Basis and dimension

Let $S := (\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n)$ be a sequence in a vector space V.

- S *spans* V if every element $\mathbf{v}$ of V is a linear combination of S:

  $$\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_n\mathbf{v}_n$$

  for some sequence $x.$ of numbers.

- S is *linearly independent* if whenever

  $$\mathbf{0} = x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n,$$

  each $x_i = 0$.

- S is an *ordered basis* for V is it spans V and is linearly independent.

Theorem: S is an ordered basis for V if and only if every element of v can be written uniquely as a $\mathbf{v} = x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_n\mathbf{v}_n$, for some (unique) sequence of numbers $x_.$.

Theorem: Every vector space V has a basis. Any two bases for the same V have the same number of elements. This number is called the $dimension$ of V.

Theorem: Let V be a vector space of dimension n and let $S := \{\mathbf{w}_1, \cdots \mathbf{w}_m\}$ be a subset of V.

- If S is linearly independent, it can be completed to a basis for V, and already is a basis if $m = n$.
- If S spans V it contains a basis, and already is basis if $m = n$.
- If W is a linear subspace of V of dimension m, then $m < n$, and $m = n$ if and only if $W = V$.