bar{K}$, and $\bar{K}_0 \cong F^bK/pF^bK \cong \mathrm{Gr}_F^i K$ for $i \geqslant b$, so $H^{n-1}(\mathrm{Gr}_F^b K) \cong H^{n-1}(K'_0) \cong 0$. Now the G-transversality of (K, F) implies that there is an exact sequence

$$H^{n-2}(\mathrm{Gr}_F^i K_0) \to H^{n-1}(\mathrm{Gr}_F^{i-1} K) \xrightarrow{p} H^{n-1}(\mathrm{Gr}_F^i K).$$

Thus the vanishing of $H^{n-2}(\mathrm{Gr}^i_F K_0)$ implies that $H^{n-1}(\mathrm{Gr}^{i-1}_F K)$ is contained in $H^{n-1}(\mathrm{Gr}^i_F K)$ for all i; since $H^{n-1}(\mathrm{Gr}^i_F K)$ vanishes for $i \geqslant b$, we see by descending induction on i that in fact it vanishes for all i. It then follows that the maps $H^n(F^i K) \to H^n(K)$ are injective for all i. Again, the hypotheses and conclusion are invariant under replacing K' by $\bar{K}$. ▮

Corollary 3.15. *Let (K, F) be an object of KF_g which is G-transversal to p and such that $H^m(K')$ and $H^m(\bar{K})$ are p-adically separated for all m. Suppose also that n is an integer such that $H^{n-1}(\bar{K}_0)$, $H^{n-1}(K'_0)$, and $H^q(\mathrm{Gr}^i_F K'_0)$ vanish for all i and all $q \neq n, n-1$. Then (K, F) is cohomologically concentrated in degree n. Furthermore,*

1. *The filtrations F on H'_0 and $\bar{F}$ on $\bar{H}_0$ induced by F and $\bar{F}$ coincide with the filtrations induced by the filtered complexes (K'_0, F) and $(\bar{K}_0, \bar{F})$, respectively.*

2. *The spectral sequences of the filtered complexes (K'_0, F) and $(\bar{K}_0, \bar{F})$ coincide with the abstract Hodge and conjugate spectral sequences of (H', F).*

4. CRYSTALS AND THEIR COHOMOLOGY

Let X/k be a smooth scheme over a perfect field k of characteristic $p > 0$, let W be the Witt ring of k, and let $u_{X/W}\colon X_{cris} \to X_{zar}$ be the standard map from X_{cris} to X_{zar} [1, Sect. 5]. Then the crystalline cohomology $H_{cris}(X/W)$ of X/W can be viewed as the cohomology of a canonical object

$$C^{\cdot}_{X/W} := Ru_{X/W*}\mathcal{O}_{X/W}$$

in the derived category of the abelian category $\mathscr{A}_X$ of sheaves of W-modules on X_{zar}. If X can be embedded as a closed subscheme of a smooth formal scheme Y/W and if D is the divided power envelope of X in Y, then $C^{\cdot}_{X/W}$ is canonically isomorphic to the De Rham complex $\Omega^{\cdot}_{D/W}$ [1, 6.4], a bounded complex of p-torsion free objects of $\mathscr{A}_X$. In fact $C^{\cdot}_{X/W}$ admits canonical filtrations, which allow one to construct objects in suitable filtered derived categories. In particular, one has:

1. $(C^{\cdot}_{X/W}, F_p)$, where F_p is the canonical p-adic filtration (1.1.2).

2. $(C^{\cdot}_{X/W}, F_{con})$, where F_{con} (the conjugate filtration), is the décalé of F_p (3.3).

3. $(C^{\cdot}_{X/W}, F_{std})$, where F_{std} is the conjugate of F_{con} (3.3).

4. $(C^{\cdot}_{X/W}, F_{spd})$, where F_{spd} is the divided power saturation of F_{std}, defined by $F^k_{spd} := F^{\bar{\varepsilon}_k}$, where ε_k is the maximal tame gauge which vanishes at k [7, 4.2.4].

5. $(C^{\cdot}_{X/W}, F_{Hdg})$, where F_{Hdg} is the Hodge filtration. If $X \subseteq Y/W$ as above, then F_{Hdg} corresponds to the filtration on $\Omega^{\cdot}_{D/W}$ given by

$$F^i_{Hdg}\Omega^q_{D/W} = J_D^{[i-q]}\Omega^q_{D/W},$$

where J_D is the ideal of X in D.

Remark 4.1. The reduction modulo p $C^{\cdot}_{X/k}$ of $C^{\cdot}_{X/W}$ can be identified with the De Rham complex $\Omega^{\cdot}_{X/k}$, and the filtration induced on $C^{\cdot}_{X/k}$ by F_{con} is the "canonical filtration" of [2, 1.4.6]. Thus:

$$\mathrm{Gr}^i_{F_{con}} C^{\cdot}_{X/k} \cong H^{-i}(\Omega^{\cdot}_{X/k}).$$

If one is interested in cohomology in weight n, it is more usual to shift the filtration by n. For $r \geqslant 1$, the E_r term of the spectral sequence associated with F_{con} identifies with the E_{r+1} term of the usual "conjugate spectral sequence." See [2, 1.3.4].

In fact it is convenient to consider filtrations indexed not just by the integers, but by Mazur's gauges [7, Sect. 4], as we have already mentioned in (1.6). Thus, $(C^{\cdot}_{X/W}, F_{con})$ and $(C^{\cdot}, F_{std})$ can be regarded as objects in the filtered derived category $D^+F_1(\mathscr{A}_X)$ of sheaves of W-modules on X, with filtrations indexed by the 1-gauges, and $(C^{\cdot}_{X/W}, F_{Hdg})$ is an object in the filtered derived category $D^+F_{tg}(\mathscr{A}_X)$ of tame gauges. (See [7, Sect. 4.3] for the definition of these.)

Let X' be the pull-back of X/k via the Frobenius endomorphism F_k of k, so that there is a commutative diagram

$$\begin{array}{ccccc} X & \xrightarrow{F_{X/k}} & X' & \xrightarrow{\pi_{X/k}} & X \\ & \searrow & \downarrow & & \downarrow \\ & & \mathrm{Spec}\, k & \longrightarrow & \mathrm{Spec}\, k \end{array}$$

Then the relative Frobenius morphism $F_{X/k}$ induces a morphism

$$\Phi\colon C^{\cdot}_{X'/W} \to F_{X/k*}C^{\cdot}_{X/W}.$$

Since $F_{X/k}$ is a homeomorphism, the derived functor of $F_{X/k*}$ exists and can be identified with $F_{X/k*}$.

THEOREM 4.2. *Let X/k be a smooth scheme and let $F_{X/k}: X \to X'$ be its relative Frobenius morphism. Then $F_{X/k}$ induces an isomorphism*

$$\Psi: (C^{\cdot}_{X'/W}, F_{H\,dg}) \to F_{X/k*}(C^{\cdot}_{X/W}, F_{std}) \qquad (in\ D^{+}F_{tg}(\mathscr{A}_{X'}))$$

Consequently:

1. *There is a filtered quasi-isomorphism*

$$\Psi: (C^{\cdot}_{X'/W}, F_{H\,dg}) \to F_{X/k*}(C^{\cdot}_{X/W}, F_{spd}) \qquad in \quad D^{+}F(\mathscr{A}_{X'}).$$

2. *The morphism $\Phi: C^{\cdot}_{X'/W} \to F_{X/k*}C^{\cdot}_{X/W}$ can be identified with the span associated to $F_{X/k*}(C^{\cdot}_{X/W}, F_{std}) \in D^{+}F_{g}(\mathscr{A}_{X'})$.*

3. *There are isomorphisms in the filtered derived categories $D^{+}F(\mathscr{A}_{X'})$ and $D^{+}F(\mathscr{A}_{X})$:*

$$(C^{\cdot}_{X'/k}, F_{H\,dg}) \cong F_{X/k*}(Q(C^{\cdot}_{X/W}, F_{std})) \cong F_{X/k*}(C^{\cdot}_{X/k}, F_{std}) \qquad and$$

$$(C^{\cdot}_{X/k}, F_{con}) \cong Q(C^{\cdot}_{X/W}, F_{con}),$$

where Q is the construction (3.11).

Proof. Theorem (7.3.1) of [7] asserts that $F^{*}_{X/W}$ induces a filtered quasi-isomorphism:

$$\Psi: (C^{\cdot}_{X'/W}, F_{H\,dg}) \to F_{X/k*}(C^{\cdot}_{X/W}, F'_{con}),$$

where F'_{con} is the filtration sending a gauge ε to $F^{\bar{\varepsilon}}_{con}C^{\cdot}_{X/W}$. By (1.6),

$$F^{\bar{\varepsilon}}_{con}C^{\cdot}_{X/W} = F^{\varepsilon}_{std}C^{\cdot}_{X/W},$$

and this proves the main statement of the theorem. If we apply this to the maximal tame gauge which vanishes at i, we see that for every i, Ψ induces a quasi-isomorphism

$$\Psi: (C^{\cdot}_{X'/W}, F^{i}_{H\,dg}) \to F_{X/k*}(C^{\cdot}_{X/W}, F^{i}_{spd}),$$

proving consequence (1).

Let d be the dimension of X. Then F_{std} is quasi-good, with quasi-level in $[0, d]$, so the associated span is the morphism

$$F^{0}_{std}C^{\cdot}_{X/W} \to \bar{F}^{-d}_{std}C^{\cdot}_{X/W} = F^{-d}_{con}C^{\cdot}_{X/W} \cong C^{\cdot}_{X/W}.$$

Pushing forward by the homeomorphism $F_{X/W*}$ and composing with the quasi-isomorphism $C^{\cdot}_{X'/W} \to F_{X/W*}F^{0}_{std}C^{\cdot}_{X/W}$, we obtain consequence (2).

For (3), define 1_k by

$$1_k(i) := \begin{cases} 1 & \text{if } \ i < k \\ 0 & \text{if } \ i \geqslant k \end{cases}$$

and let $c_1(i) := 1$ for all i. Then 1_k and c_1 are tame gauges, and if F is F_{Hdg} or F_{std}, there is a distinguished triangle

$$\to F^{c_1}C \to F^{1_k}C \to F^k C_0 \to,$$

where C_0 is the reduction of C modulo p. It follows that Ψ induces a filtered quasi-isomorphism

$$(C^{\cdot}_{X'/k}, F_{Hdg}) \cong F_{X/k*}(C^{\cdot}_{X/k}, F_{std}).$$

Furthermore, $(C^{\cdot}_{X/k}, F_{std}) \cong (Q(C^{\cdot}_{X/W}, F_{std}))$ because F_{std} is G-transversal to p (3.11). The statement with F_{con} is proved in the same way. ∎

Next we pass to cohomology. It follows from (3.2) that there is a functor $R^+\Gamma_g \colon DF_g^+(\mathscr{A}_X) \to DF_g^+(\mathscr{A})$, where $\mathscr{A}$ is the category of W-modules.

Remark 4.3. Let D be the PD-envelope of X in a smooth formal scheme Y/W, let $\mathscr{U}$ be an affine hypercovering of Y, and let $(K_{\mathscr{U}}, F)$ denote the simple complex associated to the Cech double complex of $(C^{\cdot}_{D/W}, F)$ with respect to $\mathscr{U}$, where $F = F_{con}$ or F_{std}. Then $R^+\Gamma(C^{\cdot}_{X/W}, F) \cong (K_{\mathscr{U}}, F) \in D^+F_{sat}(W)$. To see this, it suffices to prove that the terms in the above complexes have no higher cohomology. Thus we need to prove that if X is affine, $H^q(X, F^iC^j_{X/W}) = 0$ for $q > 0$. When $F = F_{con}$, it seems easiest to use induction on i. For $i \ll 0$, $F^i_{con}\Omega^j_{D/W} \cong \Omega^j_{D/W}$, whose higher cohomology vanishes, and because the filtration is G-transversal to p we have an exact sequence:

$$0 \to F^{i-1}_{con}\Omega^j_{D/W} \xrightarrow{p} F^i_{con}\Omega^j_{D/W} \to F^i_{con}\Omega^j_{D_0/k} \to 0$$

But $F^i_{con}\Omega^j_{D_0/k}$ is quasi-coherent when viewed as an $\mathcal{O}_{D_0}$-module via the Frobenius map, so it too has no higher cohomology. The result for F_{std} follows by conjugating.

The following result is then an immediate consequence of Theorem (4.2), together with the fact that $R^+\Gamma_g$ commutes with conjugation, formation of spans, and (as is easily seen), the construction Q.

COROLLARY 4.4. *Let X/k be a smooth scheme. Then there are isomorphisms*:

$$R^+\Gamma_{sat}(C^{\cdot}_{X/W}, F_{std})^- \cong R^+\Gamma_g(C^{\cdot}_{X/W}, F_{con}) \quad in \quad D^+F_g(\mathscr{A})$$
$$QR^+\Gamma_{sat}(C^{\cdot}_{X/W}, F_{std}) \cong R^+\Gamma(C^{\cdot}_{X'/k}, F_{Hdg}) \quad in \quad D^+F(\mathscr{A})$$
$$QR^+\Gamma_{sat}(C^{\cdot}_{X/W}, F_{con}) \cong R^+\Gamma(C^{\cdot}_{X/k}, F_{con}) \quad in \quad D^+F(\mathscr{A}).$$

Furthermore, the Frobenius morphism $R^+\Gamma(C^{\cdot}_{X'/W}) \to R^+\Gamma(C^{\cdot}_{X/W})$ *can be identified with the span associated to* $R^+\Gamma_g(C^{\cdot}_{X/W}, F_{std})$.

THEOREM 4.5. *Let* X/k *be a smooth proper scheme, let* n *be an integer, let* $H' := H^n_{cris}(X'/W)$ *and* $H := H^n_{cris}(X/W)$, *and let* $\Phi\colon F^*_W H' \to H$ *be the map induced by the relative Frobenius morphism* $X \to X'$. *Suppose that* $H^n_{cris}(X/W)$ *and* $H^{n+1}_{cris}(X/W)$ *are torsion free and that the maps*

$$H^{n+1}(X', F^i_{Hdg}\Omega^{\cdot}_{X/k}) \to H^{n+1}(X', \Omega^{\cdot}_{X'/k})$$
$$H^{n+1}(X, F^i_{con}\Omega^{\cdot}_{X/k}) \to H^{n+1}(X, \Omega^{\cdot}_{X/k})$$

are injective for all i.

1. *There are natural filtered isomorphisms*

$$(H^n_{cris}(X/W) \otimes k, F_{std}) \cong (H^n_{DR}(X'/k), F_{Hdg}),$$
$$(H^n_{cris}(X'/W) \otimes k, F_{con}) \cong (H^n_{DR}(X/k), F_{con}).$$

2. *For each* i,

$$q_i(H^n_{DR}(X'/k), F_{Hdg}) \geqslant q_i(\Phi) \geqslant l_i(H^n_{DR}(X/k), F_{con}),$$

and

$$q_i(H_{DR}(X/k), F_{con}) \geqslant q_i(\Phi) \geqslant l_i(H^n_{DR}(X'/k), F_{Hdg}).$$

3. *Suppose* k *is an integer such that, in the Hodge spectral sequence of* X/k, $e^{i,n-i}_{\infty} = e^{i,n-i}_1$ *for all* $i \geqslant k$. *Then* $h^{i,n-i}(\Phi) = h^{i,n-i}(X/k)$ *for all* $i \geqslant k$.

Proof. Let $(K, F) := R^+\Gamma_g(C^{\cdot}_{X/W}, F_{std})$, and let $(K_0, F) := R^+\Gamma(C^{\cdot}_{X/k}, F_{std})$. It follows from (4.4) that $(K_0, F) \cong R^+\Gamma(C^{\cdot}_{X'/k}, F_{Hdg})$, and hence that we can identify the Hodge spectral sequence of X'/k with the spectral sequence of the filtered complex (K_0, F). Then the maps $H^{n+1}(F^i_{std}K_0) \to H^{n+1}(K_0)$ are injective for all i, and $H^n(K)$ and $H^{n+1}(K)$ are torsion free. By Proposition (3.6), the filtration induced by F_{std} on $H^n(K_0) \cong H^n(K)_0$ is the same as the filtration of $H^n(K_0)$ induced by the filtration F_{std} of the filtered complex $H^n(K_0)$, i.e., the Hodge filtration of $H^n_{DR}(X'/k)$. A similar argument works with the conjugate filtration. This proves statement (1) of the theorem. Moreover, by (4.4), the Frobenius morphism $F_W^* H^n(X'/W) \to H^n(X/W)$ can be identified with the nth cohomology of the span of complexes associated to the object $R^+\Gamma_g(C^{\cdot}_{X/W}, F_{std})$. Then Corollary (3.7) implies statement (2) of the current theorem. Finally, (3) is proved in the same way as Theorem (2.4), using (2) and the fact that (as a consequence of the Cartier isomorphism), $e^{i,j}_{1,Hdg}(X'/k) = e^{j,i}_{1,con}(X/k)$. (Note that our indexing of the conjugate spectral sequence is shifted by n from the usual indexing.) ▮

As a consequence of statement (1) of Proposition (3.14), we have:

COROLLARY 4.6. *The hypotheses and conclusions of the previous result hold if n is an integer such that* $H^j(X, \Omega^i_{X/k}) = 0$ *whenever* $i + j = n + 1$ *and* $H^n(X/W)$ *is torsion free.*

More generally, let X/k be a fine saturated log scheme which is smooth and of Cartier type [7] (also called "perfectly smooth" in [7]). The main result of [7] holds also for X/k and hence so do Theorem (4.2) and its corollaries (4.5) and (4.6). In fact, these results also hold with coefficients in an F-crystal Φ: $F^*_{X/k}E' \to E$, provided one gives the proper definitions of the Hodge and conjugate filtrations. Recall from [7] that M_Φ is the inverse image via Φ of the p-adic filtration on E and N_Φ is the image via Φ of the conjugate of M_Φ, as in (1.7). Thus N_Φ is a good filtration of E by subcrystals and M_Φ is a good filtration of $F^*_{X/k}E'$ by subcrystals. The construction of the filtration A_Φ of E' by subsheaves in the crystalline topos requires descent through Frobenius and is more delicate, and we have to refer to the discussion of [7, Sect. 5]. Let $\bar{F}_\Phi$ denote the décalé of the filtration N_Φ of $Ru_{X/W*}(E)$, let F_Φ be the conjugate of $\bar{F}_\Phi$, and let F_{Hdg} denote the filtration of $Ru_{X'/W*}E'$ coming from A_Φ. For example if $E = \mathcal{O}_{X/W}$ with its standard structure of an F-crystal, M_Φ and N_Φ are both just the p-adic filtration of $\mathcal{O}_{X/W}$, while A_Φ is given by the PD powers of the ideal $J_{X/W}$; on $Ru_{X/W*}\mathcal{O}_{X/W}$, $\bar{F}_\Phi = F_{con}$ and $F_\Phi = F_{std}$.

Let us state the following consequence explicitly:

THEOREM 4.7. *Let X/k be a fine saturated and perfectly smooth log scheme, let Φ: $F^*_X E \to E$ be an admissible F-crystal on X/W, and let $E' := \pi^*_{X/k}E$. Let $C^{\cdot}_{E'/W}$ and $C^{\cdot}_{E/W}$ denote the De Rham complexes of E' and E, respectively, with the filtrations F_{Hdg} and F_Φ described above. Suppose that n is an integer such that $H^q_{cris}(X'/k, \mathrm{Gr}^i_{F_{Hdg}} E'_0)$ and $H^{n-1}_{cris}(X'/k, E'_0)$ vanish for all i and all $q \neq n, n-1$. Then the results of (4.5) hold for the cohomology of X with coefficients in E in degree n. Moreover, $R^+\Gamma_g(C^{\cdot}_{E'/W}, F_\Phi)$ is cohomologically concentrated in degree n, and its*

Hodge and conjugate spectral sequences coincide with the Hodge and conjugate spectral sequences of X/k with coefficients in E and with the Hodge and conjugate spectral sequences (1.7.3) *of the filtered object* $(H^n_{cris}(X'/W, E), F_\Phi))$.

Proof. This theorem follows immediately from the above remarks and Proposition (3.14). Notice that

$$H^{n-1}_{cris}(X'/k, E'_0) \cong F_k^* H^{n-1}_{cris}(X/k, E_0)$$

so that the vanishing of $H^{n-1}_{cris}(X'/k, E'_0)$ is equivalent to the vanishing of $H^{n-1}_{cris}(X/k, E_0)$. ∎

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