Classification of phase trajectories of ODE's

Let us consider the trajectories of the vector-valued functions X such that $X' = AX$, where $A := \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is a 2×2 matrix. Of course, the constant trajectory $X(t) = 0$ for all t is one such solution. How the others look depend on the matrix A, and in particular on its eigenvalues. The characteristic polynomial of A is

$$p_A(\lambda) = \lambda^2 - (a+d)\lambda + (ad - bc),$$

so the eigenvalues of A are given by

$$\lambda = \frac{a + d \pm \sqrt{(a-d)^2 + 4bc}}{2}$$

Here are the main cases:

Saddle point: This is the case in which the eigenvalues are real and have opposite sign, which is true if and only if $\det A > 0$ and $(a-d)^2 + 4bc > 0$. In this case the trajectories are asymptotic to the positive eigenspace as t approaches infinity, and to the negative eigenspace as t approaches negative infinity.

Node: This is the case in which the eigenvalues are nonzero, real and distinct and have the same sign, which is ture if and only if $\det A < 0$ and $(a-d)^2 + 4bc > 0$. If the eigenvalues are positive, the trajectories move away from the origin with increasing time, and the node is said to be "unstable." If the eigenvalues are negative, the the trajectories approach the origin as t approaches infinity, and the node is said to be "stable." In either case, the trajectories are tangent to the eigenspace corresponding to the eigenvalue with smaller absolute value as they approach the origin.

Degenerate Node (A diagonalizable) There are various special cases called "degenerate nodes." For example, if the eigenvalues are equal and not zero and the matrix is diagonal, the trajectories are rays emanating from the origin. If one of the eigenvalues is zero, the trajectories are rays parallel to the nonzero eigenspace, but also each point on the zero eigenspace is a trajectory corresponding to a constant equilibrium solution. If both eigenvalues are zero and the matrix is diagonalizable, it is the zero matrix, and every trajectory is constant.

Spiral point: This is the case arising from complex eigenvalues. Thus $(a-d)^2 + 4bc$ is negative, and so b and c have opposite signs. There are several possibilities:

- If $(a+d) > 0$, the trajectories spiral out with increasing time. If $(a+d) < 0$, they spiral in with increasing time. If $a + d = 0$, they are periodic, and form ellipses.

- If $c > 0$ and $b < 0$, the trajectories move counterclockwise with time. If $c < 0$ and $b > 0$, they move clockwise with time. This is easy to remember if you draw the relevant part of the vector field: for example, the vector $\begin{pmatrix} a \\ c \end{pmatrix}$, placed with its origin at $(1, 0)$, will point you in the right direction.

Improper node This corresponds to nondiagonalizable matrices, and is an interesting transitional case. In this case there is a single eigenvalue λ and a one dimensional eigenspace, that is, a line L. If λ is positive, the trajectories move away from the origin as time increases and are unbounded. The slope of the trajectory approaches the slope of the line L (but is not asymptotic to it, contrary to what it says in the book). As time approaches negative infinity, the trajectory approaches the origin and becomes tangent to the line L. The trajectories make a semiloop, as you can see from the graph (except of course for the trajectory which lies on the line L.)

For example, suppose $A = \begin{pmatrix} \lambda & 1 \\ 0 & \lambda \end{pmatrix}$, with $\lambda > 0$. Then L is the x-axis. Let $N := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}$. The fundamental solutions are given by the columns of the matrix

$$e^{tA} = e^{\lambda t} e^{tN} = \begin{pmatrix} e^{\lambda t} & te^{\lambda t} \\ 0 & e^{\lambda t} \end{pmatrix}.$$

Thus, the solutions are given by $X = e^{tA} \begin{pmatrix} c_1 \\ c_2 \end{pmatrix}$, i.e.:

$$\begin{aligned} x_1 &= c_1 e^{\lambda t} + c_2 t e^{\lambda t} \\ x_2 &= c_2 e^{\lambda t} \end{aligned}$$

Then

$$\begin{aligned} x_1' &= c_1\lambda e^{\lambda t} + c_2\lambda t e^{\lambda t} + c_2 e^{\lambda t} \\ x_2' &= c_2\lambda e^{\lambda t} \end{aligned}$$

Note that x_1 and x_2 are unbounded as t approaches infinity. The slope of the trajectory is given by the slope of the vector $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix}$, i.e., by

$$x_2'/x_1' = \frac{c_2}{(c_1 + c_2 t + \lambda^{-1} c_2)}$$

As t approaches plus or minus infinity, this tends to zero. In other words, the trajectory is tangent to the x-axis as it comes in toward the origin, and gradually becomes parallel to the x-axis as t approaches infinity. Note that if $c_2 \neq 0$, the slope changes its sign at some point. Thus x_2 is always increasing, but x_1 changes from decreasing to increasing at some t. This explains the semiloop.