Second order linear differential equations

Notation: $I := (a,b) := \{t \in \mathbf{R} : a < t < b\}$.
$C^k(I) := \{f\colon I \to \mathbf{R} : f, f' \cdots f^{(k)}$ exist and are continuous $\}$.

Theorem: Given $p, q \in C^0(I)$ and $f \in C^2(I)$, let $L(f) := f'' + pf' + qf$.

1. L is a linear transformation: $C^2(I) \to C^0(I)$.

2. Given any $g \in C^0(I)$, $t_0 \in I$, $y_0, y_1 \in \mathbf{R}$, there exists a unique $f \in C^2(I)$ such that

   (a) $L(f) = g$
   (b) $f(t_0) = y_0$ and $f'(t_0) = y_1$.

Corollary: L is surjective, and its nullspace has dimension 2.

Corollary: If (f_1, f_2) is a linearly independent pair of elements in $NS(L)$, then (f_1, f_2) forms a basis for $NS(L)$.

Corollary: Let (f_1, f_2) be a pair of elements of $NS(L)$, and let $W(f_1, f_2) := f_1 f_2' - f_2 f_1'$. Then if $W(f_1, f_2)(t_0) \neq 0$ for any $t_0 \in I$, then (f_1, f_2) is a basis for $NS(L)$ and $W(f_1, f_2)(t) \neq 0$ for all $t \in I$.

Theorem: If (f_1, f_2) is a pair of elements of $NS(L)$ and $W := W(f_1, f_2)$, then $W' + pW = 0$. Hence $W = ce^{-P}$, where c is some constant and $P' = p$.

Constant coefficients

If p and q are constant and $g = 0$, a fundamental solution set to the equation can be found easily. Thus we fix constants b and c and consider the equation

$$f'' + bf' + cf = 0.$$

One method is to try exponential solutions of the form e^{rt}.

Claim: Suppose $r^2 + br + c = 0$. Then e^{rt} is a solution of $f'' + bf' + cf = 0$.

If the equation $x^2 + bx + c = 0$, has distinct roots r_1 and r_2, one gets a fundamental solution set $(e^{r_1 t}, e^{r_2 t})$. If these are complex, one can use Euler's formula $e^{x+iy} = e^x(\cos y + i \sin y)$. If there is only one root r, then (e^{rt}, te^{rt}) is a fundamental solution set.