Matrix Inversion, Elementary matrices

Definition 1. *Let A be an $n \times n$ matrix. Then A is $invertible$ if there exists a matrix A^{-1} such that $AA^{-1} = I_n$ and $A^{-1}A = I_n$.*

If A^{-1} exists, it is unique; this follows from the associative property of matrix multiplication.

Example: elementary matrices are invertible.

Definition 2. *An $elementary\ matrix$ is a matrix obtained by applying an elementary row operation to the identity matrix.*

The following result is an easy computation but very much worth remembering.

Formula 3. *Let R denote an ERO (elementary row operation) and let $E := (RI_m)$ denote the corresponding $m \times m$ elementary matrix. Then for any $A \in M_{mn}$,*

$$EA = R(A).$$

That is, the matrix product EA is obtained by applying the row operation R to A.

Since any ERO R can be reversed by an ERO R' of the same type, it follows that the inverse of an elementary matrix E is an elementary matrix E' of the same type.

Theorem 4. *Let A be an $n \times n$ matrix. Then the following are equivalent.*

1. *A is invertible*
2. *A is nonsingular: $AX = 0$ implies $X = 0$.*
3. *A is row equivalent to I_n.*
4. *A can be written as a product of elementary matrices.*

The key steps in the proof are (2) implies (3) and (3) implies (4), which give a fairly efficient procedure for computing A^{-1} when it exists.

Corollary 5. *Suppose $A \in M_{n\times n}$ and there exists $A' \in M_{n\times n}$ such that $A'A = I_n$. Then A is invertible and $A' = A^{-1}$.*

Proof: Suppose $AX = 0$. Then

$$X = I_n X = (A'A)X = A'(AX) = A'0 = 0.$$

This shows that A is nonsingular, hence by the theorem it is invertible. Let A^{-1} be its inverse.

$$A' = A'I_n = A'(AA^{-1}) = (A'A)A^{-1} = I_n A^{-1} = A^{-1}.$$

Computing matrix inverses

Suppose A is invertible and $R_1, \cdots R_n$ is a sequence of ERO's which puts A into reduced row echelon form. Then if E_i is the elemenatary matrix corresponding to R_i,

$$E_n E_{n-1} \cdots E_1 A = I_n, \quad \text{hence}$$

$$A^{-1} = E_n E_{n-1} \cdots E_1 = R_n R_{n-1} \cdots R_1 I_n.$$

An easy way to do this computationally is to apply the row operations simulataneously to A and to I_n. Equivalently, put I_n to the right of A to make an $n \times 2n$ matrix and row reduce that.

Example: $\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & 2 \\ -2 & 1 & 1 \end{pmatrix}^{-1} = \begin{pmatrix} -1 & 1 & -1 \\ -4 & 3 & -2 \\ 2 & -1 & 1 \end{pmatrix}$

$$\left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ -2 & 1 & 1 & 0 & 0 & 0 \end{array}\right)$$

$$\xrightarrow{E_{31}(2)} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 1 & 3 & 2 & 0 & 1 \end{array}\right)$$

$$\xrightarrow{E_{32}(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 2 & 0 & 1 & 0 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{E_{23}(-2)} \left(\begin{array}{ccc|ccc} 1 & 0 & 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & -4 & 3 & -2 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right)$$

$$\xrightarrow{E_{13}(-1)} \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & -1 & 1 & -1 \\ 0 & 1 & 0 & -4 & 3 & -2 \\ 0 & 0 & 1 & 2 & -1 & 1 \end{array}\right)$$