Systems of linear differential equations

Notation: $I := (a, b) := \{t \in \mathbf{R} : a < t < b\}$.
$C_2^k := \{\begin{pmatrix} x_1 \\ x_2 \end{pmatrix} : x_1, x_2 \in C^k(I)\}$.

Theorem: Let $P := \begin{pmatrix} p_{11} & p_{12} \\ p_{21} & p_{22} \end{pmatrix}$ be a 2×2 matrix of continuous functions on I. If $X := \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ is a vector of differentiable functions on I, let $L(X) := X' - PX$.

1. L is a linear transformation.

2. Given any $Q \in C_2^0(I)$, any $t_0 \in I$, and any vector $Y_0 := \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \in \mathbf{R}^2$, there exists a unique X such that

   (a) $L(X) = Q$, i.e, $X' = PX + Q$
   (b) $X(t_0) = Y_0$.

Corollary: $L : C_2^1(I) \to C_2^0(I)$ is surjective, and its nullspace has dimension 2.

Corollary: If (X_1, X_2) is a linearly independent pair of elements in $NS(L)$, then (X_1, X_2) forms a basis for $NS(L)$.

Corollary: Let (X_1, X_2) be a pair of elements of $NS(L)$, and let $W(X_1, X_2)$ be the determinant of the matrix whose columns are X_1, X_2. If $W(X_1, X_2)(t_0) \neq 0$ for any $t_0 \in I$, then (X_1, X_2) is a basis for $NS(L)$ and $W(X_1, X_2)(t) \neq 0$ for *all* $t \in I$.

Theorem: If (X_1, X_2) is a pair of elements of $NS(L)$ and $W := W(X_1, X_2)$, then $W' = \mathrm{tr}(P)W$. Hence $W = ce^{\alpha}$, where c is some constant and $\alpha' = \mathrm{tr}(P)$.

Constant coefficients

If the coefficients of P are constant and $Z = 0$, a fundamental solution set can be found easily. One method is to try exponential solutions of the form e^{rt}.

Claim: Suppose P is constant and ξ_0 is an eigenvector of P with eigenvalue λ. Then $X := e^{\lambda t}\xi_0$ satisfies $X' = PX$.

Thus if P is diagonalizable and ξ_1, ξ_2 is a basis of eigenvectors with corresponding eigenvalues λ_1, λ_2, then $(e^{\lambda_1 t}\xi_1, e^{\lambda_2 t}\xi_2)$ is a fundamental solution set. The general solution to the differential equation is then

$$X(t) = c_1 e^{\lambda_1 t}\xi_1 + c_2 e^{\lambda_2 t}\xi_2.$$