Jordan Normal form of $2 \times 2$ matrices

**Theorem:** Let $A$ be a $2 \times 2$ matrix. Then exists an invertible matrix $S$ such that $A = SBS^{-1}$, where $B$ has one of the following forms:

1. $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$
2. $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

The matrix $B$ is called the *Jordan normal form* of $A$.

**Formula:** The characteristic polynomial $p_A(X)$ of $A$ is given by

$$p_A(X) = X^2 - X\text{tr}A + \text{det}A.$$

The eigenvalues of $A$ are given by

$$\lambda = \frac{\text{tr}(A) \pm \sqrt{(\text{tr}A)^2 - 4 \det A}}{2}.$$
Case 1: \((\text{tr} A)^2 \neq 4 \det A\)

In this case, the eigenvalues are distinct, and the matrix is diagonalizable. If \((\text{tr} A)^2 > 4 \det A\), the eigenvalues are real; otherwise they are complex.

Case 2: \((\text{tr} A)^2 = 4 \det A\).

In this case, there is a unique eigenvalue, namely \(\lambda = 1/2(\text{tr} A)\), and there are two subcases.

Case 2a: \(A\) is already diagonal: \(A = \lambda I\).

Case 2b: \(A\) is not diagonal.

In this case \(A\) is not diagonalizable. To find \(S\) as in the theorem, we proceed as follows. Let \(N = A - \lambda I\). Then \(N\) has zero as its unique eigenvalue.

**Lemma:** Let \(N\) be a \(2 \times 2\) matrix whose only eigenvalue is 0. Then \(N^2 = 0\).

Proof: The characteristic polynomial \(p_N(X)\) of \(N\) must be \(X^2\), since 0 is its only root. Thus \(\text{tr} N = \det N = 0\). If \(N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}\), then \(a + d = 0\) and
ad = bd, so

\[ N^2 = \begin{pmatrix} a^2 + bc & ab + bd \\ ac + cd & bc + d^2 \end{pmatrix} = \begin{pmatrix} a^2 + ad & ab + bd \\ ac + cd & ad + d^2 \end{pmatrix} = \begin{pmatrix} a(a + d) & b(a + d) \\ c(a + d) & d(a + d) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}. \]

At this point, we can already conclude the following

**Corollary:** Let \( A \) be a \( 2 \times 2 \) matrix which is not diagonalizable. Then there exist matrices \( D \) and \( N \), where \( D \) is diagonal and \( N \) is nilpotent, with \( A = D + N \).

To find the Jordan normal form we proceed as follows. Since \( A \) is not diagonal, \( N \) is not zero, so one of its columns is not zero.

**Subcase:** If the second column is not zero, let \( \mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \). Note that \( \mathbf{v}_1 := N\mathbf{v}_2 \) is the second column of \( N \).

**Subcase:** If the second column is zero, let \( \mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \).
Then \( v_1 := Nv_2 \) is the first column of \( N \), which is not zero.

In either case, let \( S \) be the matrix obtained by concatenating the columns \( v_1 \) and \( v_2 \). Note that \( Nv_1 = N^2v_2 = 0 \), and since \( v_1 = Nv_2 \neq 0 \), it follows that \( v_2 \) is not a multiple of \( v_1 \). Then \( S \) is invertible, and it is easy to check that \( NS = S \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \). From this it follows that

\[
N = S \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} S^{-1}
\]

and hence that

\[
A = S \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix} S^{-1}
\]