Spans

Definition: Let S be a nonempty subset of V. Then the *span* of S is the set of all linear combinations of elements of S, i.e., the set of all elements of V that can be written

$$\mathbf{v} = a_1\mathbf{w}_1 + a_2\mathbf{w}_2 + \cdots a_n\mathbf{w}_n$$

where each $a_i \in \mathbf{R}$ and each $\mathbf{w}_i \in S$. The span of the empty set is the set containing just the zero vector.

Theorem: If S is any subset of V, the span of S is the smallest linear subspace of V containing S.

Example Let $S := (\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3)$, where

$$\mathbf{v}_1 := \begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix}, \mathbf{v}_2 := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \mathbf{v}_3 := \begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}.$$

Then $\begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} \in span(\ S)$, since

$$\begin{aligned} \begin{pmatrix} 5 \\ -2 \\ 5 \end{pmatrix} &= 2\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} + 1\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 3\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}, \text{or} \\ &= 5\begin{pmatrix} 1 \\ 0 \\ 1 \end{pmatrix} - 2\begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix} + 0\begin{pmatrix} 1 \\ -1 \\ 1 \end{pmatrix}. \end{aligned}$$

This is because $\mathbf{v}_3 = \mathbf{v}_1 - \mathbf{v}_2$, so

$$\begin{aligned} \mathbf{v} &= 2\mathbf{v}_2 + \mathbf{v}_2 + 3\mathbf{v}_3 \\ &= 2\mathbf{v}_2 + \mathbf{v}_2 + 3(\mathbf{v}_1 - \mathbf{v}_2) \\ &= 5\mathbf{v}_1 - 2\mathbf{v}_2. \end{aligned}$$

The ambiguity is caused by the fact that $\mathbf{v}_1, \mathbf{v}_2, \mathbf{v}_3$ are not "linearly independent."

Definition A sequence $(\mathbf{v}_1, \cdots \mathbf{v}_n)$ is *linearly dependent* if for some i, $\mathbf{v}_i$ is in the span of the remaining vectors $(\mathbf{v}_1, \cdots \mathbf{v}_{i-1}, \mathbf{v}_{i+1}, \cdots \mathbf{v}_n)$. Otherwise, the sequence is *linearly independent*.

Thus the above sequence is not linearly independent.

Theorem: A sequence $(\mathbf{v}_1, \cdots \mathbf{v}_n)$ in a vector space V is linearly dependent if and only if there is a sequence $(x_1, \cdots x_n)$ of numbers, not all zero, such that

$$x_1\mathbf{v}_1 + \cdots x_n\mathbf{v}_n = \mathbf{0}.$$

As an application, here is a useful lemma.

Lemma: If $(\mathbf{w}_1, \cdots \mathbf{w}_m)$ is a linearly independent sequence in V and $v \in V$ but $v \notin Span\ (\mathbf{w}_1, \cdots \mathbf{w}_m)$, then the sequence $(\mathbf{w}_1, \cdots \mathbf{w}_m, \mathbf{v})$ is linearly independent.

Proof: Suppose $x_1\mathbf{w}_1 + x_2\mathbf{w} + 2 + \cdots x_m\mathbf{w}_m + x\mathbf{v} = \mathbf{0}$. If $x \neq 0$, this equation can be solved for $\mathbf{v}$, showing that $\mathbf{v} \in$ span$(w_1, \cdots \mathbf{w}_m)$, a contradiction. Hence $x = 0$. Hence $x_1\mathbf{w}_1 + x_2\mathbf{w}_2 + \cdots x_m\mathbf{w}_m = \mathbf{0}$. Since $(\mathbf{w}_1, \cdots \mathbf{w}_m)$ is linearly independent, each $x_i = 0$.

Example: Let A be an $m \times n$ matrix and let $(\mathbf{v}_1, \cdots \mathbf{v}_n)$ be its columns, a sequence of n vectors in $\mathbf{R}^m$. Then $(\mathbf{v}_1, \cdots \mathbf{v}_n)$ is linearly dependent if and only if the nullspace of A contains a nonzero vector.

For example, in the example we started with, the equation $\mathbf{v}_1 - \mathbf{v}_2 - \mathbf{v}_3$ translates into the matrix equation

$$\begin{pmatrix} 1 & 0 & 1 \\ 0 & 1 & -1 \\ 0 & 1 & -1 \end{pmatrix} \begin{pmatrix} 1 \\ -1 \\ -1 \end{pmatrix} = \mathbf{0}.$$

Corollary: Any sequence of n vectors in $\mathbf{R}^m$ with $n > m$ is linearly dependent.

Proof: Arranging the sequence as the columns of a matrix A, we find an $m \times n$ matrix with $n > m$. We claim that $N(A)$ contains a nonzero vector. Find a row echelon matrix A' which is row equivalent to A. Then $N(A') = N(A)$. Furthermore, the number m' of nonzero rows of A' is less than or equal to m and hence less than n. But $n - m' > 0$ is the number of free variables, and hence there is at least one of these. Thus $N(A')$ contains a nonzero vector.

Key theorem

Theorem: Let $(\mathbf{w}_1, \mathbf{w}_2, \cdots \mathbf{w}_m)$ and $(\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n)$ be sequences in a vector space V. Suppose that $(\mathbf{w}_1, \mathbf{w}_2, \cdots \mathbf{w}_m)$ spans V and $(\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n)$ is linearly independent. Then $n \leq m$.

Proof: We prove that if $n > m$, then $(\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n)$ must be linearly dependent.

Since $(\mathbf{w}_1, \mathbf{w}_2, \cdots \mathbf{w}_m)$ spans V, each $\mathbf{v}_j$ can be written as a linear combination of $(\mathbf{w}_1, \mathbf{w}_2, \cdots \mathbf{w}_m)$. Thus there exists a sequence $(a_{1j}, a_{2j}, \cdots a_{mj})$ such that $\mathbf{v}_j = a_{1j}\mathbf{w}_1 + \cdots a_{mj}\mathbf{w}_m$. Then (a_{ij}) forms an $m \times n$ matrix A, with $n > m$. Hence its nullspace is not trivial: there exists an $X \in M_{n1}$, with $X \neq \mathbf{0}$, such that $AX = 0$. That is, there exists a sequence $(x_1, \cdots x_n)$, with x_i not all zero, such that

$$a_{i1}x_1 + a_{i2}x_2 + \cdots a_{in}x_n = 0$$

for all i. Let us compute $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_n\mathbf{v}_n$.

$$\begin{aligned}
x_1\mathbf{v}_1 &= x_1a_{11}\mathbf{w}_1 + x_1a_{21}\mathbf{w}_2 + \cdots x_1a_{m1}\mathbf{w}_1 \\
x_2\mathbf{v}_2 &= x_2a_{12}\mathbf{w}_1 + x_2a_{22}\mathbf{w}_2 + \cdots x_2a_{m2}\mathbf{w}_2 \\
& \cdots \\
x_n\mathbf{v}_n &= x_na_{1n}\mathbf{w}_1 + x_na_{2n}\mathbf{w}_2 + \cdots x_na_{mn}\mathbf{w}_m \\
\sum x_j\mathbf{v}_j &= y_1\mathbf{w}_1 \quad + \quad y_2\mathbf{w}_2 \; + \; \cdots \quad y_m\mathbf{w}_m
\end{aligned}$$

where

$$\begin{aligned}
y_1 &= x_1a_{11} + x_2a_{12} + \cdots x_na_{1n} = 0 \\
y_2 &= x_1a_{21} + x_2a_{22} + \cdots x_na_{2n} = 0 \\
y_m &= x_1a_{m1} + x_2a_{m2} + \cdots x_na_{mn} = 0
\end{aligned}$$

Hence $x_1\mathbf{v}_1 + x_2\mathbf{v}_2 + \cdots x_n\mathbf{v}_n = \mathbf{0}$, and $(\mathbf{v}_1, \mathbf{v}_2, \cdots \mathbf{v}_n)$ is linearly depedendent.

Theorem: Let V be a vector space of dimension n.

1. Every linearly independent sequence S in V can be extended to a basis for V. If S has n elements, it is already a basis for V.

2. Every spanning sequence S in V contains a basis for V. If S has n elements, it is already a basis for V.

3. If W is a linear subspace of V, then $\dim W \leq \dim V$. If $\dim W = \dim V$, then $W = V$.

Remark: Actually carrying this out in practice may depend on how much information you have about V.

Now suppose that $(\mathbf{w}_1, \cdots \mathbf{w}_m)$ is a linearly independent sequence in V (eg with $m = 0$). If $(\mathbf{w}_1, \cdots \mathbf{w}_m)$ spans V, it is a basis. If not, there is a $\mathbf{v} \in V$ but not in the span, and then the new sequence obtained by adjoining v will still be linearly independent. By the key theorem, this process cannot produce a sequence with more than n elements, and will eventually produce a basis for V. If W is a linear subspace of V, we can start the process by choosing vectors from W as long as possible, until the first m of them span W, and then keep on until we get a basis for all of V. This proves:

Lemma: If W is a linear subspace of V, then there is a basis $(\mathbf{v}_1, \cdots \mathbf{v}_n)$ for V and an integer $m \leq n$ such that $(\mathbf{v}_1, \cdots \mathbf{v}_m)$ is a basis for W.

Now part 3 of the theorem is clear. Necessarily $m \leq n$, and if $m = n$. then $(\mathbf{v}_1, \cdots \mathbf{v}_n)$ spans both W and V, so they must be equal.