Inverses of 2×2 matrices

Let us look at the 2×2 case. What does it mean for $A' := \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix}$ to be the inverse of $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$? We can just write this out brutally:

$$I_2 = AA' = A'A$$

$$\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} = \begin{pmatrix} a' & b' \\ c' & d' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

$$\begin{aligned} 1 &= aa' + bc' & \qquad 1 &= a'a + b'c \\ 0 &= ab' + bd' & \qquad 0 &= a'b + b'd \\ 0 &= ca' + dc' & \qquad 0 &= c'a + d'c \\ 1 &= cb' + dd' & \qquad 1 &= c'b + d'd \end{aligned}$$

Note how different the sets of equations on the left and the right look. If we regard the matrix A as given, then we have four unkowns a', b', c', d', with eight equations. Quite a mess!

If we write this out as a system of linear equations, we get (putting the equations on the right at the bottom):

$$\begin{pmatrix} a & 0 & b & 0 \\ 0 & a & 0 & b \\ c & 0 & d & 0 \\ 0 & c & 0 & d \\ a & c & 0 & 0 \\ b & d & 0 & 0 \\ 0 & 0 & a & c \\ 0 & 0 & b & d \end{pmatrix} \begin{pmatrix} a' \\ b' \\ c' \\ d' \end{pmatrix} = \begin{pmatrix} 1 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 1 \\ 1 \end{pmatrix}.$$

It turns out that the above equations aren't independent: the ones on the right (now the bottom) are a consequence of the ones on the left. The analogous statement is true for $n \times n$ matrices, but we won't prove this until later.

In the 2×2 case we can see it using the following trick, which is worth memorizing.

Theorem 1. *Let* $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a* 2×2 *matrix. Then* A *is invertible if and only if* $ad - bc \neq 0$*, and if this is the case,*

$$A^{-1} = (ad - bc)^{-1} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$$

Proof: Let $\tilde{A} := \begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$. We just make the following computation:

$$A\tilde{A} = \tilde{A}A = (ad - bc)I_n, \quad \text{i.e:}$$

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} d & -b \\ -c & a \end{pmatrix} = (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$
$$\begin{pmatrix} d & -b \\ -c & a \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = (ad - bc) \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Now if $ad - bc \neq 0$, it is clear that $(ad - bc)^{-1}\tilde{A}$ satisfies the definition of A^{-1}. For the converse, suppose that A is invertible. In fact, just assume there is a matrix A' such that $A'A = I$. Then so $A'(A\tilde{A}) = A'(ad-bc)I_n = (ad-bc)A'$. Using the associativity, we find this is also equal to $(A'A)\tilde{A} = I_2\tilde{A} = \tilde{A}$. We conclude from this that $(ad-bc)A' = \tilde{A}$. If $ad-bc$ were zero, this would imply that $\tilde{A} = 0$, hence $a = b = c = d = 0$, which obviously can't be true if A' exists. This shows that if $A'A = I$, then $ad - bc \neq 0$, and hence by the previous calculation, A is invertible.

See if you can fill in the proof of the following result.

Theorem 2. *Let* $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *be a* 2×2 *matrix. Then the following are equivalent.*

1. $ad - bc \neq 0$.

2. A *is invertible.*

3. *There exists an* A' *such that* $A'A = I_2$.

4. *There exists an* A' *such that* $AA' = I_2$.