Initial Value problem

Recall that the existence and uniqueness theorem says the following:

Theorem 1: Let p, q and be real numbers. Let

$$W := \{y : y'' + py' + qy = 0\}.$$

1. W is a linear subspace of the space of all functions, and has dimension 2.

2. For any t_0, the map

$$W \to \mathbf{R}^2 : y \mapsto \begin{pmatrix} y(t_0) \\ y'(t_0) \end{pmatrix}$$

   is an isomorphism. That is, it is linear, and its matrix representative with respect to any basis is invertible.

Boundary value problems

Now choose t_0, t_1 with $t_1 > t_0$, and consider the boundary value mapping

$$B\colon W \to \mathbf{R}^2 : y \mapsto \begin{pmatrix} y(t_0) \\ y(t_1) \end{pmatrix}$$

This is again a linear transformation, but it *not* always an isomorphism.

Theorem 2: In the above situation, let $\theta := \sqrt{4q - p^2}$ and let $\ell := t_1 - t_0$. Then there are two possible cases:

1. B has rank two, and so is an isomorphism. This happens whenever $p^2 \geq 4q$ or whenever $4q > p^2$ and $\ell \neq n\pi/\theta$ for some integer n. In either of these cases, given any y_0, y_1, there is a unique solution $y \in W$ with $y(t_i) = y_i$.

2. B has rank one. This happens when $4q > p^2$ and ℓ is an integer n times π/θ. In this case, the set of

(y_0, y_1) in $\mathbf{R}^2$ for which a solution exists is a one-dimensional linear subspace of $\mathbf{R}^2$, and the set of all y such that $y(t_0) = 0 = y(t_1)$ is a one-dimensional linear subspace of W, with basis $\sin(\theta t) = \sin(\frac{n\pi}{\ell} t)$.

See the text for a discussion of the boundary value problem given by taking the derivatives of y at t_0 and t_1 in place of its values.

Periodic boundary value problem

Here is another boundary value problem we can consider. Let I denote the interval (t_0, t_1), and for $y \in W$ let

$$\Delta_I(y) := \begin{pmatrix} y(t_1) - y(t_0) \\ y'(t_1) - y'(t_0) \end{pmatrix}$$

and let $P_I := \{y \in W : y(t_0) = y(t_1), y'(t_0) = y'(t_1)\}$ be the nullsapce of Δ_I.

Theorem 3: In the above situation, either:

1. $q \neq 0$, $p = 0$, and $\ell = \frac{2n\pi}{\sqrt{q}}$ for some integer n. In this case Δ_I is zero, and $(\cos(\frac{2n\pi}{\ell}t), \sin(\frac{2n\pi}{\ell}t))$ is a basis for P_I

2. $q = 0$ and $p \neq 0$. In this case, Δ_I has rank one, and the constant solution 1 is a basis for P_I.

3. In all other cases, Δ_I is an isomorphism and $P_I = \{0\}$.

Proof of Theorem 2

Recall that the roots of characteristic equation are given by

$$(*) \qquad \lambda = \frac{-p \pm \sqrt{p^2 - 4q}}{2}$$

For simplicity we assume that $t_0 = 0$. First suppose that $p^2 \neq 4q$. In this case, the roots λ_1 and λ_2 of $(*)$ are distinct, and $(e^{\lambda_1 t}, e^{\lambda_2 t})$ is a basis for W. The matrix for B with respect to this basis is $\begin{pmatrix} 1 & 1 \\ e^{\lambda_1 \ell} & e^{\lambda_2 \ell} \end{pmatrix}$ and its determinant is $e^{\lambda_2 \ell} - e^{\lambda_1 \ell}$. This is zero if and only if $e^{\ell\lambda_2 - \ell\lambda_1} = 1$. By $(*)$, $\lambda_2 - \lambda_1 = 2i\theta$, so we see that B is an isomorphism unless $e^{2i\theta\ell} = 1$. This is true if and only if $\ell\theta$ is an integer multiple of π.

There remains the possibility that $\theta = 0$. In this case a fundamental solution set is $(e^{\lambda t}, te^{\lambda t})$, and the matrix for B is $\begin{pmatrix} 1 & 0 \\ e^{\ell\lambda} & \ell e^{\lambda\ell} \end{pmatrix}$ is always invertible.

Proof of theorem 3

Again we first treat the case in which the roots λ_1 and λ_2 are distinct, so $(e^{\lambda_1 t}, e^{\lambda_2 t})$ is a basis for W. The determinant of the matrix for Δ_I is then

$$\begin{vmatrix} e^{\lambda_1 \ell} - 1 & e^{\lambda_2 \ell} - 1 \\ \lambda_1(e^{\lambda_1 \ell} - 1) & \lambda_2(e^{\lambda_2 \ell} - 1) \end{vmatrix} = (\lambda_2 - \lambda_1)(e^{\lambda_1 \ell} - 1)(e^{\lambda_2 \ell} - 1).$$

Since $\lambda_2 \neq \lambda_1$, this is zero if and only if λ_1 or λ_2 satisfies $e^{\lambda} = 1$, so $\lambda = 2n\pi i$ for some integer n.

Suppose this is the case. If $n > 0$ then λ_1 and λ_2 are purely imaginary, so $p = 0$ and $\lambda_i = \pm i\sqrt{q}$. Then the matrix Δ_I is zero, so $P_I = W$, which has real basis $(\cos(\frac{2n\pi}{\ell}), \sin(\frac{2n\pi}{\ell}))$. If $n = 0$, then one root is zero and the other nonzero and real. In this case $q = 0$ and the matrix for Δ_I has rank one. Hence the P_I consists just of the constant solutions.

If there is only one eigenvalue λ, then a basis for W is

$(e^{\lambda t}, te^{\lambda t})$, and the computation is the following:

$$\begin{vmatrix} e^{\lambda\ell} - 1 & \ell e^{\lambda\ell} \\ \lambda(e^{\lambda\ell} - 1) & \lambda\ell e^{\lambda\ell} + e^{\lambda\ell} - 1 \end{vmatrix} =$$

$$(e^{\lambda\ell}-1)(\lambda\ell e^{\lambda\ell}+e^{\lambda\ell}-1)-\ell e^{\lambda\ell}(\lambda e^{\lambda\ell}-1) = (e^{\lambda\ell}-1)^2.$$

Since λ is real and $\ell > 0$, this cannot be zero unless $\lambda = 0$.