Inner products

An inner product space is a vector space with a notion of magnitude and othogonality. Precisely:

Definition: Let V be a vector space. An *inner product* on V is an operation which assigns to any pair $(\mathbf{v}, \mathbf{w})$ of elements of V a number $(\mathbf{v}|\mathbf{w}) \in \mathbf{R}$ such that:

1. $(\mathbf{v}|\mathbf{w}) = (\mathbf{w}|\mathbf{v})$ for $\mathbf{v}, \mathbf{w} \in V$.
2. $(\mathbf{v} + \mathbf{v}'|\mathbf{w}) = (\mathbf{v}|\mathbf{w}) + (\mathbf{v}'|\mathbf{w})$, for $\mathbf{v}, \mathbf{v}', \mathbf{w} \in V$
3. $(a\mathbf{v}|\mathbf{w}) = a(\mathbf{v}|\mathbf{w}) = (\mathbf{v}|a\mathbf{w})$ for $a \in \mathbf{R}, \mathbf{v}, \mathbf{w} \in V$.
4. $(\mathbf{v}|\mathbf{v}) > 0$ for $\mathbf{v} \in V$ and $\mathbf{v} \neq 0$.

If $\mathbf{v} \in V$, the *magnitude* (or *norm* or *length*) of $\mathbf{v}$ is by definition

$$||\mathbf{v}|| := \sqrt{(\mathbf{v}|\mathbf{v})}.$$

Two vectors $\mathbf{v}, \mathbf{w} \in W$ are said to be *orthogonal* if $(\mathbf{v}|\mathbf{w}) = 0$.

If S is a subset of V,

$$S^{\perp} := \{v \in V : (v|s) = 0 \text{ for all } s \in S\},$$

and is a linear subspace of V.

Fundamental formulas

Let $\mathbf{v}$ and $\mathbf{w}$ be vectors in an inner product space V.

1. If $\mathbf{v} \perp \mathbf{w}$, $||\mathbf{v}+\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2$.
   (Pythagorean theorem)

2. $||\mathbf{v}+\mathbf{w}||^2 = ||\mathbf{v}||^2 + ||\mathbf{w}||^2 + 2(\mathbf{v}|\mathbf{w})$.
   (Law of cosines).

3. $|(\mathbf{v}|\mathbf{w})| \leq ||\mathbf{v}||||\mathbf{w}||$.
   (Cauchy-Schwartz inequality.)

4. $||\mathbf{v}+\mathbf{w}|| \leq ||\mathbf{v}|| + ||\mathbf{w}||$.
   (Triangle inequality.)