Jordan Normal form of 2×2 matrices

Theorem: Let A be a 2×2 matrix. Then exists an invertible matrix S such that $A = SBS^{-1}$, where B has one of the following forms:

1. $B = \begin{pmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{pmatrix}$

2. $B = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$

The matrix B is called the *Jordan normal form* of A.

Formula: The characteristic polynomial $p_A(X)$ of A is given by

$$p_A(X) = X^2 - X\mathrm{tr}A + \det A.$$

The eigenvalues of A are given by

$$\lambda = \frac{\mathrm{tr}(A) \pm \sqrt{(\mathrm{tr}A)^2 - 4\det A}}{2}.$$

Case 1: $(\mathrm{tr}A)^2 \neq 4\det A$

In this case, the eigenvalues are distinct, and the matrix is diagonalizable. If $(\mathrm{tr}A)^2 > 4\det A$, the eigenvalues are real; otherwise they are complex.

Case 2: $(\mathrm{tr}A)^2 = 4\det A$.

In this case, there is a unique eigenvalue, namely $\lambda = 1/2(\mathrm{tr}A)$, and there are two subcases.

Case 2a: A is already diagonal: $A = \lambda I$.

Case 2b: A is not diagonal.

In this case A is not diagonalizable. To find S as in the theorem, we proceed as follows. Let $N = A - \lambda I$. Then N has zero as its unique eigenvalue.

Lemma: Let N be a 2×2 matrix whose only eigenvalue is 0. Then $N^2 = 0$.
Proof: The characteristic polynomial $p_N(X)$ of N must be X^2, since 0 is its only root. Thus $\mathrm{tr}N = \det N = 0$. If $N = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, then $a + d = 0$ and

$ad = bd$, so

$$N^2 = \begin{pmatrix} a^2+bc & ab+bd \\ ac+cd & bc+d^2 \end{pmatrix} = \begin{pmatrix} a^2+ad & ab+bd \\ ac+cd & ad+d^2 \end{pmatrix}$$

$$= \begin{pmatrix} a(a+d) & b(a+d) \\ c(a+d) & d(a+d) \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.$$

At this point, we can already conclude the following

Corollary: Let A be a 2×2 matrix which is not diagonalizable. Then there exist matrices D and N, where D is diagonal and N is nilpotent, with $A = D + N$.

To find the Jordan normal form we proceed as follows. Since A is not diagonal, N is not zero, so one of its columns is not zero.

Subcase: If the second column is not zero, let $\mathbf{v}_2 = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$. Note that $\mathbf{v}_1 := N\mathbf{v}_2$ is the second column of N.

Subcase: If the second column is zero, let $\mathbf{v}_2 = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$.

Then $\mathbf{v}_1 := N\mathbf{v}_2$ is the first column of N, which is not zero.

In either case, let S be the matrix obtained by concatenating the columns $\mathbf{v}_1$ and $\mathbf{v}_2$. Note that $N\mathbf{v}_1 = N^2\mathbf{v}_2 = \mathbf{0}$, and since $\mathbf{v}_1 = N\mathbf{v}_2 \neq \mathbf{0}$, it follows that $\mathbf{v}_2$ is not a multiple of $\mathbf{v}_1$. Then S is invertible, and it is easy to check that $NS = S\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. From this it follows that

$$N = S\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}S^{-1}$$

and hence that

$$A = S\begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}S^{-1}$$