Linear Subspaces

Definition: Let V be a vector space. A *linear subspace* of V is a subset W of V such that:

1. W is not empty.
2. W is closed under addition.
3. W is closed under scalar multiplication.

Theorem: Let V be a vector space and let W be a subset of V. Then W is a linear subspace of V if and only if the following conditions hold

1. The zero vector belongs to W.
2. Every linear combination of every finite set of elements of W belongs to W.

Spans

Definition: Let S be a nonempty subset of V. Then the *span* of S is the set of all linear combinations of elements of S, i.e., the set of all elements of V that can be written

$$\mathbf{v} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots a_n\mathbf{w}_n$$

where each $a_i \in \mathbf{R}$ and each $\mathbf{w}_i \in S$. The span of the empty set is the set containing just the zero vector.

Theorem: If S is any subset of V, the span of S is the smallest linear subspace of V containing S.