Fourier Series: Summary

Fix $L > 0$ and let $\ell := 2L$ and $I := [-L, L]$.

$$\mathcal{P} := \{f\colon \mathbf{R} \to \mathbf{R} : f(x+\ell) = f(x) \text{ for all } x \in \mathbf{R}\}.$$

$$\mathcal{P}^0 := \{f \in \mathcal{P} : \text{ f is continuous } \}.$$

$$(f|g) := \frac{1}{L}\int_{-L}^{L} fg \quad \text{for } f, g \in C^0(I) \text{ or in } \mathcal{P}^0.$$

$$u_n(x) := \cos(\frac{n\pi}{L}x), \quad n = 0, 1, 2, \ldots$$

$$v_n(x) := \sin(\frac{n\pi}{L}x), \quad n = 1, 2, 3 \ldots$$

$$W_N := span\{u_0, u_1, \ldots u_N, v_1, v_2, \ldots v_N\}$$

Theorem: W_N is a linear subspace of $\mathcal{P}^0$ of dimension $2N + 1$, and $(u_0, \ldots u_N, v_0, \ldots v_N)$ is an orthogonal basis. Moreover

$$||u_n||^2 = ||v_n||^2 = 1 \text{ for } n > 0, \text{ and } ||u_0||^2 = 2.$$

Corollary: If $f \in C^0(I)$, let

$$a_n := (f|u_n) = \frac{1}{L}\int_{-L}^{L} f(x)u_n(x)dx$$

$$b_n := (f|v_n) = \frac{1}{L}\int_{-L}^{L} f(x)v_n(x)dx$$

$$\pi_N(f) := \frac{a_0}{2}u_0 + a_1u_1 + \cdots a_Nu_N + b_1v_1 + b_n \cdots v_N.$$

Then $\pi_N(f)$ is the orthogonal projection of f onto W_N.

Proofs

Let

$$\epsilon_n(x) := u_n(x) + iv_n(x) = e^{\frac{in\pi x}{L}}$$

for $n = 0, \pm 1, \pm 2, \ldots$. Note that $\epsilon_n \epsilon_m = \epsilon_{n+m}$ and $\epsilon_{-n} = \bar{\epsilon}_n$.

Lemma:

$$(\epsilon_m | \epsilon_n) = \begin{cases} 0 & \text{if } n \neq m \\ 2 & \text{if } n = m. \end{cases}$$

Proof:

$$\begin{aligned} (\epsilon_m | \epsilon_n) &= \frac{1}{L} \int_{-L}^{L} \epsilon_m(x) \bar{\epsilon}_n(x) dx \\ &= \frac{1}{L} \int_{-L}^{L} e^{\frac{\pi i (m-n) x}{L}} dx \end{aligned}$$

If $n = m$, this is $\frac{1}{L} \int_{-L}^{L} dx = 2$. If $n \neq m$, it is

$$\frac{1}{L} \frac{L}{(m-n)\pi i} e^{\frac{\pi i (m-n) x}{L}} \Big|_{-L}^{L} = 0.$$

Hence if $n \neq m$,

$$
\begin{aligned}
(u_n|u_m) &= (\frac{\epsilon_n + \epsilon_{-n}}{2}|\frac{\epsilon_m + \epsilon_{-m}}{2}) \\
&= \frac{1}{4}((\epsilon_n|\epsilon_m) + (\epsilon_n|\epsilon_{-m} + (\epsilon_{-n}|\epsilon_m) + (\epsilon_{-n}|\epsilon_{-m})) \\
&= 0.
\end{aligned}
$$

In the same way, one can show that $(u_n|v_m) = (v_n|v_m) = 0$, assuming $n \neq m$.

Now if $n > 0$,

$$
\begin{aligned}
(u_n|v_n) &= (\frac{\epsilon_n + \epsilon_{-n}}{2}|\frac{\epsilon_n - \epsilon_{-n}}{2i}) \\
&= \frac{-1}{4i}((\epsilon_n|\epsilon_n) + (\epsilon_{-n}|\epsilon_n) - (\epsilon_n|\epsilon_{-n}) - (\epsilon_{-n}|\epsilon_{-n})) \\
&= (2 + 0 + 0 - 2) \\
&= 0.
\end{aligned}
$$

Finally, if $n > 0$, $\epsilon_n \perp \epsilon_{-n}$, so

$$
||u_n||^2 = \frac{1}{4}||\epsilon_n + \epsilon_{-n}||^2
$$

$$
\begin{aligned}
&= \frac{1}{4}||\epsilon_n||^2 + ||\epsilon_{-n}||^2 \\
&= \frac{1}{4}(2+2) \\
&= 1
\end{aligned}
$$

The proof that $||v_n||^2 = 1$ is similar.