The Cauchy Schwartz Inequality

Theorem. Let V be an inner product space. Then if $\mathbf{v}, \mathbf{w} \in V$,

$$|(\mathbf{v}|\mathbf{w})| \leq ||\mathbf{v}||||\mathbf{w}||.$$

Proof: If $\mathbf{w} = \mathbf{0}$, this is clearly true, since $(\mathbf{v}|\mathbf{0}) = 0$. Suppose $\mathbf{w} \neq \mathbf{0}$. We choose $t \in \mathbf{R}$ so that $t\mathbf{w}$ is the orthogonal projection of $\mathbf{v}$ on the line W spanned by $\mathbf{w}$. That is, we choose t so that

$$\mathbf{v}' := \mathbf{v} - t\mathbf{w} \in \mathbf{w}^{\perp}.$$

This just says that $(\mathbf{v} - t\mathbf{w}|\mathbf{w}) = 0$, that is

$$(\mathbf{v}|\mathbf{w}) = t(\mathbf{w}|\mathbf{w}).$$

Taking absolute values, we find

$$* \quad |(\mathbf{v}|\mathbf{w})| = |t| \, ||\mathbf{w}||^2.$$

Now $\mathbf{v} = t\mathbf{w} + \mathbf{v}'$ with $\mathbf{v}' \perp t\mathbf{w}$. Hence by the Pythagorean theorem,

$$||\mathbf{v}|| \geq ||t\mathbf{w}|| = |t|\,||\mathbf{w}||.$$

Multiplying both sides by $||\mathbf{w}||$ and using *, we find

$$||\mathbf{v}||||\mathbf{w}|| \geq |t|\,||\mathbf{w}||^2 = |(\mathbf{v}|\mathbf{w})|.$$