Orthogonal projection

Theorem: Let V be an inner product space, let $\mathbf{v}$ be a vector in V, and let W be a finite dimensional linear subspace of V. Then there is a unique vector $\mathbf{w} := \pi_W(\mathbf{v}) \in W$ such that $\mathbf{v} - \mathbf{w} \in W^\perp$. Furthermore:

- $\pi_W(\mathbf{v})$ is the vector in W which is closest to $\mathbf{v}$.

- Let $\pi_W^\perp(\mathbf{v}) := \mathbf{v} - \pi_W(\mathbf{v})$. Then

$$\mathbf{v} = \pi_W(\mathbf{v}) + \pi_W^\perp(\mathbf{v}),$$

and $\pi_W(\mathbf{v}) \in W$ and $\pi_W^\perp(\mathbf{v}) \in W^\perp$.

- The maps

$$\pi_W : V \to W \quad \text{a}nd \quad \pi_W^\perp : V \to W^\perp$$

are linear transformations.

Orthogonality

Definition: A sequence of vectors $(\mathbf{w}_1, \ldots \mathbf{w}_m)$ in an inner product space V is *orthogonal* if $(\mathbf{w}_i|\mathbf{w}_j) = 0$ whenever $i \neq j$.

Theorem: Let $S := (\mathbf{w}_1, \ldots \mathbf{w}_m)$ be an orthogonal sequence of nonzero vectors in V.

• The sequence S is automatically linearly independent, hence is an ordered basis for $W := \operatorname{span} S$.

• If $\mathbf{v}$ in V, then

$$\pi_W(\mathbf{v}) = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots a_m\mathbf{w}_m, \quad \text{where}$$

$$a_i := \frac{(\mathbf{v}|\mathbf{w}_i)}{(\mathbf{w}_i|\mathbf{w}_i)}$$

Gram-Schmidt

Algorithm: Let V be an inner product space, let $(\mathbf{v}_1, \dots \mathbf{v}_m)$ be a linearly independent sequence of vectors in V, and let $W_i := \mathrm{span}(\mathbf{v}_1, \dots \mathbf{v}_i)$ for i with $1 \leq i \leq m$. Then there is an *orthogonal* sequence $(\mathbf{w}_1, \dots \mathbf{w}_m)$ such that for each i, $(\mathbf{w}_1, \dots \mathbf{w}_i)$ is an orthogonal basis for W_i.

This is computed step by step, starting by taking $\mathbf{w}_1 := \mathbf{v}_1$. Now assume that $(\mathbf{w}_1, \dots \mathbf{w}_i)$ is already computed. Then we can compute π_{W_i} and $\pi^{\perp}_{W_i}$. Hence we can compute

$$\mathbf{w}_{i+1} := \pi^{\perp}_{W_i}(\mathbf{v}_{i+1}) = \mathbf{v}_{i+1} - (a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots a_i\mathbf{w}_i)\,,$$

$$\text{where} \qquad a_i := \frac{(\mathbf{v}|\mathbf{w}_i)}{(\mathbf{w}_i|\mathbf{w}_i)}.$$

It is easy to check that this works.