Gauss Elimination

Row Echelon Form

Let A be an $m \times n$ matrix. Then A has m rows, which we denote by $R_1, R_2, \cdots R_m$. We say that a row R_i is **zero** if each of its entries is zero, otherwise we say that it is **nonzero**. Let R_i be a nonzero row of A, say $R_i = (a_{i1}, a_{i2}, \cdots a_{in})$. The **pivot**, or **leading entry**, of R_i is its first nonzero entry, starting from the left. More important than its value is the column in which it occurs:

Definition: Let R_i be a nonzero row of a matrix A. Then ℓ_i, the **leading index** of R_i, is the smallest j such that $a_{ij} \neq 0$:

$$R_i = (0\ 0 \cdots a_{i,\ell_i} a_{i,\ell_{i+1}} \cdots a_{in})$$

Definition: Let A be an $m \times n$ matrix. Then A is in **row echelon form** if

1. All the nonzero rows are at the top. Thus, for some integer m', with $0 \leq m' \leq m$, the rows $R_1, \cdots R_{m'}$ are nonzero, but the rows $R_{m'+1}, \cdots R_m$ are zero.

2. The leading indices of the nonzero rows are increasing:

$$\ell_1 < \ell_2 < \cdots \ell_{m'}.$$

Theorem: Let A be the $m \times n + 1$ augmented matrix corresponding to a system of m linear equations in n unknowns. Assume A is in row echelon form. Then the solution set $SS(A)$ can be determined as follows.

1. If the pivot of the last nonzero row $R_{m'}$ is in the last column, that is if $\ell_{m'} = n + 1$, then the equations are inconsistent, and the solution set is empty.

2. Suppose that $\ell_{m'} \neq 0$. Then $SS(A)$ is not empty, and can be computed by back subsitution. Expliclty, m' of the variables, namely the variables $x_{\ell_1} \cdots x_{\ell_{m'}}$ corresponding to the leading indices, are "dependent variables" and can be solved for in terms of the remaining $n - m'$

"independent variables," starting at the bottom and working up. For example the last nonzero row $R_{m'}$ gives an equation for the dependent variable $x_{\ell_{m'}}$ in terms of the remaining independent variables x_j with $j > \ell_{m'}$ (if any). Working from the bottom up, one then uses each nonzero row R_i to find the dependent variable x_{ℓ_i} in terms of the variables x_j with $j > \ell_i$. Note that such an x_j is either free (if j is not a leading index), or is already computed (if j is a leading index of some row).

Elementary Row Operations

Let A be an $m \times n$ matrix, with rows $R_1, \cdots R_m$. There are three types of elementary row operations.

1. $E_{ij}(\lambda)$: Given two distinct integers i and j and a number λ, add λ times R_j to R_i.

2. E_{ij}: Given two integers i and j, interchange rows R_i and R_j.

3. $E_i(\lambda)$: Given an integer i and a nonzero number λ, multiply R_i by λ.

Theorem:

1. If a matrix A' can be obtained from a matrix A by a sequence of elementary row operations, then the solution sets of the systems of equations corresponding to A and A' are the same.

2. If A is any matrix, then there is a sequence of elementary row operations which, when applied to A, yields a matrix in row echelon form.